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A chain of evolution algebras

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ABSTRACT

We introduce a notion of chain of evolution algebras. The sequence of matrices of the structural constants for this chain of evolution algebras satisfies an analogue of Chapman–Kolmogorov equation. We give several examples (time homogenous, time non-homogenous, periodic, etc.) of such chains. For a periodic chain of evolution algebras we construct a continuum set of non-isomorphic evolution algebras and show that the corresponding discrete time chain of evolution algebras is dense in the set. We obtain a criteria for an evolution algebra to be baric and give a concept of a property transition. For several chains of evolution algebras we describe the behavior of the baric property depending on the time. For a chain of evolution algebras given by the matrix of a two-state evolution we define a baric property controller function and under some conditions on this controller we prove that the chain is not baric almost surely (with respect to Lebesgue measure). We also construct examples of the almost surely baric chains of evolution algebras. We show that there are chains of evolution algebras such that if it has a unique (resp. infinitely many) absolute nilpotent element at a fixed time, then it has unique (resp. infinitely many) absolute nilpotent element any time; also there are chains of evolution algebras which have not such property. For an example of two dimensional chain of evolution algebras we give the full set of idempotent elements and show that for some values of parameters the number of idempotent elements does not depend on time, but for other values of parameters there is a critical time t_c such that the chain has only two idempotent elements if time $t \geqslant t_c$ and it has four idempotent elements if time $t < t_c$.

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1. Introduction

In this paper we consider some classes of non-associative algebras. There exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose investigation has provided a number of significant contributions to theoretical population genetics. Such classes have been defined at different times by several authors, and all the algebras belonging to these classes are generally called "genetic". Etherington introduced the formal language of abstract algebra to study of genetics in his series of seminal papers [2–4]. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. The most comprehensive references for the mathematical research done in this area are [11,12,15,16].

In [11] an evolution algebra $\mathcal{A}$ associated to the free population is introduced and using this non-associative algebra many results are obtained in explicit form, e.g. the explicit description of stationary quadratic operators, and the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection.

In [15] a new type of evolution algebra is introduced. This evolution algebra is defined as follows. Let $(E, \cdot)$ be an algebra over a field K. If it admits a basis $e_1, e_2, \dots$, such that $e_i \cdot e_j = 0$, if $i \neq j$ and $e_i \cdot e_i = \sum_k a_{ik} e_k$, for any i, then this algebra is called an *evolution algebra*.

In this paper by the term *evolution algebra* we will understand a finite dimensional evolution algebra E (as mentioned above) over field $\mathbb{R}$.

Evolution algebras have the following elementary properties (see [15]): Evolution algebras are not associative, in general; they are commutative, flexible, but not power-associative, in general; direct sums of evolution algebras are also evolution algebras; Kronecker products of evolutions algebras are also evolution algebras.

The concept of evolution algebras lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebra; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, stochastic processes, mathematical physics, etc.

In the book [15], the foundation of evolution algebra theory and applications in non-Mendelian genetics and Markov chains are developed, with pointers to some further research topics.

In [13] the algebraic structures of function spaces defined by graphs and state spaces equipped with Gibbs measures by associating evolution algebras are studied. Results of [13] also allow a natural introduction of thermodynamics in studying of several systems of biology, physics and mathematics by theory of evolution algebras.

The paper is organized as follows. In section we give main definitions related to a chain of evolution algebras. Therein we give several examples (time homogenous, time non-homogenous, periodic, etc.) of such chains. For a periodic chain of evolution algebras we construct a continuum set of non-isomorphic evolution algebras and show that the corresponding discrete time chain of evolution algebras is dense in the set. In Section 3 we obtain a criteria for an evolution algebra to be baric. The concept of a property transition is introduced in Section 4. This section also contains several chains of evolution algebras for which we describe the behavior of the baric property depending on the time. For a chain of evolution algebras given by the matrix of a two-state evolution we define a baric property controller function and under some conditions on this controller we prove that the chain is not baric almost surely (with respect to Lebesgue measure). We also construct examples of the almost surely baric chains of evolution algebras. We show that there are chains of evolution algebras such that if it has a unique (resp. infinitely many) absolute nilpotent element at a fixed time, then it has unique (resp. infinitely many) absolute nilpotent element any time; also there are chains of evolution algebras which have not such property. In the last subsection for an example of two dimensional chain of evolution algebras we give the full set of idempotent elements and show that for some values of parameters the number of idempotent elements does not depend on time, but for other values of parameters there is a critical time t_c such that the chain has only two idempotent elements if time $t \geqslant t_c$ and it has four idempotent elements if time $t < t_c$.

