Asymptotic Stability and Smooth Lyapunov Functions

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We establish that differential inclusions corresponding to upper semicontinuous multifunctions are strongly asymptotically stable if and only if there exists a smooth Lyapunov function. Since well-known concepts of generalized solutions of differential equations with discontinuous right-hand side can be described in terms of solutions of certain related differential inclusions involving upper semicontinuous multifunctions, this result gives a Lyapunov characterization of asymptotic stability of either Filippov or Krasovskii solutions for differential equations with discontinuous right-hand side. In the study of weak (as opposed to strong) asymptotic stability, the existence of a smooth Lyapunov function is rather exceptional. However, the methods employed in treating the strong case of asymptotic stability are applied to yield a necessary condition for the existence of a smooth Lyapunov function for weakly asymptotically stable differential inclusions; this is an extension to the context of Lyapunov functions of Brockett’s celebrated “covering condition” from continuous feedback stabilization theory.

Key Words: Differential inclusion; strong asymptotic stability; converse Lyapunov theorem; smooth Lyapunov pair; Filippov and Krasovskii solutions; weak asymptotic stability; necessary covering condition.

1. INTRODUCTION

The intensive development of Lyapunov function methods during the 1950s has produced a vast body of fundamental results and techniques on the stability of solutions of the ordinary differential equation

\[ \dot{x}(t) = f(x(t)), \]  

(1)
for continuous $f$, as well as functional–differential equations and some general dynamical systems. Excellent expositions of these results can be found in Antosiewicz [3], Hahn [17], Krasovskii [22], Zubov [37] as well as Lakshmikantham and Leela [24]. Nevertheless, the problem of Lyapunov characterization of stability of differential equations with discontinuous right-hand side has remained open, although N. N. Krasovskii has pointed out [22] as early as 1959 the desirability of such characterizations. A principal obstacle to progress is that the best known concepts of generalized solutions of differential equations with discontinuous right-hand side are formulated in terms of solutions of differential inclusions with upper semicontinuous (but not continuous) multifunctions, making impossible the straightforward application of existing results. The main result of the present paper is a converse Lyapunov function theorem for the strong asymptotic stability of differential inclusions exhibiting merely upper semicontinuous behavior. At first glance, a somewhat surprising aspect of this result is the fact that the Lyapunov function produced is smooth. In particular, this leads to the characterization of the asymptotic stability of certain generalized solutions of discontinuous differential equations in terms of the existence of smooth Lyapunov functions.

Consider the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad (2)$$

where $x(t) \in \mathbb{R}^n$. Here $F$ is a multifunction whose values are subsets of $\mathbb{R}^n$. As usual, a solution of (2) on an interval $[a, b]$ is an absolutely continuous function $x: [a, b] \to \mathbb{R}^n$ such that (2) holds a.e. on $[a, b]$. Our standing hypotheses on $F$, referred to as (H) for brevity, are the following:

(H1) $F(x)$ is a nonempty compact convex subset of $\mathbb{R}^n$ for every $x$ in $\mathbb{R}^n$.

(H2) The multifunction $F$ is upper semicontinuous; that is, given $x \in \mathbb{R}^n$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x'| < \delta \Rightarrow F(x') \subseteq F(x) + \varepsilon B,$$

where $B$ denotes the open unit ball.

The books by Aubin and Cellina [5], Clarke [10], Deimling [15] or Clarke, Ledyaev, Stern, and Wolenski [13] can be used as general references on multifunctions and differential inclusions.

It is well known (see [5, 13, 15]) that (H) provides for local existence of solutions of (2); that is, for every $x_0 \in \mathbb{R}^n$ there exists a solution $x(t)$ of
(2) satisfying $x(0) = x_0$, on an interval $[0, T)$ for some maximal $T > 0$. If $T < \infty$, then
\[ \lim_{t \uparrow T} |x(t)| = \infty; \]
that is, finite time blow-up occurs. (In certain problems, finite time blow-up can be precluded by means of Lyapunov functions.)

Definition 1.1. A pair of continuous functions $(V, W)$ on $\mathbb{R}^n$, with $V \in C^\omega(\mathbb{R}^n)$ and $W \in C^\omega(\mathbb{R}^n \setminus \{0\})$ constitutes a $C^\omega$-smooth strong Lyapunov pair for $F$ provided that the following conditions are satisfied:

(L1) **Positive Definiteness.** $V(x) > 0$ and $W(x) > 0$ for all $x \neq 0$. In addition, $V(0) = 0$.

(L2) **Properness.** The sublevel sets
\[ \{x \in \mathbb{R}^n : V(x) \leq a\} \]
are bounded for every $a > 0$.

(L3) **Strong Infinitesimal Decrease.**
\[ \max_{v \in F(x)} \langle \nabla V(x), v \rangle \leq -W(x) \quad \forall x \neq 0. \]

By using standard arguments from stability theory [17, 22, 24], one can prove that if such a pair $(V, W)$ exists, then for any initial point $x_0 \in \mathbb{R}^n$, every solution $x(\cdot)$ of (2) with $x(0) = x_0$ is defined on the entire interval $[0, \infty)$ and is attracted to the origin in a uniform and stable manner, a property which we shall refer to as strong asymptotic stability of $F$, to be precisely defined in Section 2 below. This is the “easy” or “non-converse” part of our main result, Theorem 1.2 below, and it follows that 0 is necessarily an equilibrium of $F$; i.e., $0 \in F(0)$. The qualifier “strong” in our terminology pertains to the fact that the asymptotic stability property applies to all solutions of the differential inclusion. A corresponding weak or “control” form (wherein “all” is replaced by “some”) will also enter our discussions later, but our present interest is in the strong version.

The following main result of this article includes a converse theorem to the above.

**Theorem 1.2.** Let the multifunction $F$ satisfy hypotheses (H). Then $F$ is strongly asymptotically stable iff there exists a $C^\omega$-smooth strong Lyapunov pair $(V, W)$. 
This result can be viewed as a generalization of Kurzweil’s converse Lyapunov function theorem for the ordinary differential equation (1) [23], where \( f \) is continuous (and where, as in the case of differential inclusions, nonuniqueness of solutions is not precluded.) Kurzweil’s result in turn generalizes a first converse theorem for local asymptotic stability due to Massera [27, 28] for the case of smooth \( f \). Furthermore, the global aspects of Theorem 1.2 relate it to the converse theorem of Barbashin and Krasovskii [7] for global asymptotic stability of (1), where the concept of global asymptotic stability (“asymptotic stability in the large”) was first introduced.

Another feature of Theorem 1.2 is that it generalizes converse Lyapunov theorems for dynamical systems under disturbance. Such systems are described by a differential equation

\[
\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U,
\]

where the function \( u(\cdot) \) is viewed as a disturbance or noise in the dynamics, and where \( u(\cdot) \) is valued in some restraint set \( U \). Upon defining the multifunction

\[
F(x) := f(x, U),
\]

the system (4) becomes a specially parametrized differential inclusion of the form (2). In this setting, strong asymptotic stability of \( F \) can naturally be interpreted as a “robust” asymptotic stability property of (4). Problems of stability under disturbance have been studied since the 1940s; see e.g. [21].

Recent results on the strong stability of system (4) are given in the article by Li, Sontag and Wang [26]; see also Tsinias [35] for related work.

The main results proven by those authors provide for global asymptotic stability of (4) to a (possibly unbounded) closed invariant set and not simply the origin, as in the present article. On the other hand, the assumptions in [26] make direct use of the fact that \( F \) is given by (5), and require that \( F \) be locally Lipschitz on \( \mathbb{R}^n \), assumptions which we do not impose in the present work.

As mentioned above, our main result, Theorem 1.2, also bears upon the asymptotic stability of solutions of the ordinary differential equation (1) when \( f \) is discontinuous. Since classical solutions of such systems can fail to exist, generalized solution concepts have been developed in the past few decades; see Hajek [18] and Deimling [15] for an overview of this topic. A Krasovskii solution is a solution of the differential inclusion

\[
\dot{x}(t) \in \bigcap_{\delta > 0} \partial f(x + \delta B),
\]
while a Filippov solution is a solution of the differential inclusion
\[ \dot{x}(t) \in \bigcap_{\delta > 0} \bigcap_{\text{meas}(\delta) = 0} \mathcal{C}(x + \delta B \setminus N), \]
where the second intersection is taken over all subsets \( N \) of \( \mathbb{R}^n \) with Lebesgue measure zero. The Filippov solution concept for discontinuous differential equations was the first one formulated in terms of a differential inclusion [16]; for an early predecessor of Filippov solutions see [2]. If \( f \) is bounded on bounded sets and (for the case of Filippov solutions) if \( f \) is also measurable, then the multivalued right-hand sides in (6) and (7) satisfy hypotheses (H) [16, 18] and therefore Theorem 1.2 is applicable. Specifically, if we define the strong asymptotic stability of the differential equation (1) in the sense of Krasovskii or Filippov solutions to mean the strong asymptotic stability of the multifunctions in (6) and (7), respectively, we obtain the following:

**Theorem 1.3.** Assume that \( f \) is bounded on bounded subsets of \( \mathbb{R}^n \). Then

(a) Krasovskii solutions of (2) are strongly asymptotically stable iff there exists a \( C^\infty \)-smooth pair of functions \( (V, W) \) satisfying (L1), (L2) and
\[ \limsup_{y \to x} \langle \nabla V(x), f(y) \rangle \leq -W(x) \quad \forall x \neq 0. \]

(b) If, in addition, \( f \) is measurable, then Filippov solutions of (1) are strongly asymptotically stable iff there exists such a pair satisfying
\[ \text{ess lim sup}_{y \to x} \langle \nabla V(x), f(y) \rangle \leq -W(x) \quad \forall x \neq 0. \]

We remark that another converse Lyapunov theorem for differential equations with discontinuous right-hand side, with respect to the Filippov solution concept, was given by Rosier. In [29], the existence of a locally Lipschitz Lyapunov function was deduced from the asymptotic stability of the Filippov solutions of the differential equation \( \dot{x}(t) = f(t, x(t)) \) with measurable right-hand side. (See also Bacciotti and Rosier [6] for other related work.) An interesting application of these results to the proof of the existence of a smooth control Lyapunov function in a problem of stabilization by discontinuous feedback can be found in Coron and Rosier [14].

The natural counterpart of strong asymptotic stability is weak asymptotic stability. The multifunction \( F \) is said to be weakly asymptotically stable if for any \( x_0 \) there is at least one solution of the differential inclusion (2) starting at \( x_0 \) satisfying attractiveness and stability, and provided this occurs in a certain uniform way with respect to \( x_0 \). In the control system case where \( F \) is given by (5), weak asymptotic stability is equivalent to asymptotic controllability.
of (4); that is, for any initial state \( x_0 \) there exists a control function \( u : [0, \infty) \to U \) which drives the state \( x(\cdot) \) to the origin in a stable and uniform manner. By results in Sontag [32] and Sontag and Sussmann [33], under the assumption that \( F(x) = f(x, U) \) satisfies (H), one has that asymptotic controllability is equivalent to the existence of a continuous pair of functions \((V, W)\) satisfying (L1)–(L2) and the “weak” infinitesimal decrease condition

\[
\min_{v \in F(x)} DV(x; v) \leq -W(x) \quad \forall x \neq 0,
\]

where the Dini subderivative of \( V \) at \( x \) in the direction \( v \) is defined as

\[
DV(x; v) := \liminf_{w \to v, t \downarrow 0} \frac{V(x + tw) - V(x)}{t}.
\]

Note that since the extended real valued-function \( v \to DV(x; v) \) is lower semicontinuous, the use of “min” as opposed to “inf” in (10) is justified. Furthermore, if convexity of the sets \( F(x) = f(x, U) \) is not assumed, the result holds true with \( \overline{co} F(x) \) replacing \( F(x) \) in (10).

**Definition 1.4.** A pair of continuous functions \((V, W)\) on \( \mathbb{R}^n \), with \( V \in C^1(\mathbb{R}^n) \) and \( W \in C^1(\mathbb{R}^n \setminus \{0\}) \) constitutes a \( C^1 \)-smooth weak Lyapunov pair for \( F \) provided that (L1) and (L2) hold, as well as

\[
\text{(L4) Weak Infinitesimal Decrease.}
\]

\[
\min_{v \in F(x)} \langle VV(x), v \rangle \leq -W(x) \quad \forall x \neq 0.
\]

In the case of a control system (5), a weak Lyapunov pair has been called a control Lyapunov pair [32].

**Remark 1.5.** Note that in view of Theorem 1.2, the existence of a \( C^1 \)-smooth weak Lyapunov pair \((V, W)\) implies the existence of a \( C^\infty \)-smooth weak Lyapunov pair. Namely, define

\[
\tilde{F}(x) := \{ v \in F(x) : \langle VV(x), v \rangle \leq -W(x) \}.
\]

It is easy to verify that \( \tilde{F} \) satisfies (H) and that \((V, W)\) is a \( C^1 \)-smooth strong Lyapunov pair for it. This implies that \( \tilde{F} \) is strongly asymptotically stable (by the proof of the non-converse part of Theorem 1.2 below.) Then it follows from Theorem 1.2 that there exists a \( C^\infty \)-smooth strong Lyapunov pair for \( \tilde{F} \), which is a \( C^\infty \)-smooth weak Lyapunov pair for \( F \).