2. Definition and examples of CEA

Consider a family $\left\{E^{[s,t]}\ :\ s,t\in\mathbb{R},\ 0\leqslant s\leqslant t\right\}$ of n-dimensional evolution algebras over the field $\mathbb{R}$, with basis $e_1,\ldots,e_n$ and multiplication table

$$e_ie_i=M_i^{[s,t]}=\sum_{j=1}^{n}a_{ij}^{[s,t]}e_j,\quad i=1,\ldots,n;\quad e_ie_j=0,\quad i\neq j. \tag{2.1}$$

Here parameters s,t are considered as time.

Denote by $\mathcal{M}^{[s,t]}=\left(a_{ij}^{[s,t]}\right)_{i,j=1,\ldots,n}$-the matrix of structural constants.

Definition 2.1. A family $\left\{E^{[s,t]}\ :\ s,t\in\mathbb{R},\ 0\leqslant s\leqslant t\right\}$ of n-dimensional evolution algebras over the field $\mathbb{R}$ is called a chain of evolution algebras (CEA) if the matrix $\mathcal{M}^{[s,t]}$ of structural constants satisfies the Chapman–Kolmogorov equation

$$\mathcal{M}^{[s,t]}=\mathcal{M}^{[s,\tau]}\mathcal{M}^{[\tau,t]},\quad\text{for any}\quad s<\tau<t. \tag{2.2}$$

If ρ_i is a projection map of $E^{[s,t]}$, which maps every element of $E^{[s,t]}$ to its e_i component, then Eq. (2.2) can be written as

$$M_i^{[s,t]}=\sum_{j=1}^{n}\rho_j(M_i^{[s,\tau]})M_j^{[\tau,t]},\quad\text{for any}\quad s<\tau<t. \tag{2.3}$$

Definition 2.2. A CEA is called a time-homogenous CEA if the matrix $\mathcal{M}^{[s,t]}$ depends only on $t-s$. In this case we write $\mathcal{M}^{[t-s]}$.

Definition 2.3. A CEA is called periodic if its matrix $\mathcal{M}^{[s,t]}$ is periodic with respect to at least one of the variables s,t, i.e. (periodicity with respect to t) $\mathcal{M}^{[s,t+P]}=\mathcal{M}^{[s,t]}$ for all values of t. The constant P is called the period, and is required to be non-zero.

Remark 2.4. In general, an algebra $\mathcal{A}^{[s,t]}$ can be given by a cubic matrix $\mathcal{M}^{[s,t]}=\left(a_{ijk}^{[s,t]}\right)_{i,j,k=1,\ldots,n}$ of structural constants. Our Definition 2.1 can be extended to $\mathcal{A}^{[s,t]}$ using analogues of the Chapman–Kolmogorov equations for quadratic operators (see [6,7,14]). Since in the general case there are two types of the Chapman–Kolmogorov equations: type A and type B [6], one also can define two types of chain of (general) algebras using the Chapman–Kolmogorov equations of type A and type B, respectively. In this paper we shall only consider CEA, which is more simple than general case, because it is defined by quadratic matrices.

The CEA corresponding to a Markov process

Let $\left\{\mathcal{M}^{[s,t]},\quad 0\leqslant s\leqslant t\right\}$ be a family of stochastic matrices which satisfies the Eq. (2.2), then it defines a Markov process. Thus we have

Theorem 2.5. *For each Markov process, there is a CEA whose structural constants are transition probabilities of the process, and whose generator set (basis) is the state space of the Markov process.*

If $\mathcal{M}^{[s,t]}$ does not depend on time (i.e. $=\mathcal{M}$) then the CEA contains only one evolution algebra E. Note that for a Markov chain defined by $\mathcal{M}$ the corresponding E has been studied in [15].

Now we shall give several concrete examples of CEA.

Example 1. To show a time dependent CEA we use the following example of time homogenous Markov process (see [10]) : for $n = 3$ consider

$$a_{ii}^{[t]} = \frac{2}{3}e^{-\frac{3}{2}At}\cos(\alpha t) + \frac{1}{3}, \quad i = 1, 2, 3;$$

$$a_{12}^{[t]} = a_{23}^{[t]} = a_{31}^{[t]} = e^{-\frac{3}{2}At}\left(\frac{1}{\sqrt{3}}\sin(\alpha t) - \frac{1}{3}\cos(\alpha t)\right) + \frac{1}{3};$$

$$a_{21}^{[t]} = a_{32}^{[t]} = a_{13}^{[t]} = -e^{-\frac{3}{2}At}\left(\frac{1}{\sqrt{3}}\sin(\alpha t) + \frac{1}{3}\cos(\alpha t)\right) + \frac{1}{3},$$

where $A > 0, \alpha = \frac{\sqrt{3}}{2}A$.

Let $E^{[t]}$, $t \geqslant 0$ be the corresponding CEA. It is easy to see that $E^{[t]}$ has an oscillation behavior depending on time t. Moreover $\lim_{t\to+\infty} E^{[t]} = E$, where E is an evolution algebra with the multiplication table

$$e_1^2 = e_2^2 = e_3^2 = \frac{1}{3}(e_1 + e_2 + e_3), \quad e_ie_j = 0, i \neq j.$$

The CEA corresponding to a family of matrices which do not define a process.

Example 2. We shall give a time homogenous CEA which are different from CEAs corresponding to Markov processes. For $n = 2$ take

$$a_{11}^{[t]} = a_{22}^{[t]} = a^{[t]}; \quad a_{12}^{[t]} = a_{21}^{[t]} = b^{[t]}.$$

Then Eq. (2.2) is equivalent to

$$a^{[t]} = a^{[\tau]}a^{[t-\tau]} + b^{[\tau]}b^{[t-\tau]};$$

$$b^{[t]} = a^{[\tau]}b^{[t-\tau]} + b^{[\tau]}a^{[t-\tau]}.$$

Denote $f(t) = a^{[t]} + b^{[t]}$, $\varphi(t) = a^{[t]} - b^{[t]}$, then the last system of functional equations can be written as

$$f(t) = f(\tau)f(t-\tau), \quad \varphi(t) = \varphi(\tau)\varphi(t-\tau).$$

Both these equations are known as exponential Cauchy equation and the system of equations has solution $f(t) = \lambda^t, \varphi(t) = \mu^t$, where $\lambda, \mu \geqslant 0$. Consequently, $a^{[t]} = \frac{1}{2}(\lambda^t + \mu^t)$, $b^{[t]} = \frac{1}{2}(\lambda^t - \mu^t)$. But this solution does not define any Markov process, in general.

Let $E^{[t]}$, $t \geqslant 0$ be the corresponding CEA. Depending on parameters λ and μ we get distinct behavior of $E^{[t]}$ for $t \to +\infty$, i.e. we have

$$\lim_{t\to+\infty} E^{[t]} = \begin{cases} E_0 \text{ if } 0 < \lambda, \mu < 1, \\ E_1 \text{ if } \lambda = \mu = 1, \\ E_{1/2} \text{ if } \lambda = 1, 0 \leqslant \mu < 1, \\ E_{-1/2} \text{ if } \mu = 1, 0 \leqslant \lambda < 1, \\ E_\infty \text{ otherwise}, \end{cases}$$

where E_0 is an evolution algebra with zero multiplication; E_1 is an evolution algebra with multiplication table

$$e_1^2 = e_1, e_2^2 = e_2, \quad e_1e_2 = 0;$$

$E_{1/2}$ is an evolution algebra with multiplication table

$$e_1^2 = e_2^2 = \frac{1}{2}(e_1 + e_2), \quad e_1e_2 = 0;$$

$E_{-1/2}$ is an evolution algebra with multiplication table

$$e_1^2 = \frac{1}{2}(e_1 - e_2), \quad e_2^2 = -\frac{1}{2}(e_1 - e_2), \quad e_1e_2 = 0;$$

and E_∞ is a vector space which has "infinity multiplication", or we can say that in E_∞ an algebra structure is not defined. This example shows that a limit of a CEA can be non-evolution algebra.

Example 3. *A two-state evolution.* Now we shall give an example of time non-homogeneous CEA, the matrix of structural constants of which also does not define any (time non-homogenous) Markov process in general. Consider $n = 2$ and matrix $\mathcal{M}^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,2}$ with

$$\begin{aligned} a_{11}^{[s,t]} &= \tfrac{1}{2}\left(1 + \alpha(s,t) + \beta(s,t)\right), \quad a_{12}^{[s,t]} = \tfrac{1}{2}\left(1 - \alpha(s,t) - \beta(s,t)\right), \\ a_{21}^{[s,t]} &= \tfrac{1}{2}\left(1 + \alpha(s,t) - \beta(s,t)\right), \quad a_{22}^{[s,t]} = \tfrac{1}{2}\left(1 - \alpha(s,t) + \beta(s,t)\right). \end{aligned} \tag{2.4}$$

In this case the Eq. (2.2) is equivalent to (see [9])

$$\begin{aligned} \alpha(s,t) &= \alpha(\tau,t) + \alpha(s,\tau)\beta(\tau,t), \\ \beta(s,t) &= \beta(s,\tau)\beta(\tau,t), \quad s < \tau < t. \end{aligned} \tag{2.5}$$