It is noteworthy that the existence of a \( C^1 \)-smooth weak Lyapunov pair for weakly asymptotically stable differential inclusions is the exception.
rather than the rule. It will be shown in Theorem 6.1 that for a multifunc-
tion $F$ satisfying (H), the existence of such a pair implies that for any given $\gamma > 0$ there exists $A > 0$ such that

$$AB \subseteq F(\gamma B).$$

(12)

This conclusion is closely related to the main result in Ryan [31], which in turn generalizes Brockett’s covering condition [8], which is well known in feedback stabilization theory. Brockett’s result asserts that in the control system case (4) with $f(\cdot, \cdot)$ assumed to be $C^1$-smooth, stabilizability by means of a continuous feedback law implies that the image of $f$ contains an open neighborhood of the origin. The following asymptotically controllable system, called the “non-holonomic integrator” (see [8]),

$$\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1u_2 - x_2u_1
\end{align*}$$

provides an example of a differential inclusion of type (5) with

$$U := \{(u_1, u_2); u_1^2 + u_2^2 \leq 1\},$$

which is weakly asymptotically stable and which does not satisfy the covering condition (12). This implies that there is no smooth weak Lyapunov pair for this differential inclusion. Also, since Brockett’s condition does not hold for the non-holonomic integrator, there is no continuous stabilizing feedback for this system. A well-known result of Artstein [4] affirms that for systems affine in the control (which is the case of the non-holonomic integrator), the existence of a smooth Lyapunov function is equivalent to smooth stabilizability. For such systems then, our Theorem 6.1 is equivalent to Brockett’s. In general, however, it can be viewed as a variant in Lyapunov function terms (rather than stabilizing feedback) of Brockett’s result.

The initial belief that the Filippov solution concept was adequate for using discontinuous feedback laws to achieve stabilizability was invalidated by the aforementioned result of Ryan [31]; see also Coron and Rosier [14]. A solution concept for discontinuous feedback under which asymptotic controllability and feedback stabilization are equivalent was introduced in Clarke, Ledyaev, Sontag and Subottin [11]. The main result of the present paper, Theorem 1.2, was used in [25] to show that this discontinuous stabilizing feedback is robust with respect to measurement error if and only if there exists a smooth control Lyapunov function.
Adaptations of the methods of [33] yield the following fact: When (H) holds, with the additional assumption that $F$ be continuous (in the Hausdorff sense), weak asymptotic stability is equivalent to the existence of a continuous pair $(V, W)$ satisfying (L1)-(L2) and (10). As was pointed out in [33], in the case of merely upper semicontinuous $F$, similar methods yield the fact that such a pair $(V, W)$ exists, but where $V$ can only be taken to be lower semicontinuous; for a survey of results on nonsmooth Lyapunov functions for differential inclusions see also Deimling [15]. In fact, an elegant example in [33] displays a system where $F$ is only upper semicontinuous, where weak asymptotic stability holds, but for which no continuous $V$ satisfying (10) can possibly exist. Sontag and Sussman [34] have also posed the question, still open to our knowledge, of the existence of a locally Lipschitz $V$ for a locally Lipschitz $F$ which is weakly asymptotically stable. To summarize, the type of continuity of the weak Lyapunov function $V$ depends upon the type of continuity of the underlying weakly asymptotically stable multifunction $F$. That is, in general only a continuous $V$ exists for continuous $F$, while only lower semicontinuous $V$ exists for upper semicontinuous $F$. This is in sharp contrast to the main result on strong asymptotic stability given in this paper, Theorem 1.2, since we obtain a $C^\infty$-smooth $V$ for a merely upper semicontinuous $F$.

This article is organized as follows. In Section 2 we provide the basic definitions involved in Theorem 1.2, and some auxiliary results which will be required. Then in the next three sections we deal with the proof of the theorem, with the main effort devoted to the more difficult implication; namely, the necessity of the existence of a $C^\infty$-smooth Lyapunov pair when strong asymptotic stability holds. In Section 3, we establish that there is a positive Lipschitz function $\delta: \mathbb{R}^n \to [0, \infty)$ such that the perturbed differential inclusion

$$\dot{x}(t) \in \partial \int_0^t F(x + \delta(x) \dot{B}) + \delta(x) \dot{B}$$

remains strongly asymptotically stable. The important and new feature here is that a perturbation of the original multifunction $F$ is made “from the inside.” This result is then applied to derive the existence of a multifunction $F_L$ which satisfies (H), is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, strongly asymptotically stable, and which is an upper estimate of $F$ in the sense that

$$F(x) \subseteq F_L(x) \quad \forall x.$$ 

We wish to emphasize that the construction of a locally Lipschitz multifunction $F_L$ which remains strongly asymptotically stable is the essential and new contribution of this article, since it allows us to follow the classical scheme for constructing a smooth Lyapunov function, while of course taking into account the specific features of the problem under consideration. It is clear
that a smooth strong Lyapunov pair for $F$ will be a smooth strong Lyapunov pair for $F$. In Section 4 we assume that $F$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, and produce a pair $(V, W)$, locally Lipschitz on $\mathbb{R}^n$, which satisfies (L1)-(L2) and the strong infinitesimal decrease condition

$$\sup_{v \in F(x)} DV(x; v) \leq -W(x) \quad \forall x \neq 0. \quad (13)$$

The function $V$ is constructed as the optimal value function for a certain infinite-horizon optimal control problem. The maximizing cost functional is close to the construction used by Massera [27]. The proof of the necessity part of Theorem 1.2 is then completed in Section 5 via a smoothing procedure which is a modification of a technique in Li, Sontag, and Wang [26], which in turn extends methods of Kurzweil [23] and Wilson [36]. The sufficiency part is dealt with in the same section, completing the proof. As an application, we also provide the proof of Theorem 1.3. In Section 6 we prove the necessary covering condition mentioned above.

We shall employ the following notations throughout: $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$, $| \cdot |$ the corresponding Euclidean norm, $B$ the open unit ball in $\mathbb{R}^n$, $\overline{B}$ its closure, and $\overline{S}$ denotes the closure of the convex hull of a set $S$. The Euclidean distance from a point $x$ to a set $S$ will be denoted $d(x, S)$, and the Hausdorff distance between two closed sets $S_1$ and $S_2$ is denoted $h(S_1, S_2)$. For a compact set $S \subseteq \mathbb{R}^n$, we shall denote $\| S \| := \max \{ |s| : s \in S \}$.

Let us also recall some standard topological facts and definitions in $\mathbb{R}^n$. Let $\Omega$ be an open subset of $\mathbb{R}^n$, and let $\{ \Gamma_\alpha \}_{\alpha \in A}$ be an open covering of $\Omega$. Then there exists a locally finite countable refinement $\{ U_i \}_{i=1}^\infty$ of $\{ \Gamma_\alpha \}_{\alpha \in A}$, with each $U_i$ bounded. This means that for each $i$, $U_i \subseteq \Gamma_\alpha$ for some $\alpha \in A$, and for each $x \in \Omega$, there exists $\rho > 0$ such that the ball $x + \rho B$ intersects only a finite number of the sets $U_i$. Furthermore, there exists a $C^\infty$ partition of unity $\{ \psi_i(\cdot) \}_{i=1}^\infty$ subordinate to the covering $\{ U_i \}_{i=1}^\infty$. That is, each $\psi_i(\cdot)$ is in $C^\infty(\mathbb{R}^n)$, and for each $x \in \Omega$ one has the following:

(a) $0 \leq \psi_i(x) \leq 1$ $\forall i$ and $\sum \psi_i(x) = 1$;

(b) For every $i$, $\psi_i(x) > 0$ implies $x \in U_i$.

2. PRELIMINARIES

With regard to making precise the statement of Theorem 1.2, we posit the following definition of strong asymptotic stability which emphasizes the physically meaningful uniform character of attractiveness and boundedness of solutions, and includes their Lyapunov stability.
Definition 2.1. The differential inclusion (2) (or simply the multifunction $F$) is strongly asymptotically stable provided that no solution exhibits finite time blow-up, and provided that the following hold:

(a) **Uniform Attraction.** For any $r > 0$, $R > 0$, there exists $T = T(r, R)$ such that for any solution $x(\cdot)$ of (2) with $|x(0)| \leq R$, one has

$$|x(t)| \leq r \quad \forall t \geq T.$$  \hspace{1cm} (14)

(b) **Uniform Boundedness.** There is a continuous nonincreasing function $m: (0, \infty) \to (0, \infty)$ such that for any solution $x(\cdot)$ of (2) with $|x(0)| \leq R$ one has

$$|x(t)| \leq m(R) \quad \forall t \geq 0.$$  \hspace{1cm} (15)

(c) **Lyapunov Stability.**

$$\lim_{R \to 0} m(R) = 0.$$  \hspace{1cm} (16)

It is clear that the last condition (16) together with (15) imply that the following classical Lyapunov stability property holds for $F$:

(i) **Lyapunov Stability.** For any given $\varepsilon > 0$ there exists $\delta > 0$ such that any solution $x(\cdot)$ with $|x(0)| < \delta$ satisfies

$$|x(t)| < \varepsilon \quad \forall t \geq 0.$$ 

Further, the uniform attraction in Definition 2.1 evidently implies the following attractiveness property of solutions of $F$:

(ii) **Attractiveness.** For each individual solution $x(\cdot)$, one has

$$\lim_{t \to \infty} x(t) = 0.$$  \hspace{1cm} (17)

In the classical stability theory of differential equations, the notion of asymptotic stability of solutions of (1) is comprised of global existence of all solutions together with their attractiveness and Lyapunov stability. It has long been recognized that these properties imply uniform attractiveness and uniform boundedness of solutions of (1). Hence the classical notion of asymptotic stability is rigorously equivalent to the type of stability for (1) given in Definition 2.1. An analogous fact also holds for the differential inclusion (2), as we now see.

**Proposition 2.2.** When $F$ satisfies (H), the differential inclusion (2) is strongly asymptotically stable iff no solution exhibits finite time blow-up, and $F$ satisfies properties (i)-(ii) above.
The proof of this proposition requires the following compactness property of solutions of differential inclusions (see e.g. [5, 10, 15]).

**Lemma 2.3.** Let $F$ satisfy hypotheses (H). Then for any sequence $\delta_k \to 0$ and sequence $x_k(\cdot)$ of absolutely continuous and uniformly bounded functions on $[a, b]$ satisfying

$$x_k(t) \in \partial F(x_k(t) + \delta_k B) + \delta_k B,$$

there exists a subsequence $x_{k_i}(\cdot)$ converging uniformly to some solution $x(\cdot)$ of (2) on $[a, b]$.

**Proof of Proposition 2.2.** The “only if” part of the statement is immediate; we therefore turn to the “if” part. Due to the attraction property (i), each individual solution $x(\cdot)$ of (2) is bounded on $[0, \infty)$. Let us now show that for every $R > 0$, all solutions of (1) with $|x(0)| \leq R$ are uniformly bounded. On the contrary, suppose that this did not hold for some fixed $R$. Then for any integer $k > R$, the instant

$$t_k := \sup \{t': |x(t')| \leq k \forall t' \in [0, t'), \forall solutions x(\cdot) of (2) with |x(0)| \leq R \}$$

is finite. It is clear that the sequence $\{t_k\}$ is strictly increasing and that there is a sequence of solutions $x_k(\cdot)$ such that

$$|x_k(t_k)| = k, \quad |x_k(t)| \leq k \quad \forall t \in [0, t_k]. \quad (18)$$

Let

$$i := \lim_{k \to \infty} t_k.$$

In view of (18) and Lemma 2.4 we can assume without loss of generality that $x_k(\cdot)$ converges uniformly on every compact subinterval of $[0, i)$ to some solution $x(\cdot)$ of (1) with $|x(0)| \leq R$.

It follows from Lyapunov stability that for fixed $\varepsilon > 0$ there is $\delta > 0$ such that any solution of (1) which enters the ball $\delta B$ will stay in the ball $\varepsilon B$ thereafter. There are now two cases to consider.

**Case 1.** $i = \infty$.

In view of the attractiveness property (ii) for $x(\cdot)$, there exists a moment $T > 0$ such that

$$|x(T)| < 4\delta. \quad (19)$$
for every solution of (2) starting in $R\bar{B}$. Since $x_k(\cdot)$ converges uniformly to $\bar{x}(\cdot)$ on $[0, T]$, there exists $K_T$ such that
\[ |x_k(T)| < \delta \quad \forall k \geq K_T. \tag{20} \]
Because of Lyapunov stability of $F$ this implies
\[ |x_k(t)| \leq \epsilon \quad \forall t \geq T, \quad \forall k \geq K_T. \tag{21} \]
Let $K_*$ be the least integer $k$ such that $t_k > T$. Due to (18), we then have that
\[ |\bar{x}(t)| \leq K_* \quad \forall t \in [0, T], \quad \forall k \geq K_. \tag{22} \]
It follows from (21)-(22) that the sequence $x_k(\cdot)$ is uniformly bounded on $[0, \infty)$. This contradicts (18) for large $k$.

Case 2. $i < \infty$.

Since the solutions $x_k(\cdot)$ converge to $x(\cdot)$ we can assume without loss of generality that for any $k$ there exists $t' \in (0, t_k]$ such that
\[ |x_k(t) - x(t)| \leq 1 \quad \forall t \in [0, t']. \]
Denote by $t'_k$ the supremum of such $t'_k$. Since $x_k(\cdot)$ converges uniformly to $x(\cdot)$ on $[0, T]$ for every $T \in (0, i)$, we deduce that $t'_k > T$ for $k$ large enough, and therefore $t'_k \rightarrow i$ as $k \rightarrow \infty$. Let
\[ M := \max_{t \in (0, i)} |x(t)|. \]
Then
\[ |x_k(t)| \leq M + 1 \quad \forall t \in [0, t'_k]. \]
Now denote
\[ A := \max_{|x| \leq M + 2} |F(x)|. \]
It is readily seen that
\[ |x_k(t)| \leq M + 2 \quad \forall t \in \left[ t'_k, t'_k + \frac{1}{A} \right]. \]
But the fact that $t'_k \rightarrow i$ then implies that
\[ |x_k(t)| \leq M + 2 \quad \forall t \in [0, i] \]

for all $k$ large enough which provides a contradiction to (18) for such $k$. Hence we have shown that for every $R > 0$, all solutions of (2) with $|x(0)| \leq R$ are uniformly bounded on $[0, \infty)$.

Let
$$R_k := 2^k, \quad k = 0, 1, 2, \ldots,$$
(23)
and let $m_k$ denote an upper bound for $|x(\cdot)|$ on $[0, \infty)$ for solutions with $|x(0)| \leq R_k$. Without loss of generality we can assume that the sequence $\{m_k\}$ is strictly increasing, and because of Lyapunov stability (i) we have that
$$\lim_{k \to \infty} m_k = 0.$$
Denote $s_k := (m_{k+1} - m_k)(R_k - R_{k-1})$ and define the continuous function
$$m(R) := m_k + s_k(R - R_{k-1}), \quad R \in [R_{k-1}, R_k),$$
which is strictly increasing and satisfies (15) and (16).

It remains to verify the uniform attraction property (14). Suppose to the contrary that there exist positive numbers $r < R$, a sequence $T_k \to \infty$, and a sequence of solutions $x_k(\cdot)$ of (2) such that $x_k(0) \leq R$ and
$$|x_k(T_k)| > r \quad \forall k = 1, 2, \ldots$$
(24)
We have already proved the uniform boundness of the sequence of solutions $x_k(\cdot)$ on $[0, \infty)$. Then in view of Lemma 2.3, we can without loss of generality assume that $x_k(\cdot)$ converges to $x(\cdot)$ uniformly on every compact interval $[0, T]$. Because of Lyapunov stability (property (i)) with $\varepsilon = r$, we can find $\delta > 0$ such that every solution of (2) will stay in $\delta B$ after entering $B$. Due to the attraction property (ii) for the particular solution $x(\cdot)$, we have the existence of a moment $T > 0$ such that (19) holds. Then by the uniform convergence of $x_k(\cdot)$ to $x(\cdot)$ on $[0, T]$ we obtain that (20) holds for some $K_T$. This implies that $x_k(t)$ stays in $\delta B$ for all $t \geq T$ and all large $k$, which contradicts (24). Thus, solutions of (1) have the uniform attraction property of Definition 2.2 and $F$ is strongly asymptotically stable.

We will employ the preceding proposition to derive the following preparatory lemma.

**Lemma 2.4.** Suppose that $F$ satisfies (H) and that $F$ is strongly asymptotically stable. Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be a continuous function such that $\varphi(x) > 0$ whenever $x \neq 0$. Then the multifunction $F_\varphi$ defined by
$$F_\varphi(x) := \varphi(x) F(x)$$
satisfies (H) and is strongly asymptotically stable.
Proof. That $F_s$ satisfies (H) is straightforward. In accordance with previous proposition, it is enough to verify that solutions of the differential inclusion

$$x(t) \in F_s(x(t))$$

(25)

have the Lyapunov stability property (i) and the attraction property (ii).

Consider an arbitrary solution $x(\cdot)$ of (25) with $x(0) \neq 0$, and let $\ell > 0$ be the first time $t$ such that $x(t) = 0$ if such a time exists; otherwise, let $\ell = \infty$. Then the function $\rho(\cdot)$ defined by

$$\rho(t) := \int_0^t x(s) \, ds$$

is strictly increasing on $[0, \ell)$, with inverse denoted by $\gamma(\cdot)$. By a straightforward application of the chain rule, it is seen that the function $z(\cdot)$ defined via

$$z(t) = x(\gamma(t)), \quad t \in [0, \ell),$$

(26)

is a solution of (2) on $[0, \ell)$. Thus, for every solution $x(\cdot)$ of (25) there exists a solution $z(\cdot)$ of (2) such that

$$x(t) = z(\rho(t)), \quad t \in [0, \ell).$$

(27)

Then the Lyapunov stability of solutions of (25) follows immediately from this representation and its obvious consequence that for any solution $x(\cdot)$ of (25) we have $x(t) = 0$ for all $t \geq \ell$. (The last relation follows from the fact that $F$ is Lyapunov stable (i) which implies that the unique solution of (2) with $x(0) = 0$ is $x(t) \equiv 0$.)

Let us now verify the attraction property (17) for $x(\cdot)$. It suffices to consider the case $t = \infty$. By way of contradiction, let us suppose that (17) did not hold. Then there would exist $r > 0$ and a sequence $t_k \uparrow \infty$ such that $|x(t_k)| > r$ for each $k$. Note that $|z(\rho(t))|$ and consequently $|x(t)|$ are bounded by some constant $M$ for all $t \geq 0$, since $F$ is globally asymptotically stable and $z(\cdot)$ is a solution of (2). This implies that $x(\cdot)$ is Lipschitz, namely, there is a constant $C$ such that for any non-negative $t$ and $t'$ one has

$$|x(t) - x(t')| \leq C |t - t'|.$$

(One can take $C$ to be the maximum of $\|F(x)\|$ over the ball $MB$.) This implies that for each $k$, $|x(t)| > r/2$ for all $t \in [t_k, t_k + A]$ with $A = r/2C$. Thus, we have that

$$\int_0^\infty x(t) \, dt \geq \sum_{k=1}^\infty \tau_k A,$$
where
\[ x_0 := \min \left\{ x(x): \frac{r}{2} \leq |x| \leq M \right\}. \]

Since \( x_0 > 0 \) we obtain that the integral above is divergent and consequently that
\[ \lim_{t \to \infty} \rho(t) = \infty. \]

Then (17) follows from the representation (27) (with \( \ell = \infty \)) and the fact that \( z(\rho(t)) \to 0 \) as \( t \to \infty \), due to the global asymptotic stability of \( F \).

Most of the following auxiliary results are well known in stability theory. We place them here to make the exposition self-contained and the references convenient. An elementary but useful result is the following:

**Lemma 2.5.** Let the function \( \varphi: [0, \infty) \to (0, \infty) \) be nondecreasing (non-increasing). Then there exist strictly increasing (strictly decreasing) \( C^\infty \)-functions \( \varphi_i: (0, \infty) \to (0, \infty) \), \( i = 1, 2 \), such that
\[ \varphi_1(r) \leq \varphi(r) \leq \varphi_2(r) \quad \forall r \geq 0. \]

Furthermore, if \( \varphi \) is nondecreasing and satisfies the limit relations
\[ \lim_{r \downarrow 0} \varphi(r) = 0, \quad \lim_{r \uparrow \infty} \varphi(r) = \infty, \]
or if \( \varphi \) is nonincreasing and satisfies the limit relations
\[ \lim_{r \downarrow 0} \varphi(r) = \infty, \quad \lim_{r \uparrow \infty} \varphi(r) = 0, \]
then \( \varphi_1 \) and \( \varphi_2 \) can be specified to satisfy these relations as well.

**Sketch of Proof.** We only will outline the proof of part of the assertion, which is indicative of the general technique. To prove the existence of the upper function \( \varphi_2(\cdot) \) when \( \varphi \) is nondecreasing, for example, consider the partition \( \{ R_k \} \) of \( (0, \infty) \) and define quantities
\[ \varphi_k := \begin{cases} \varphi(R_k) + 2^k, & k \leq 0 \\ \left( \varphi(R_k) + 2 \right) 2^{-k}, & k > 0 \end{cases}, \quad s_k := \frac{\varphi_{k+1} - \varphi_k}{R_k - R_{k-1}}, \]
and a function \( \phi_2(\cdot) \) on successive intervals via the formula

\[
\phi_2(r) := \varphi_k + s_k (r - R_{k-1}) \quad r \in [R_{k-1}, R_k).
\]

Then \( \phi_2(r) \) is piecewise linear, continuous and strictly increasing on \([0, \infty)\), and \( \phi_2(r) > \varphi(r) \) for all \( r \). By an appropriate “smoothing of the corners” of \( \phi_2(\cdot) \), we obtain the desired function \( \varphi(\cdot) \).

By using this lemma it is an easy exercise to establish for a continuous function \( V \) satisfying both \((L1)\) and \((L2)\), the existence of continuous positive definite increasing functions \( \phi_i : [0, \infty) \to [0, \infty) \), \( i = 1, 2 \), such that

\[
\phi_i(|x|) \leq V(x) \leq \phi_2(|x|).
\]  

(28)

The following standard “decay estimate” gives a characterization of strong asymptotic stability. Decay estimates of this general type have often proven their usefulness in stability theory [17] (for recent applications see [1, 26]).

**Lemma 2.6.** \( F \) is strongly asymptotically stable iff global existence holds and there exists a function \( \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \) such that for each fixed \( t \) the function \( \beta(t, \cdot) \) is nondecreasing, for each fixed \( R \) the function \( \beta(\cdot, R) \) is nonincreasing,

\[
\lim_{t \to \infty} \beta(t, R) = 0, \quad \lim_{R \to \infty} \beta(0, R) = \infty, \quad \beta(t, 0) = 0 \quad \forall t \geq 0,
\]

and for any solution \( x(\cdot) \) of (2) with \( x(0) = x \) one has

\[
|x(t)| \leq \beta(t, |x|).
\]  

(29)

It is easy to see that if the decay estimates hold, then \((15)-(16)\) hold with \( m(R) = \beta(0, R) \). On the other hand, if strong asymptotic stability holds, one can define

\[
\beta(t, R) := \max_{|x(0)| \leq R} \max_{t \geq \tau} |x(\tau)|,
\]

where the outer maximum is taken over all solutions of (2) starting in the ball \( RB \), and judiciously employ Lemma 2.3 in order to check that this function is well defined (i.e., the maxima are indeed attained) and that it satisfies the specified requirements. We leave it to the reader to prove by using this construction and Lemma 2.5 that the function \( \beta \) in Lemma 2.6 can be chosen to be continuous in \( t \) and in \( R \). The following two lemmas will play an essential role:

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Lemma 2.7. If $F$ is strongly asymptotically stable, then there exists a continuous function $T: \mathbb{R}^n \to [0, \infty)$ and a strictly decreasing continuous function $\varphi \in C^\omega(0, \infty)$, such that the right-hand derivative $\varphi'(0)$ exists,

$$\lim_{t \to \infty} \varphi(t) = 0,$$

and such that every solution $x(\cdot)$ of (2) with $x(0) = x$ satisfies

$$|x(t)| \leq \varphi(t - T(x)) \quad \forall t \geq T(x).$$

Proof. First note that strong asymptotic stability implies that for a given solution $x(\cdot)$ of (2) one has

$$|x(t')| \leq 1$$

at some moment $t' \geq 0$, and therefore because of Lemma 2.6

$$|x(t)| \leq \beta(t - t', 1) \quad \forall t \geq t'.$$

Now Lemma 2.5 yields the existence of a strictly decreasing $C^\omega$-function $\varphi: [0, \infty) \to (0, \infty)$, such that (30) holds and

$$\beta(t, 1) \leq \varphi(t) \quad \forall t \geq 0.$$

It is clear that it can be arranged that $\varphi'(0)$ exists. Also, if (31) holds, we have

$$|x(t)| \leq \varphi(t - t') \quad \forall t \geq t'$$

for any solution $x(\cdot)$ of (2) and any $t'$ such that $|x(t')| \leq 1$.

Now we shall construct a function $T$ such that

$$|x(T(x))| \leq 1$$

for any solution $x(\cdot)$ with $x(0) = x$; this clearly will complete the proof. Let $R_k$ be given by (23). By the assumption of strong asymptotic stability, there exists $\tau_0 > 0$ such that any solution $x(\cdot)$ of (2) with $|x(0)| \leq R_0$ satisfies $|x(t)| \leq 1$ whenever $t \geq \tau_0$, and for each integer $k > 1$ there exists $\tau_k > 0$ such that for any solution $x(\cdot)$ with $R_k \leq |x(0)| \leq R_{k+1}$, one has $|x(t)| \leq 1$ whenever $t \geq \tau_k$. Of course, we can assume that the sequence $\{\tau_k\}_{k=0}^\infty$ is nondecreasing. Now consider the nondecreasing step function $\tilde{\tau}(\cdot)$ defined via

$$\tilde{\tau}(r):=\begin{cases} \tau_0 & \text{if } 0 \leq r \leq 1, \\ \tau_k & \text{if } R_{k-1} \leq r \leq R_k, \quad k = 1, 2, \ldots. \end{cases}$$
An obvious variant of Lemma 2.5 implies that there exists a continuous function \( T: [0, \infty) \to [0, \infty) \) such that \( \tau(r) \geq \tilde{\tau}(r) \) for \( r \geq 1 \). We define
\[
T(x) = \tau(|x|).
\]
It is easy to check that this function is continuous and satisfies (32), as required.

**Lemma 2.8.** If \( F \) is strongly asymptotically stable, then there exists a continuous function \( \tilde{T}: \mathbb{R}^n \to (0, \infty) \) such that for any solution \( x(\cdot) \) of (2) with \( x(0) = x \) one has
\[
|x(t)| \geq \frac{1}{2}|x| \quad \forall t \in [0, \tilde{T}(x)].
\]

**Proof.** Let \( R_k \) be a sequence as in the proof of Lemma 2.7, and consider a solution of (2) such that the initial point \( x(0) = x \) satisfies
\[
R_{k-1} \leq |x| < R_k.
\]
Then due to the uniform boundedness condition (15), we have that \( x(\cdot) \) remains in the closed ball of radius \( m(R_k) \). Note that the hypotheses on \( F \) imply that \( |F(x)| \) is bounded above on this ball, say by \( p_k \). For any \( t, t' \geq 0 \) we have
\[
|x(t) - x(t')| \leq p_k |t - t'|.
\]
Let us denote
\[
\tau_k := \frac{1}{2} R_{k-1} p_k.
\]
Then for any \( t \in [0, \tau_k] \) one has
\[
|x(t)| = |x(0)| - \tau_k p_k \geq R_{k-1} - \frac{1}{2} R_{k-1} = \frac{1}{2} R_{k-1} = \frac{1}{4} R_k \geq \frac{1}{2} |x|.
\]
Without loss of generality, we can assume that the sequence \( \tau_k \) is increasing for integers \( k \leq 0 \) and that the sequence \( \tau_k \) is strictly decreasing for integers \( k \geq 0 \). Now define a function \( \tilde{\tau}(\cdot) \) as follows:
\[
\tilde{\tau}(r) := \tau_k \quad \text{if} \quad r \in [R_{k-1}, R_k), \quad k = 0, \pm 1, \pm 2, \ldots.
\]
This step function is positive and nondecreasing on \( (0, 1] \), positive and nonincreasing on \( [1, \infty) \), and \( \tilde{\tau}(r) \to 0 \) as \( r \to 0 \). By a variant of Lemma 2.5, there exists a continuous function \( \tau: [0, \infty) \to [0, \infty) \) such that
\[
\tau(r) \leq \tilde{\tau}(r), \quad \tau(0) = 0.
\]
It is easy to check that the function
\[ \tilde{T}(x) := \tau(|x|) \]
has the required properties.

3. ROBUSTNESS WITH RESPECT TO PERTURBATIONS OF \( F \)

Proposition 3.1 below asserts that if a multifunction \( F \) satisfying (H) is strongly asymptotically stable, then it is possible to produce a specific kind of “inflation” of \( F \) which also satisfies (H) and is strongly asymptotically stable. In this sense, the strong asymptotic stability property is robust with respect to a certain class of perturbations. This result is then used in Proposition 3.5 in order to construct an upper approximation of \( F \) which is locally Lipschitz on \( \mathbb{R}^n \setminus \{0\} \) and strongly asymptotically stable.

**Proposition 3.1.** Let \( F \) satisfy (H) and be strongly asymptotically stable. Then there exists a Lipschitz function \( \delta: \mathbb{R}^n \to [0, \infty) \) with Lipschitz constant 1 such that \( \delta(0) = 0 \), \( \delta(x) > 0 \) for all \( x \neq 0 \), and such that the multifunction
\[ \bar{F}(x) := \bar{F}(x + \delta(x)B) + \delta(x)B \]
satisfies (H) and is strongly asymptotically stable.

It is easy to verify that for a continuous (and in particular, Lipschitz) function \( \delta: \mathbb{R}^n \to [0, \infty) \), the multifunction (33) satisfies (H). The proof of the rest of the assertion will rely on the next three lemmas. For a given constant \( \delta > 0 \), we shall make reference to the differential inclusion
\[ \dot{x}(t) \in F_d(x(t)) := \bar{F}(x(t) + \delta B) + \delta B. \quad (34) \]

**Lemma 3.2.** Let \( T, R, \varepsilon \) be given positive numbers. Then there exists \( \delta'(T, R, \varepsilon) > 0 \) such that the following hold:

(a) For any \( \delta \in (0, \delta') \), every solution \( x(\cdot) \) of (843) with \( |x(0)| \leq R \) does not blow-up on \([0, T]\) and satisfies
\[ |x(t)| < 2m(R) \quad \forall t \in [0, T]. \]

(b) Furthermore, there exists a solution \( \tilde{x}(\cdot) \) of (2) on \([0, T]\) such that
\[ |x(t) - \tilde{x}(t)| < \varepsilon \quad \forall t \in [0, T]. \quad (36) \]
Proof. If finite-time blow-up occurred, then there would exist sequences \( \delta_i, T_i, x_i(\cdot) \) corresponding solutions \( x_i(\cdot) \) of (34) on \( [0, T_i] \) with \( \delta = \delta_i \) and \( |x_i(0)| \leq R \), such that
\[
|x_i(T_i)| = 2m(R), \quad |x_i(t)| < 2m(R) \quad \forall t \in [0, T_i]. \tag{37}
\]
Due to Lemma 2.3 on compactness of solutions we can assume that \( x_i(\cdot) \) converges to some solution \( x(\cdot) \) of (2) on every compact subinterval of \( [0, T') \). Because of (37) this implies that \( |x(T')| = 2m(R) \), which gives a contradiction to (15) and verifies part (a) of the assertion. Also, (35) is valid. Indeed, if this were not so, then a sequence \( T_i \) as above would exist, and the preceding arguments yield a contradiction.

In order to prove part (b), suppose to the contrary that for a sequence \( \delta_i \downarrow 0 \) there exists a sequence of solutions \( x_i(\cdot) \) of (34) on \( [0, T'] \), such that
\[
|x_i(0)| \leq R
\]
and
\[
\max_{t \in [0, T]} |x_i(t) - \tilde{x}(t)| \geq \varepsilon
\]
for every trajectory \( \tilde{x}(\cdot) \) of (2). In view of the uniform boundedness of the sequence \( \{x_i(\cdot)\} \) (due to part (a)), Lemma 2.3 is applicable and readily yields a contradiction.

Let \( R_k \) be defined as in (23). Because of the uniform attractiveness of the origin for solutions of (2) (property (14)) there exists an instant \( T_k \) such that for any solution \( \tilde{x}(\cdot) \) of (2) with \( |\tilde{x}(0)| \leq R_{k+1} \), one has
\[
|\tilde{x}(T_k)| < R_{k-1}.
\]
Take \( \delta_k = R_{k-1} \) and define \( A_k = A(T_k, R_{k+1}, \delta_k) \) as in Lemma 3.2. It then follows from the previous inequality and (36) that for any solution \( x(\cdot) \) of (34) with \( \delta \in (0, A_k] \) and \( |x(0)| < R_{k+1} \), one has
\[
|x(T_k)| < R_{k-1} + \delta_k = R_k.
\]
Furthermore, by (35), for all \( t \in [0, T_k] \)
\[
|x(t)| < 2m(R_{k+1}) \tag{38}
\]
Upon defining the set
\[
G_k := \{ x \in \mathbb{R}^n : R_k \leq |x| < R_{k+1} \},
\]
we arrive at the following:
Lemma 3.3. For any integer $k$ there exist positive numbers $T_k, A_k$ such that if $\delta \in (0, A_k']$ and $x(\cdot)$ is any solution of (34) with $x(0) \in G_k$, the following hold:

(a) $x(\cdot)$ does not blow-up on $[0, T_k]$ and (38) holds for all $t \in [0, T_k]$.
(b) There exists an instant $\tau \in (0, T_k)$ such that $x(\tau) \in G_{k-1}$.

For any integer $i$, define

$$k_i := \min \{ k : k \leq i, R_i \leq 2m(R_{i+1}) \}.$$

Note that the set over which this minimum is taken is nonempty, since for any $i$ one always has $R_i < m(R_{i+1})$. Also, due to the Lyapunov stability property (16), we have $k_i \geq -\infty$. We now define a function $A : \mathbb{R}^n \to [0, \infty)$ via

$$A(x) := \begin{cases} \min \{ A_k : k \leq i \} & \text{if } x \in G_i, \quad i = 0, \pm 1, \pm 2, \ldots; \\ 0 & \text{if } x = 0. \end{cases}$$

The next lemma concerns solutions of the following differential inclusion:

$$\dot{x}(t) \in \mathbb{T} F(x(t) + A(x(t))\bar{B}) + A(x(t))\bar{B}. \quad (39)$$

Lemma 3.4. For a given integer $k$, let $x(\cdot)$ be any solution of (39) with $x(0) \in G_k$. Then there exists $\tau^* \in (0, T_k)$ such that $x(\cdot)$ does not blow-up on $[0, \tau^*]$, (38) holds for all $t \in [0, \tau^*]$ and

$$x(\tau^*) \in G_{k-1}. \quad (40)$$

Proof. Consider the positive instant

$$\hat{\tau} := \sup \{ \tau \in [0, T_k] : R_k \leq |x(t)| \leq 2m(R_{k+1}) \forall t \in [0, \tau] \}.$$

For any $t \in [0, \hat{\tau}]$ there exists $i \geq k$ such that $x(t) \in G_i$, which means $R_i \leq 2m(R_{i+1})$. Therefore $k \geq k_i$ and $x(t) \in G_i$. Hence $x(\cdot)$ is a solution of (34) on $[0, \hat{\tau}]$, with $\delta = A_k$. Then we obtain that (38) holds for $t = \hat{\tau}$. Let us assume that (40) fails for all $\tau^* \in (0, \hat{\tau}]$. Then we obtain that $\hat{\tau} = T_k$. But this contradicts Lemma 3.3.

We are now in position to complete the proof of the proposition.
Proof of Proposition 3.1. It is not difficult to verify that the “inf convolution” function

$$\delta(x) := \inf_{y \neq 0} [A(y) + |y - x|]$$

is positive on $\mathbb{R}^n \setminus \{0\}$, Lipschitz with constant 1, $\delta(0) = 0$, and

$$\delta(x) \leq A(x) \quad \forall x \in \mathbb{R}^n.$$  (41)

Since the multifunction $\hat{F}$ satisfies (H), solutions of the differential inclusion

$$\dot{x}(t) \in \hat{F}(x(t))$$  (42)

locally exist, and in view of (41), are also solutions of the differential inclusion (39). To prove that this differential inclusion is strongly asymptotically stable, we define integers $K = K(r)$ and $N = N(R)$ as

$$K(r) := \max\{k : 2m(R_{k+1}) \leq r\}, \quad N(R) := \min\{k : R < R_{k+1}\},$$

and quantities

$$\hat{T}(r, R) := T_N + T_{N-1} + \cdots + T_{N-K}, \quad \hat{m}(R) = 2m(R_{N+1}),$$

where, as before, the function $m$ is as in Definition 2.1 and $T_k$ is as defined in Lemma 3.3. We will show that solutions of (42) satisfy Definition 2.1 with the above defined functions $\hat{T}(r, R)$ and $\hat{m}(R)$. Consider an arbitrary solution $x(\cdot)$ starting from the ball of radius $R$. Then $x(0) \in G_k$ for some $k \leq N$. By Lemma 3.4 there exists a positive $t_1 \leq T_k$ such that $x(\cdot)$ does not blow-up on $[0, t_1]$, satisfies (38) on this interval, and $x(t_1) \in G_{k+1}$. Then we apply Lemma 3.4 to any solution $x(\cdot)$ starting from the point $x(t_1)$ and so on. Thus we obtain an strictly increasing sequence $\{t_i\}_{i \geq 0}$ with $t_0 = 0$ such that

$$|x(t)| < 2m(R_{k-i+1}), \quad \forall t \in [t_i, t_{i+1}],$$

$$x(t_i) \in G_{k-i}, \quad t_{i+1} - t_i \leq T_{k-i}, \quad i \geq 0.$$

It follows that, for $t \geq t_{k-K}$,

$$|x(t)| \leq 2m(R_{k+1}) < r.$$  

This implies that $x(\cdot)$ is defined on the entire interval $[0, \infty)$ and satisfies (14) with $T = \hat{T}(r, R)$ since $t_{k-K} \leq \hat{T}(r, R)$. This means that solutions of (42) have the uniform attractiveness property. It follows from the previous relations that $|x(t)|$ is bounded by $\hat{m}(R)$ for any $t \geq 0$, which implies the uniform boundedness property for (42). It is clear that $N(R) \to -\infty$ when
Then we have from the definition of \( m \) that \( \lim_{R \to 0} \tilde{m}(R) = 0 \), which implies the Lyapunov stability property for solutions of (42). Thus, this differential inclusion is strongly asymptotically stable.

A multifunction \( F \) is said to be locally Lipschitz on \( \mathbb{R}^n \setminus \{0\} \) provided that to every compact set \( S \subseteq \mathbb{R}^n \setminus \{0\} \) there corresponds \( K > 0 \) such that

\[
F(x_1) \subseteq F(x_2) + K|x_1 - x_2|B \quad \forall x_1, x_2 \in S.
\]

We shall require the following result.

**Proposition 3.5.** Let \( F \) satisfy (H) and be strongly asymptotically stable. Then there exists a strongly asymptotically stable multifunction \( F_L \) satisfying (H) which is locally Lipschitz on \( \mathbb{R}^n \setminus \{0\} \), and such that

\[
F(x) \subseteq F_L(x). \quad (43)
\]

**Proof.** Let the function \( \delta \) be as in Proposition 3.1. For every \( x \neq 0 \) we define the open set

\[
W_x = \{ y : |y - x| < \frac{1}{3} \delta(x) \}. \quad (44)
\]

The family \( \{ W_x \} \) is an open covering of \( \mathbb{R}^n \setminus \{0\} \). Denote by \( \{ U_i \}_{i=1}^\infty \) a locally finite refinement of this cover, and associate with it a subordinated \( C^\infty \) partition of unity \( \{ \psi_i \} \). For each \( i \), choose \( x_i \) such that \( U_i \subseteq W_{x_i} \).

For \( x \neq 0 \), define

\[
F_L(x) := \begin{cases} \sum \psi_i(x) \circ F(x_i + \frac{1}{3} \delta(x_i)B) & \text{if } x \neq 0; \\ F(0) & \text{if } x = 0. \end{cases}
\]

It is readily checked that \( F_L \) satisfies (H) and what is more, \( F_L \) is locally Lipschitz on \( \mathbb{R}^n \setminus \{0\} \).

For an arbitrary \( x \neq 0 \), consider \( i \) such that

\[
\psi_i(x) > 0,
\]

which means that \( x \in U_i \subseteq W_{x_i} \), and in view of (44), that

\[
|x - x_i| < \frac{1}{3} \delta(x_i).
\]

This implies

\[
F(x) \subseteq F(x_i + \frac{1}{3} \delta(x_i)B)
\]

We shall require the following result.
for every i such that (45) holds, which implies (43). Since $\delta$ is Lipschitz with rank 1, we have

$$\delta(x_i) - \delta(x) < \frac{1}{\lambda} \delta(x_i),$$

or

$$\frac{2}{\lambda} \delta(x_i) < \delta(x),$$

which implies

$$x_i + \frac{1}{\lambda} \delta(x_i) B \subseteq x + \frac{2}{\lambda} \delta(x_i) B \subseteq x + \delta(x) B.$$

We deduce that

$$\overline{\overline{\delta}} F(x_i + \frac{1}{\lambda} \delta(x_i) B) \subseteq \overline{\overline{\delta}} F(x + \delta(x) B)$$

for every i satisfying (45). Thus

$$F_L(x) \subseteq \overline{\overline{\delta}} F(x + \delta(x) B),$$

and in view of Proposition 3.1, we also conclude that $F_L$ is strongly asymptotically stable.

We will subsequently require the following result, which applies to the multifunction $F_L$ of the preceding proposition.

**Proposition 3.6.** Let $F$ be a multifunction which is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. Then there exists a $C^1$ function $\ell: (0, \infty) \to (0, \infty)$ such that for any $x \neq 0$ one has

$$F(x_1) \subseteq F(x_2) + \ell(|x|) |x_1 - x_2| B$$

(46)

for all $x_1, x_2$ sufficiently near $x$, and

$$\lim_{r \to 0} \ell(r) = \lim_{r \to \infty} \ell(r) = -\lim_{r \to 0} \ell'(r) = \lim_{r \to \infty} \ell'(r) = \infty.$$  (47)

**Proof.** It is clear that $F$ satisfies a Lipschitz condition on every compact set $G_\delta$ with some Lipschitz constant $L_\delta$. Without loss of generality it can be assumed that the sequence $\{L_\delta\}$ is strictly increasing for $k \geq 0$ and strictly decreasing for $k \leq 0$, and that

$$\lim_{k \to -\infty} L_k = \lim_{k \to \infty} L_k = \infty.$$
Now define a function $\tilde{L}: (0, \infty) \to (0, \infty)$ via
$$
\tilde{L}(r) := L_k \quad \text{if} \quad r \in [R_k, R_{k+1}), \quad k = 0, \pm 1, \pm 2, \ldots
$$
A construction similar to that pointed out in sketching the proof of Lemma 2.5 then produces a function $\tilde{c} \in C^1((0, \infty))$, majorizing $\tilde{L}$, which has has the required properties.

4. CONSTRUCTION OF A LOCALLY LIPSCHITZ STRONG LYAPUNOV PAIR

In this section we shall consider the differential inclusion (2) with the multifunction $F$ satisfying (H) and the additional assumption of local Lipschitz behavior away from the origin, as is the case for $F_L$ in Proposition 3.5 above. The main result to be proven in this section is the following converse Lyapunov theorem: Under these conditions, if $F$ is strongly asymptotically stable, then there exists a locally Lipschitz strong Lyapunov pair, in accordance with the following definition:

**Definition 4.1.** A pair of continuous functions $V, W: \mathbb{R}^n \to [0, \infty)$ is called a locally Lipschitz strong Lyapunov pair for $F$ provided that

(a) $V$ and $W$ are locally Lipschitz on $\mathbb{R}^n$;

(b) (L1)–(L2) as well as the strong infinitesimal decrease condition (13) hold.

**Proposition 4.2.** Let the multifunction $F$ satisfy (H) and in addition be locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. Suppose that $F$ is strongly asymptotically stable. Then there exists a locally Lipschitz strong Lyapunov pair $(V, W)$ for $F$.

**Proof.** We shall define $V: \mathbb{R}^n \to [0, \infty)$ as the value function of a certain infinite horizon optimal control problem, as follows:

$$
V(x) := \sup_{x(0) = x} \int_0^\infty w(|x(t)|) \, dt \quad \text{(48)}
$$

where the supremum is taken over all solutions $x(\cdot)$ of (2) with $x(0) = x$. Here the function $w: [0, \infty) \to [0, \infty)$ is defined as

$$
w(r) := \int_0^r \Phi(p) \, dp, \quad \text{(49)}
$$

where $\Phi: [0, \infty) \to [0, \infty)$ is specified below, in (51).
We shall invoke the following temporary assumption:

\((\text{TA})\) \(F\) is globally Lipschitz with Lipschitz constant \(K \geq 1\) on \(\mathbb{R}^n\setminus\{0\}\).

Later we shall show that the general case of local Lipschitz behavior is reducible to this one.

Let \(\varphi\) be as in Lemma 2.7, and let \(\eta\) be the inverse of \(\varphi\). Then \(\eta: (0, \varphi(0)] \to [0, \infty)\) is a strictly decreasing function satisfying \(\lim_{r \to 0} \eta(r) = \infty\) and

\[
\eta(\varphi(t)) = t \quad \forall t \geq 0.
\]

It is easy to see that there is a continuous strictly decreasing extension of \(\eta\) to the entire interval \((0, \infty)\) such that for all \(r > \varphi(0)\) sufficiently large, one has

\[
\eta(r) < \frac{\ln T_r}{2K}, \tag{50}
\]

where

\[
T_r = \min \{ \tilde{T}(x): |x| = 8r \},
\]

and the function \(\tilde{T}\) is as defined in Lemma 2.8. Note that \(T_r\) depends continuously upon \(r\), due to the definition of \(T\).

Now let

\[
h(r) := \begin{cases} 
1 + |\varphi'(\eta(r))| & \text{if } 0 < r < \varphi(0); \\
1 + |\varphi'(0)| & \text{if } r \geq \varphi(0).
\end{cases}
\]

Then \(h\) is continuous on \((0, \infty)\), as is the function

\[
\Phi(r) := \frac{e^{-2K\eta(r)}}{h(r)}. \tag{51}
\]

Let \(\Phi(0) := 0\). Then \(\Phi\) is continuous on \([0, \infty)\), and therefore the integral in (49) is well defined.

We now define \(W: \mathbb{R}^n \to [0, \infty)\) as

\[
W(x) := w(|x|).
\]

Since \(w\) is locally Lipschitz on \([0, \infty)\), it follows that \(W\) is locally Lipschitz on \(\mathbb{R}^n\). Also, observe that both \(V(x)\) and \(W(x)\) are positive for nonzero \(x\), and \(V(0) = 0\) since the only solution of (2) with \(x(0) = 0\) is \(x(t) \equiv 0\). Hence the pair \((V, W)\) satisfies the positive definiteness requirement (L1).
LEMMA 4.3.

\[ \int_0^{\infty} w(\varphi(t)) \, dt \leq 1. \]  \tag{52}

Proof. For any \( T > 0 \) one has by integration by parts

\[ \int_0^T w(\varphi(t)) \, dt = w(\varphi(T)) T - \int_0^T t(w(\varphi(t)))' \, dt \]
\[ = w(\varphi(T)) T - \int_0^T t\Phi(\varphi(t)) \varphi'(t) \, dt. \]

Here we have used the observation

\[ \left( w(\varphi(t)) \right)' = w'(\varphi(t)) \varphi'(t) = \Phi(\varphi(t)) \varphi'(t). \]

Then by L’Hôpital’s rule,

\[ \lim_{T \to \infty} w(\varphi(T)) T = \lim_{T \to \infty} \frac{(w(\varphi(T)))'}{-T^{-2}} = -\lim_{T \to \infty} \frac{T^2 e^{-K_T(\varphi(T))} \varphi'(T)}{h(\varphi(T))} \]
\[ = -\lim_{T \to \infty} \frac{T^2 e^{-K_T} \varphi'(T)}{h(\varphi(T))} = 0. \]

It is readily checked that we also have

\[ \left| \int_0^T t\Phi(\varphi(t)) \varphi'(t) \, dt \right| \leq \int_0^Te^{-2K_T} \, dt \leq 1, \]

and it follows that (52) holds. \( \blacksquare \)

LEMMA 4.4. For any \( x \in \mathbb{R}^n \) one has

\[ V(x) \leq w(m(|x|)) T(x) + 1. \]

Proof. By uniform boundedness (15), Lemma 2.7 and the obvious monotonicity of \( w \), we have that for any solution \( x(\cdot) \) of (2) with \( x(0) = x \),

\[ w(|x(t)|) \leq \begin{cases} w(m(|x|)) & \text{if } t \leq T(x); \\ w(\varphi(t - T(x))) & \text{if } t \geq T(x). \end{cases} \]  \tag{53}
Then
\[ V(x) \leq \int_0^{T(x)} w(m(|x|)) \, dt + \int_{T(x)}^{\infty} W(\varphi(t - T(x))) \, dt, \]
and the assertion follows from (52).

**Lemma 4.5.** For any \( x \in \mathbb{R}^n \) there exists a solution \( \hat{x}(\cdot) \) of (2) with \( \hat{x}(0) = x \) such that
\[ V(x) = \int_0^\infty w(|\hat{x}(t)|) \, dt. \]  

**Proof.** Recalling the definition of \( V(x) \) as a value function, given by (48), let \( x_k(\cdot) \) be a maximizing sequence of solutions to (2) on \([0, \infty)\) with \( x_k(0) = x \), such that
\[ \int_0^\infty w(|x_k(t)|) \, dt \leq V(x). \]
Because of uniform boundedness of \( x_k(\cdot) \) and Lemma 2.3 we can without loss of generality assume that \( x_k(\cdot) \) converges pointwise to a solution \( \hat{x}(\cdot) \). Note that (53) holds for \( x(\cdot) = x_k(\cdot) \) and any \( k \). Therefore, in view of Lemma 4.3, the Lebesgue dominated convergence theorem is applicable. We conclude that
\[ \lim_{k \to \infty} \int_0^\infty w(|x_k(t)|) \, dt = \int_0^\infty w(|\hat{x}(t)|) \, dt, \]
which yields (54).

An analogous argument is used in the proof of the next lemma.

**Lemma 4.6.** The function \( V \) is upper semicontinuous at any \( x \) and continuous at 0.

**Proof.** Consider any sequence \( x_k \in \mathbb{R}^n \) converging to \( x \). Then by the preceding lemma, for each \( k \) there exists a solution \( x_k(\cdot) \) such that \( x_k(0) = x_k \) and
\[ V(x_k) = \int_0^\infty w(|x_k(t)|) \, dt. \]  

In view of strong asymptotic stability, continuity of the function $T$, and
monotonicity of $\phi$, we have for all $k$ large enough
\[
|\mathbf{x}_k(t)| \leq \begin{cases} m(|\mathbf{x}| + 1), & \text{if } t \geq 0; \\ \phi(t - T(\mathbf{x}) - 1), & \text{if } t \geq T(\mathbf{x}) + 1. \end{cases}
\]
(56)

This implies that $w(|\mathbf{x}_k(t)|)$ is bounded by an integrable function analogous
to the one on the right-hand side of (53). Similarly to the preceding lemma,
we can without loss of generality assume that $\mathbf{x}_k(\cdot)$ converges pointwise to
a solution $x(\cdot)$ of (2), and another application of the Lebesgue dominated
convergence theorem yields
\[
\limsup_{k \to \infty} V(\mathbf{x}_k) = \limsup_{k \to \infty} \int_0^{\infty} w(|\mathbf{x}_k(t)|) \, dt
\]
\[
= \int_0^{\infty} \lim_{k \to \infty} w(|\mathbf{x}_k(t)|) \, dt = \int_0^{\infty} w(|x(t)|) \, dt \leq V(x),
\]
which establishes upper semicontinuity of $V$ at $x$. For any sequence $\mathbf{x}_k$
converging to 0 we have, due to this property and positive definiteness
of $V$,
\[
0 \leq \limsup_{k \to \infty} V(\mathbf{x}_k) \leq V(0) = 0,
\]
which implies the continuity of $V$ at 0.

Let us pause to summarize some of the facts established thus far, under
the temporary hypothesis (TA), which is still in force. We know that the
pair $(V, W)$ satisfies the positive definiteness property (L1) and that $V$
is upper semicontinuous at any $x$ and continuous at the origin. Also, as pointed
out earlier, $W$ is locally Lipschitz on $\mathbb{R}^n$. The properness property (L2) of $V$
is addressed next.

**Lemma 4.7.** For every $\alpha \geq 0$, the sublevel set
\[
\{ x \in \mathbb{R}^n : V(x) \leq \alpha \}
\]
is bounded.

**Proof.** This will follow immediately upon verifying
\[
\lim_{|x| \to \infty} V(x) = \infty.
\]
To show this, consider any $x \neq 0$, and let $\hat{x}(\cdot)$ be a solution of (2) with $\hat{x}(0) = x$ and satisfying (54). Then in view of Lemma 2.8 and the monotonicity of $w(\cdot)$, we have

$$V(x) \geq \int_0^T |\hat{x}(t)| \, dt \geq w\left(\frac{|x|}{8} h(\phi(0))\right) \hat{T}(x).$$

Hence, the proof of the lemma will be completed upon showing that

$$w\left(\frac{1}{4} |x|\right) \hat{T}(|x|) \geq \frac{|x|}{8 h(\phi(0))}$$

whenever $|x|$ is sufficiently large. To see that this holds, observe that

$$w\left(\frac{1}{4} |x|\right) = \int_0^{1/4 |x|} e^{-2h(\rho)/h(\phi(0))} \, d\rho \geq \int_{1/8 |x|}^{1/4 |x|} e^{-2h(\rho)/h(\phi(0))} \, d\rho \geq e^{-2h(1/8 |x|)/h(\phi(0))} |x|$$

for any $x$ such that $|x| > 8\phi(0)$. A simple calculation, using the fact that $\eta(r)$ satisfies (50) for large enough $r$, then yields (57).

**Lemma 4.8.** The pair $(V, W)$ satisfies the strong infinitesimal decrease condition (13).

**Proof.** Let $x \neq 0$ and let $v \in F(x)$. For every $y$ we denote by $g(x) \in F(y)$ which is the unique closest point in the compact convex set $F(y)$ to $v$; of course, $g(x) = v$. The function $g: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ is continuous since $F$ is locally Lipschitz (see e.g. [5]). Let $x(\cdot)$ be a (locally defined) solution to the ordinary differential equation

$$\dot{x}(t) = g(x(t)),$$

such that $x(0) = x$; of course, this is also a solution to (2). We have

$$x(\tau) = x + \tau v + o(\tau).$$

Fix $\tau$, and recalling Lemma 4.5, consider a solution $\hat{x}(\cdot)$ of (2) with $\hat{x}(0) = x(\tau)$ such that

$$V(x(\tau)) = \int_0^\infty w(|\hat{x}(t)|) \, dt.$$  

Now define a function $z: [0, \infty) \to \mathbb{R}^n$ via

$$z(t) = \begin{cases} x(t) & \text{if } 0 \leq t < \tau; \\ \hat{x}(t - \tau) & \text{if } t \geq \tau. \end{cases}$$
Clearly \( z(\cdot) \) is a solution of (2) such that \( z(0) = x \), and
\[
V(x) \geq \int_0^\infty w(|z(t)|) \, dt = \int_0^\tau w(|x(t)|) \, dt + \int_\tau^\infty w(|\dot{x}(t-\tau)|) \, dt
\]
\[
= \int_0^\tau w(|x(t)|) \, dt + \int_0^\tau w(|\dot{x}(t)|) \, dt.
\]
By (59) we then have
\[
V(x(\tau)) - V(x) \leq -\int_0^\tau w(|x(t)|) \, dt,
\]
and because of (58),
\[
\frac{V(x + \tau v + o(\tau)) - V(x)}{\tau} \leq -\int_0^\tau w(|x(t)|) \, dt
\]
for small \( \tau > 0 \). This implies
\[
DV(x; v) \leq -w(|x|) = -W(x),
\]
and since \( v \) was an arbitrary vector in \( F(x) \), the strong infinitesimal decrease condition (13) holds.

Now we shall turn to proving that \( V \) is a locally Lipschitz function on \( \mathbb{R}^n \), under the assumption that (TA) holds. We shall employ the following
infinitesimal necessary and sufficient condition (in Dini subderivative terms) due to Clarke, Stern and Wolenski [12] for local Lipschitz behavior
of a function \( f: \mathbb{R}^n \to (-\infty, \infty] \), assumed \textit{a priori} only to be lower semi-
continuous: \( f \) is Lipschitz with Lipschitz constant \( M \) on an open convex set \( U \subseteq \mathbb{R}^n \) iff
\[
Df(x; v) \leq M \, |v| \quad \forall x \in U, \quad \forall v \in \mathbb{R}^n.
\]
In light of Lemma 4.6, the function \(-V(\cdot)\) is lower semicontinuous and
continuous at 0. Then due to (60), a sufficient condition for it to be locally
Lipschitz on \( \mathbb{R}^n \) is the existence of a function \( L: \mathbb{R}^n \to \mathbb{R} \), positive on \( \mathbb{R}^n \setminus \{0\} \), bounded on bounded subsets of \( \mathbb{R}^n \), and such that
\[
D(-V)(x; v) \leq L(x) \, |v| \quad \forall v \in \mathbb{R}^n, \quad \forall x \neq 0.
\]

**Lemma 4.9.** The function \( V \) is locally Lipschitz on \( \mathbb{R}^n \).

**Proof.** We consider an arbitrary \( x \neq 0 \), and show that (61) holds for
an \( L \) as specified above. To this end, choose any \( v \in \mathbb{R}^n \), and let \( \dot{x}(\cdot) \) be a
solution of (2) with $\hat{x}(0) = x$ and satisfying (54). Given a sequence $\epsilon_k \downarrow 0$, we will show that there exists a sequence of solutions $x_k(\cdot)$ of (2) with
\[ x_k(0) = x + \epsilon_k v \] (62)
such that
\[ |x_k(t) - \hat{x}(t)| \leq \epsilon_k |v| e^{Kt} \] (63)
for all $t > 0$, where $K$ is the Lipschitz constant for $F$, as in (TA). To see this, denote by $g(t, x)$ the unique closest point in $F(x)$ to $\hat{x}(t)$. Then $g: \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ determined in this way is a Carathéodory function (see [15, p. 49]), and therefore solutions of the ordinary differential equation
\[ \dot{x}(t) = g(t, x(t)) \]
exist at least locally; we denote by $x_k(\cdot)$ such a solution, satisfying the initial condition (62). The present Lipschitz assumption on $F$ then implies
\[ |x_k(t) - \hat{x}(t)| = d(\hat{x}(t), F(x_k(t))) \leq K |x_k(t) - \hat{x}(t)| \]
for small $t > 0$. Then an application of the Gronwall inequality yields
\[ |x_k(t) - \hat{x}(t)| \leq e^{Kt} |x_k(0) - \hat{x}(0)| \]
for small $t > 0$, which implies (63), as claimed. This in turn implies that $x_k(\cdot)$ exists on the entire interval $[0, \infty)$ (since finite time blow-up has been precluded), and that (63) holds for all $t \geq 0$.

Since
\[ V(x + \epsilon_k v) \geq \int_0^\infty w(|x_k(t)|) \, dt, \]
we obtain that
\[
\begin{align*}
D(-V)(x; v) &\leq \liminf_{k \to \infty} \frac{V(x) - V(x + \epsilon_k v)}{\epsilon_k} \\
&\leq \liminf_{k \to \infty} \int_0^\infty \frac{w(|\dot{x}(t)|) - w(|x_k(t)|)}{\epsilon_k} \, dt.
\end{align*}
\] (64)

Note that $x_k(\cdot)$ satisfies the estimates (56) for sufficiently large $k$, as the solution $\hat{x}(\cdot)$ does.
The Mean Value Theorem implies that for each \( t \geq 0 \) there exists \( \zeta(t) \) in the line segment between \( |x_k(t)| \) and \( |\dot{x}(t)| \) such that

\[
|w(|x_k(t)|) - w(|\dot{x}(t)|)| = |\Phi(\zeta(t))(|x_k(t)| - |\dot{x}(t)|)\|
\leq |\Phi(\zeta(t))| |x_k(t) - \dot{x}(t)|,
\]
(65)

where we have used the fact that \( w' = \Phi \). We clearly have that \( \zeta(\cdot) \) satisfies (56) too. From the definition of \( \Phi \) it then follows that

\[
|\Phi(\zeta(t))| \leq \begin{cases} 
    e^{-2Km|x| + 1} & \text{if } t \in [0, T(x) + 1]; \\
    e^{-2K(T(x) - T(x) - 1)} & \text{if } t > T(x) + 1.
\end{cases}
\]

Here we have used the fact that

\[
\eta(\zeta(t)) \geq \eta(\varphi(t - T(x) - 1)) = t - T(x) - 1
\]

for \( t > T(x) + 1 \). These estimates together with (63) and (65) imply that the integrand occurring in (64) is bounded above by \( \chi(t, x) |w| \), where

\[
\chi(t, x) := \begin{cases} 
    e^{-2Km|x| + 1} + Kt & \text{if } t \in [0, T(x) + 1]; \\
    e^{-Kt + 2K(T(x) + 1)} & \text{if } t > T(x) + 1.
\end{cases}
\]

Then it follows from (64) that the relation (61) holds with \( L \) defined as

\[
L(x) := \int_0^\infty \chi(t, x) \, dt,
\]

which is readily seen to have the required properties. This completes the proof of the lemma.

In order to complete the proof of Proposition 4.2, it remains to relax (TA) to local Lipschitz behavior on \( \mathbb{R}^n \). With this goal in mind, we have the following. First recall Proposition 3.6, which asserted that the local Lipschitzness of \( \hat{F} \) implies the existence of \( l(\cdot) \) satisfying (46)–(47).

**Lemma 4.10.** Let the multifunction \( F \) satisfy (H) and be locally Lipschitz on \( \mathbb{R}^n \setminus \{0\} \). Then there exists a continuous function \( \alpha : [0, \infty) \to [0, \infty) \) such that \( \alpha(r) > 0 \) for \( r > 0 \) and such that the multifunction \( \hat{F}_\alpha \) defined by

\[
\hat{F}_\alpha(x) := \alpha(|x|) \hat{F}(x)
\]
(66)
satisfies (H) and is globally Lipschitz on \( \mathbb{R}^n \setminus \{0\} \).
Proof. In view of (H), Lemma 2.5 can be used to construct a continuous function $\beta : [0, \infty) \to [0, \infty)$ such that
\[ \|F(x)\| \leq \beta(x) \quad \forall x \in \mathbb{R}^n. \tag{67} \]
Define
\[ G_1(r) := \int_{0}^{r} \frac{dp}{1 + \beta(p) + p^2 + \rho^2}, \]
and
\[ G_2(r) := \int_{r}^{\infty} \frac{dp}{1 + \beta(p) + p^2 + \rho^2}. \]
Then $G_1$ is strictly increasing on $[0, \infty)$, while $G_2$ is strictly decreasing on $[0, \infty)$. We now define $\alpha$ to be the lower envelope function
\[ \alpha(r) := \min\{G_1(r), G_2(r)\}. \]
Then $\alpha$ is continuous on $[0, \infty)$, and is continuously differentiable on $(0, \infty)$ except possibly at a single point $\bar{r}$ such that $G_1(\bar{r}) = G_2(\bar{r})$ where one-sided derivatives exist.
Let us choose some $x_0 \neq 0$ and consider the function
\[ \mu(x) := h(\hat{F}_d(x), \hat{F}_d(x_0)), \]
where $h(A, C)$ denotes the Hausdorff distance between the sets $A$ and $C$; that is
\[ h(A, C) := \inf\{r > 0 : A \subset C + rB, C \subset A + rB\}. \]
It is easy to see that the multifunction $\hat{F}_d$ is globally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ with Lipschitz constant $K$ if the same is true of the scalar valued function $\mu$ with arbitrary $x_0 \neq 0$. Since $\mu$ is continuous, it is our intention to use (60) as a criterion for Lipschitzness.
Due to the triangle inequality for the Hausdorff distance, for any $v \in \mathbb{R}^n$ and $\lambda > 0$ we have
\[ \mu(x + \lambda v) - \mu(x) \leq h(\alpha(|x + \lambda v|) F(x + \lambda v), \alpha(|x|) F(x)), \]
and therefore
\[ \mu(x + \lambda v) - \mu(x) \leq h(\alpha(|x + \lambda v|) F(x), \alpha(|x|) F(x)) \]
\[ + h(\alpha(|x + \lambda v|) F(x + \lambda v), \alpha(|x + \lambda v|) F(x)). \]
This readily implies the estimate
\[ D\mu(x; v) \leq (|x'|(|x|) + \|F(x)\| + \alpha(|x|) \ell(|x|)) |v|, \]
where (by abuse of notation) \( \alpha'(|x|) \) is the derivative of \( \alpha(\cdot) \) at \( |x| \) if \( |x| \neq \bar{r} \), or denotes either the right or left derivative if \( |x| = \bar{r} \).

Note that due to the definition of \( \alpha \), one has
\[ |\alpha'(|x|)| \leq 1 \quad \forall x \neq 0. \]
Also, by L'Hôpital's rule
\[ \lim_{r \uparrow 0} \ell(r) = \lim_{r \uparrow 0} \frac{\alpha'(r)}{-\ell'(r)\ell'(r)} = \lim_{r \uparrow 0} \frac{\alpha'(r)}{-\ell'(r)} = 0. \]
These limits and the continuity of \( \alpha \) and \( \ell \) then imply that \( \alpha(|x|) / \ell(|x|) \) is bounded by some positive constant, say \( M - 1 \), for every \( x \in \mathbb{R}^n \). Then
\[ D\mu(x; v) \leq M |v| \quad \forall v \in \mathbb{R}^n. \]
Thus \( \mu(\cdot) \) is Lipschitz with constant \( M \) on \( \mathbb{R}^n \setminus \{0\} \), and as pointed out above, the same is true of the multifunction \( \hat{F}_a \).

We are now in position to complete the proof of Proposition 4.2. We assume that \( F \) satisfies (H), strong asymptotic stability and local Lipschitzness on \( \mathbb{R}^n \setminus \{0\} \), and we seek to verify the existence of a locally Lipschitz strong Lyapunov pair, as in Definition 4.1. Observe that we have completed this task already in the special case in which \( F \) is globally Lipschitz on all of \( \mathbb{R}^n \setminus \{0\} \).

It readily follows from Lemma 2.4 that the multifunction \( \hat{F}_a \) of the previous lemma, in addition to being globally Lipschitz on \( \mathbb{R}^n \setminus \{0\} \) and satisfying (H), is also strongly asymptotically stable. Then there exists a locally Lipschitz strong Lyapunov pair \( (V_a, W_a) \) for \( \hat{F}_a \), and in particular, one has the strong infinitesimal decrease condition
\[ \sup_{x \in \hat{F}_a(x)} DV_a(x; v) \leq -W_a(x) \quad \forall x \neq 0. \]  
(68)
Note that the function \( W_a/\alpha \) is continuous and positive on \( \mathbb{R}^n \setminus \{0\} \), and we can construct a positive locally Lipschitz function \( W \) on \( \mathbb{R}^n \) satisfying \( W \leq W_a/\alpha \). Since Dini subderivatives are positively homogeneous in \( v \), we obtain from (68) that the pair \( (V, W) \) with
\[ V = V_a, \quad W \leq \frac{W_a}{\alpha} \]
is a locally Lipschitz strong Lyapunov pair for \( F \).
5. COMPLETING THE PROOF

Proof of Theorem 1.2. We begin with the converse part of the theorem. Let $F$ satisfy (H) and be strongly asymptotically stable; we are to prove the existence of a $C^\infty$-smooth strong Lyapunov pair $(V, W)$.

It follows from Proposition 3.5 that there exists a multifunction $F_L$ satisfying (H), local Lipschitzness on $\mathbb{R}^n \setminus \{0\}$, strong asymptotic stability, and the containment

$$F(x) \subseteq F_L(x) \quad \forall x \neq 0.$$  

From Proposition 4.2 we obtain that there exists a locally Lipschitz strong Lyapunov pair $(V^L, W^L)$, with the strong infinitesimal decrease condition (13) being

$$\sup_{v \neq F(x)} DV^L(x; v) \leq -W^L(x) \quad \forall x \neq 0.$$  

In view of the local Lipschitzness and Rademacher's theorem, $V^L(\cdot)$ is differentiable almost everywhere, and therefore the previous inequality implies

$$\max_{x \in F_L(x)} \langle \nabla V^L(x), v \rangle \leq -W^L(x) \quad \text{a.e. in } \mathbb{R}^n \setminus \{0\}.$$  

We now turn to the construction of a smooth approximation of $(V^L, W^L)$ which forms a $C^\infty$-smooth strong Lyapunov pair for $F_L$, and therefore, in view of (69), for $F$. In this procedure we shall follow, with some modifications, that given in [26], which generalizes the methods of Kurzweil [23] and Wilson [36].

Let $\omega: \mathbb{R}^n \rightarrow [0, \infty)$ be a $C^\infty$ function with support in the closed unit ball $\bar{B}$, such that

$$\int_{\mathbb{R}^n} \omega(x) \, dx = 1.$$  

(Hereafter, integrals written without limits in this way signify integration over $\mathbb{R}^n$.)

**Lemma 5.1.** Let $\Phi: \mathbb{R}^n \rightarrow [0, \infty)$ be locally Lipschitz, $\Psi: \mathbb{R}^n \rightarrow [0, \infty)$ be continuous, with $\Phi(x) > 0$ and $\Psi(x) > 0$ whenever $x \neq 0$. Suppose that

$$\max_{v \neq F_L(x)} \langle \nabla \Phi(x), v \rangle \leq -\Psi(x) \quad \text{a.e.}$$  

(71)
For \( \sigma > 0 \) define

\[
\Phi_\sigma(x) := \int \Phi(x + \sigma y) \omega(y) \, dy,
\]

\[
\Psi_\sigma(x) := \int \Psi(x + \sigma y) \omega(y) \, dy.
\]

Then \( \Phi_\sigma \) and \( \Psi_\sigma \) are in \( C^\infty(\mathbb{R}^n) \). Furthermore, if \( S \subseteq \mathbb{R}^n \) is compact with \( 0 \notin S \), then for any positive \( \alpha, \beta \), there exists \( \sigma_0 \) such that \( \sigma \in (0, \sigma_0) \) implies

\[
|\Phi_\sigma(x) - \Phi(x)| < \alpha, \quad |\Psi_\sigma(x) - \Psi(x)| < \frac{1}{2} \beta, \quad (72)
\]

\[
\Phi_\sigma(x) > 0, \quad \Psi_\sigma(x) > 0, \quad (73)
\]

and

\[
\max_{v \in F_\sigma(x)} \langle \nabla \Phi_\sigma(x), v \rangle \leq -\Psi(x) + \beta \quad (74)
\]

for every \( x \in S \).

**Proof.** The smoothness of \( \Phi_\sigma \) and \( \Psi_\sigma \) as well as the relations (72)–(73) are standard exercises concerning regularization of functions.

To derive (74), let \( x \in S \) and let \( \ell \) be a Lipschitz constant for \( F_\sigma \) on some neighborhood of \( S \). Choose any \( v \in F_\sigma(x) \), and given any \( y \in B \), let \( g(y) \) be the closest point in \( F_\sigma(x + \sigma y) \) to \( v \). Then the function \( g: B \to \mathbb{R}^n \) is continuous, and there exists \( \sigma_1 > 0 \) such that \( \sigma \in (0, \sigma_1) \) implies

\[
g(y) \in F_\sigma(x + \sigma y), \quad |g(y) - v| \leq \ell \sigma |y| \quad \forall y \in B,
\]

where the positive constant \( \sigma_1 \) is determined only by the above-mentioned neighborhood of \( S \) and does not depend upon the choice of \( x \) and \( v \).

A straightforward argument using the Lebesgue dominated convergence theorem yields the formula

\[
\langle \nabla \Phi_\sigma(x), v \rangle = \int \langle \nabla \Phi(x + \sigma y), v \rangle \omega(y) \, dy,
\]

and therefore (71) implies the existence of \( \sigma_0 \in (0, \sigma_1) \) such that for \( \sigma \in (0, \sigma_0) \)

\[
\langle \nabla \Phi_\sigma(x), v \rangle = \int \langle \nabla \Phi(x + \sigma y), g(y) \rangle \omega(y) \, dy
\]

\[
+ \int \langle \nabla \Phi(x + \sigma y), v - g(y) \rangle \omega(y) \, dy
\]

\[
\leq -\Psi_\sigma(x) + \ell \sigma \int |\nabla \Phi(x + \sigma y)||y| \omega(y) \, dy.
\]

Since \( v \in F_\sigma(x) \) was arbitrary, (74) follows from (71) and (72).
Now let \( \{ U_i \}_{i=1}^\infty \) be a locally finite open cover of \( \mathbb{R}^n \) with \( U_i \) bounded and \( 0 \notin \mathring{U}_i \) for every \( i \), and let \( \{ \psi_i \}_{i=1}^\infty \) be a subordinated \( C^\infty \) partition of unity.

Define the quantities
\[
e_i := \frac{1}{4} \min \{ \min_{V_i} V_i, \min_{W_i} W_i \}
\]
and
\[
q_i := \max_{V_i} \| \nabla \psi_i \| \| F_L \|.
\]
In view of Lemma 5.1, for every \( i \) there exist \( C^\infty \)-smooth functions \( V_i : \mathbb{R}^n \rightarrow (0, \infty) \), \( W_i : \mathbb{R}^n \rightarrow (0, \infty) \), such that for any \( x \in U_i \) and any \( v \in F_L(x) \) one has
\[
|V^L_i(x) - V_i(x)| < \frac{e_i}{2^{1+1}(1+q_i)},
\]
\[
|W^L_i(x) - W_i(x)| < e_i,
\]
and
\[
\langle \nabla V_i(x), v \rangle < -W^L_i(x) + 2e_i \leq -\frac{1}{2}W^L_i(x).
\]
Let us define a function
\[
V(x) := \begin{cases} 
\sum_i \psi_i(x) V_i(x) & \text{if } x \neq 0; \\
0 & \text{if } x = 0.
\end{cases}
\]
Since for any \( x \in \mathbb{R}^n \)
\[
|V^L_i(x) - V_i(x)| \leq \sum_i \psi_i(x) |V^L_i(x) - V_i(x)| \leq \frac{1}{2}V^L_i(x),
\]
we obtain that the function \( V \) satisfies (L1)-(L2) (positive definiteness and properness), and continuity at the origin.

Since \( V \) is clearly \( C^\infty \)-smooth on \( \mathbb{R}^n \setminus \{0\} \) and \( F_L \) is locally Lipschitz there, in order to verify the strong infinitesimal decrease condition (3), it is clearly sufficient to verify it at almost all points; in particular, at those points \( x \neq 0 \) where the locally Lipschitz function \( V^L_i \) is differentiable. For any such point we have
\[
\langle \nabla V(x), v \rangle = \langle \nabla V^L(x), v \rangle + \sum_i \psi_i(x) \langle \nabla V_i(x) - \nabla V^L(x), v \rangle \\
+ \sum_i \langle \nabla \psi_i(x), v \rangle (V_i(x) - V^L(x)) \\
= \sum_i \psi_i(x) \langle \nabla V_i(x), v \rangle + \sum_i \langle \nabla \psi_i(x), v \rangle (V_i(x) - V^L(x)) \\
\leq \sum_i \psi_i(x) \left( -\frac{1}{2} W^L(x) \right) + \sum_i \langle \nabla \psi_i(x), v \rangle (V_i(x) - V^L(x)) \\
\leq \sum_i \psi_i(x) \left( -\frac{1}{2} W^L(x) + \varepsilon_i \right) \leq -\frac{1}{2} \sum_i \psi_i(x) W_i(x),
\]
as required.

Now let
\[
W := \frac{1}{8} \sum_i \psi_i W_i.
\]

Then \( W(x) > 0 \) for every \( x \neq 0 \), and the pair \((V, W)\) is a strong Lyapunov pair which is \( C^\infty \)-smooth on \( \mathbb{R}^n \setminus \{0\} \), with \( V \) continuous at the origin.

In order to obtain a \( V \) which is \( C^\infty \) on all of \( \mathbb{R}^n \), we apply Lemma 4.3 of [26] which asserts that there exists a function \( \beta: [0, \infty) \rightarrow [0, \infty) \) which is positive and \( C^\infty \)-smooth on \((0, \infty)\), with its derivative \( \beta'(\cdot) \) positive on \((0, \infty)\) as well, and such that the function \( \bar{V}(\cdot) := \beta(V(\cdot)) \) is \( C^\infty \)-smooth on \( \mathbb{R}^n \). Then for any \( x \neq 0 \) and any \( v \in F(x) \) one has
\[
\langle \nabla \bar{V}(x), v \rangle = \beta'(V(x)) \langle \nabla V(x), v \rangle \leq -\beta'(V(x)) W(x).
\]

Upon defining
\[
\tilde{V}(x) := \beta'(V(x)) W(x), \quad x \neq 0,
\]
it follows that the pair \((\tilde{V}, \tilde{W})\) is a \( C^\infty \)-smooth strong Lyapunov pair for \( F^L \), and therefore for \( F \) as well. Thus, the converse Lyapunov part of the theorem is proven.

We now turn to showing that the existence of a \( C^\infty \)-smooth strong Lyapunov pair for \( F \) implies strong asymptotic stability. We provide here only an outline of the proof since it is essentially standard (see [24]). First we construct a useful comparison equation for deriving the decay estimate (29) for solutions of (2). Towards this end, we define a function \( \gamma: (0, \infty) \rightarrow (0, \infty) \) via
\[
\gamma(\varepsilon) := \min \{ W(x): x \in X(\varepsilon) \},
\]
where \( X \) is the multifunction on \((0, \infty)\) given by
\[
X(v) := \{ x \in \mathbb{R}^n : V(x) = v \}.
\]
It is not difficult to show that the multifunction \( X \) is locally Lipschitz, which implies that the function \( \gamma \) is locally Lipschitz on \((0, \infty)\). Thus for any \( v > 0 \) there exists a unique solution \( \rho(\cdot; v) \) of the one-dimensional initial value problem
\[
\dot{\rho}(t) = -\gamma(\rho(t)), \quad \rho(0) = v
\]
which is defined on \([0, \infty)\), strictly decreasing to 0 in \( t \) for fixed \( v \) and is strictly increasing in \( v \) for fixed \( t \). For an arbitrary solution \( x(\cdot) \) of the differential inclusion (2) we have
\[
\frac{d}{dt} V(x(t)) = \langle \nabla V(x(t)), \dot{x}(t) \rangle \leq -W(x(t)) \leq -\gamma(V(x(t))),
\]
which implies that \( V(x(t)) \) is strictly decreasing, \( x(\cdot) \) does not blow-up and is defined on the entire interval \([0, \infty)\). It follows from the above differential inequality (see [24]) that for all \( t > 0 \)
\[
V(x(t)) \leq \rho(t; V(x(0))).
\]
Now we use the existence of positive strictly increasing functions \( \varphi_i : [0, \infty) \rightarrow [0, \infty), i = 1, 2 \) satisfying (28) to obtain from the previous inequality the fact that the decay estimate (29) is valid for \( x(\cdot) \) with the function
\[
\beta(t, R) = \varphi_1^{-1}(\rho(t, \varphi_2(R))).
\]
This implies strong asymptotic stability of \( F \) and concludes the proof of Theorem 1.2.

**Proof of Theorem 1.3.** We need to prove that condition (8) is equivalent to the strong infinitesimal decrease condition (3) with the multifunction \( F \) coinciding with the right-hand side in (6) (the Krasovskii solution case) and that (9) is equivalent to (3) with \( F \) coinciding with the right-hand side in (7) (the Filippov solution case).

Let us consider Krasovskii solutions (case (6)). It is easy to check (see also [18]) that
\[
\max_{f \in F(x)} \langle \nabla V(x), f \rangle = \lim_{y \rightarrow x} \sup_{y \in y} \langle \nabla V(x), f(y) \rangle.
\]
This relation implies that (3) is equivalent to (8) for the differential inclusion (6). In the case of Filippov solutions (7), it is not hard to verify that (see also [16, 18])

\[ \max_{f \in F(x)} \langle \nabla V(x), f \rangle = \text{ess lim sup}_{y \to x} \langle \nabla V(x), f(y) \rangle, \]

which implies the equivalence of (3) and (9) for the differential inclusion (7).

6. A NECESSARY COVERING CONDITION FOR THE EXISTENCE OF A SMOOTH WEAK LYAPUNOV PAIR

In this section we will prove the following result:

**Theorem 6.1.** The existence of a $C^1$-smooth weak Lyapunov pair for a multifunction $F$ satisfying (H) implies the following covering condition: For any given $\gamma > 0$, there exists $\delta > 0$ such that (12) holds.

Our proof of Theorem 6.1 will not have direct reliance on the concept of topological degree or the Lefschetz fixed point theorem, as do the proofs of the necessary covering conditions in Ryan [31] or Brockett [8], respectively. Instead, we will utilize the following fixed point theorem of Horn [19], which is a refined version of the Browder fixed point theorem [9]. We also refer the reader to Krasnoselskii [20] for applications of the topological degree in studying asymptotic stability of ordinary differential equations.

**Theorem 6.2.** Let $S_0 \subset S_1 \subset S_2$ be bounded convex subsets of $\mathbb{R}^n$ such that $S_0$ and $S_2$ are closed and $S_1$ is a neighborhood of $S_0$ relative to $S_2$. Let $g : S_2 \to \mathbb{R}^n$ be a continuous mapping such that for some positive integer $K$, the iterates $g^k$ of $g$ satisfy

\[ g^k(S_1) \subset S_2, \quad 1 \leq k \leq 2K - 1, \quad g^k(S_1) \subset S_2, \quad 1 \leq k \leq K - 1. \]

Then $S_0$ contains at least one fixed point of $g$.

**Proof of Theorem 6.1.** Let $\gamma > 0$ be given. Due to (L1)-(L2) there exist positive numbers $\gamma', \gamma''$ such that

\[ 2\gamma' \bar{B} \subset S := \{ x \in \mathbb{R}^n : V(x) < \gamma'' \} \subset \gamma \bar{B}. \]

We shall require the following technical lemma:
For any $\varepsilon \in (0, \gamma')$ there exists a $C^\infty$-smooth function $f_\varepsilon: \mathbb{R}^n \to \mathbb{R}^n$ such that
\[ f_\varepsilon(x) \in \overline{F}(x + \varepsilon B) \] (75)
and
\[ \langle \nabla V(x), f_\varepsilon(x) \rangle < -\frac{1}{2} W(x) \quad \forall x \notin S. \] (76)

**Proof.** Fix $\varepsilon \in (0, \gamma')$. For each $x \notin \gamma'B$, choose some vector $v_x \in F(x)$ such that
\[ \langle \nabla V(x), v_x \rangle \leq -W(x), \] (77)
and define an open set
\[ \Omega_\varepsilon := \{ y \in \mathbb{R}^n: |y - x| < \varepsilon, \langle \nabla V(y), v_x \rangle < -\frac{1}{2} W(y) \}, \]
which is non-empty due to (77). For $x \notin \gamma'B$, let $v_x$ be an arbitrary vector in $F(x)$ and let $0 \in x$. The collection $\{0 \} \cup x_{\varepsilon} B$ is an open covering of $\mathbb{R}^n$. Let $\{U_i\}_{i=1}^\infty$ be locally finite refinement, and let $\{\psi_i\}_{i=1}^\infty$ be a subordinated $C^\infty$ partition of unity. For each index $i$ pick some $x_i$ such that $U_i \subset \Omega_{x_i}$ and define
\[ f_\varepsilon(x) := \sum_i \psi_i(x) v_{x_i}. \]
It is clear that $f_\varepsilon$ is $C^\infty$-smooth. Also, for arbitrary $x$, if $i$ is such that $\psi_i(x) > 0$, then $x \in U_i \subset \Omega_{x_i}$, and in particular,
\[ |x - x_i| < \varepsilon < \gamma'. \]
This immediately implies (75). If $x \notin S$, it follows that $x \notin \gamma'B$ and
\[ \langle \nabla V(x), v_x \rangle < -\frac{1}{2} W(x). \]
Then (79) follows readily. \[ \square \]

Consider the differential equation
\[ \dot{x} = f_\varepsilon(x) - z, \] (78)
where $z$ is a constant vector.
**Lemma 6.4.** There exists positive constants $A, m,$ and $T$ such that for any $\epsilon \in (0, \gamma')$ and any $z \in A\hat{B}$, every solution $x(\cdot)$ of (78) with $|x(0)| \leq 2\gamma$ is defined on $[0, \infty)$ and satisfies

$$|x(t)| < m \quad \forall t \geq 0$$

as well as

$$|x(t)| \leq \gamma \quad \forall t \geq T.$$  

**Proof.** Let $x : [0, \infty) \to \mathbb{R}^n$ be a continuously differentiable function such that $|x(0)| \leq 2\gamma$ and

$$\langle \nabla V(x(t)), x(t) \rangle = -\frac{1}{4} W(x(t)) \forall t \quad \text{such that } V(x(t)) \geq \gamma'.$$

Then, by using the same arguments as in the second part of the proof of Theorem 1.2, we derive the existence of positive constants $m > 2\gamma$ and $T$ (which are the same for any $x(\cdot)$ considered above) such that (79) holds and $V(x(t)) \leq \gamma'$ for all $t \geq T$. The last relation implies (80).

Define

$$A := \min_{x \in \{m\} \setminus S} \frac{W(x)}{4 |\nabla V(x)|},$$

and let $x(\cdot)$ be an arbitrary solution of (78) with $|x(0)| \leq 2\gamma$ satisfying (79) on some interval $[0, T')$. Due to (76), $x(\cdot)$ also satisfies (81) on $[0, T')$. This implies that $x(\cdot)$ stays in the ball $m\hat{B}$, exists on the entire interval $[0, \infty)$ and satisfies (81) there. This implies that (80) also holds.

Consider $A$ as in Lemma 6.4, an arbitrary vector $z \in A\hat{B}$, an arbitrary integer $K > 0$, and define the continuous mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x) := x_{\tau_K}^K,$$

where $x_{\tau_K}^K$ is the (unique) solution of the differential equation (78) satisfying $x(0) = x$, and $\tau_K := T/K$.

It is easy to see from (79) and (80) that the iterates $g^k, K \leq k \leq 2K - 1,$ map the ball $S_{\epsilon} := 2\gamma \hat{B}$ into $S_0 := \gamma \hat{B}$ and $g^k(S_\epsilon) \subset S_2 := m\hat{B}$ for $1 \leq k \leq K - 1$. By Theorem 6.2 we have the existence of a fixed point $x_K$ of $g$ in $\gamma \hat{B}$. Then

$$x_K = x_{\tau_K}^K = x_K + \int_0^\tau (f(x_{\tau_K}^K; x_K) - z) \, ds.$$
This implies
\[
\frac{1}{\tau_{K}} \int_{\tau_{K}}^{\tau_{K+1}} (f_{x}(x(s; x_{K}))-z) \, ds = 0.
\]
Without loss of generality we can assume that \(x_{K}\) converges to some \(x_{*}\) in \(\gamma B\) as \(K \to \infty\). Then the previous relation implies
\[
z = f_{x}(x_{*}).
\]
In view of (75) we arrive at
\[
z \in \overline{\delta F(x_{*}+\varepsilon B)}
\]
for every positive \(\varepsilon\). Take a sequence \(\varepsilon_{i} \downarrow 0\), and assume without loss of generality that \(x_{n}\) converges to some \(\hat{x} \in \gamma \overline{B}\). Since hypothesis (H) holds for the multifunction \(\overline{\delta F}\), we obtain that \(z \in \overline{\delta F(\hat{x})} = F(\hat{x})\). Since \(z\) is an arbitrary vector from \(\Delta B\) and \(\hat{x} \in \gamma \overline{B}\), the proof of the theorem is completed.

Note that if \(F\) is strongly asymptotically stable, then by Theorem 1.2 there exists a \(C^{n}\)-smooth strong Lyapunov pair, which is obviously a \(C^{n}\)-smooth weak Lyapunov pair. Thus we have the following:

**Corollary 6.5.** Let \(F\) be strongly asymptotically stable. Then for every \(\gamma > 0\) there exists \(\Delta > 0\) such that the covering condition (12) holds.

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