The second equation of the system (2.5) is known as Cantor's second equation, it has very rich family of solutions: $\beta(s,t) = \frac{\Phi(t)}{\Phi(s)}$, where Φ is an arbitrary function with $\Phi(s) \neq 0$. Using this function β for the function α we obtain

$$\frac{\alpha(s,t)}{\Phi(t)} = \frac{\alpha(\tau,t)}{\Phi(t)} + \frac{\alpha(s,\tau)}{\Phi(\tau)}.$$

Now denote $\gamma(s,t) = \frac{\alpha(s,t)}{\Phi(t)}$ then the last equation gets the following form

$$\gamma(s,t) = \gamma(s,\tau) + \gamma(\tau,t).$$

This equation is known as Cantor's first equation which also has very rich family of solutions: $\gamma(s,t) = \Psi(t) - \Psi(s)$, where Ψ is an arbitrary function. Hence a solution $\mathcal{M}^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,2}$ to the Eq. (2.2) is given by

$$a_{11}^{[s,t]} = \frac{1}{2}\left(1 + \Phi(t)(\Psi(t) - \Psi(s)) + \frac{\Phi(t)}{\Phi(s)}\right),$$

$$a_{12}^{[s,t]} = \frac{1}{2}\left(1 - \Phi(t)(\Psi(t) - \Psi(s)) - \frac{\Phi(t)}{\Phi(s)}\right),$$

$$a_{21}^{[s,t]} = \frac{1}{2}\left(1 + \Phi(t)(\Psi(t) - \Psi(s)) - \frac{\Phi(t)}{\Phi(s)}\right),$$

$$a_{22}^{[s,t]} = \frac{1}{2}\left(1 - \Phi(t)(\Psi(t) - \Psi(s)) + \frac{\Phi(t)}{\Phi(s)}\right).$$

Let $E^{[s,t]}$, $0 \leqslant s \leqslant t$ be the corresponding to this solution CEA. This CEA varies by two parameters, for example, if $t = s$ we get $E^{[t,t]} = E$ with multiplication table $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2 = 0$. Moreover, choosing functions Φ and Ψ one can variate the limit behavior of the CEA. For example, if Φ and Ψ such that $\lim_{t\to+\infty} \Phi(t)\Psi(t) = \lim_{t\to+\infty} \Phi(t) = 0$, then for a fixed s we have $\lim_{t\to+\infty} E^{[s,t]} = E_{1/2}$, where $E_{1/2}$ is an evolution algebra with multiplication table

$$e_1^2 = e_2^2 = \frac{1}{2}(e_1 + e_2), \quad e_1e_2 = 0.$$

Example 4. *A n-dimensional time non-homogenous CEA.* Here for arbitrary n we shall give an example of time non-homogenous CEA. Let $\{A^{[t]},\ t \geqslant 0\}$ be a family of invertible (for all t), $n \times n$ matrices. Define the following matrix

$$M^{[s,t]} = A^{[s]}(A^{[t]})^{-1},$$

where $(A^{[t]})^{-1}$ is the inverse of $A^{[t]}$.

This matrix satisfies the Eq. (2.2). Indeed, using associativity of the multiplication of matrices we get

$$\mathcal{M}^{[s,\tau]}\mathcal{M}^{[\tau,t]} = A^{[s]}\left((A^{[\tau]})^{-1}A^{[\tau]}\right)(A^{[t]})^{-1} = A^{[s]}(A^{[t]})^{-1} = \mathcal{M}^{[s,t]}.$$

Thus each family (with one parameter) of invertible $n \times n$ matrices defines a CEA $E^{[s,t]}$ which is time non-homogenous, in general. But will be a time homogenous CEA, for example, if $A^{[t]}$ is equal to tth power of an invertible matrix A.

Construction of a family of invertible $n \times n$ matrices $A^{[t]}$ is not difficult, for example, one can take $A^{[t]}$ as a triangular $n \times n$ matrix of the form

$$A^{[t]} = \begin{pmatrix} a_{11}^{[t]} & 0 & 0 & \dots & 0 \\ a_{21}^{[t]} & a_{22}^{[t]} & 0 & \dots & 0 \\ a_{31}^{[t]} & a_{32}^{[t]} & \ddots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n1}^{[t]} & a_{n2}^{[t]} & \dots & a_{nn-1}^{[t]} & a_{nn}^{[t]} \end{pmatrix}.$$

which is called lower triangular matrix or one can take an upper triangular matrix. Then the matrices are invertible iff $a_{ii}^{[t]} \neq 0$, for all $i = 1, \dots, n$ and t. So this example also gives a very rich class of CEAs.

Example 5. *Periodic CEA*. To get a periodic CEA, we can consider the $E^{[s,t]}$ constructed in Example 3, and choose Φ and Ψ as periodic (non-constant) functions. Then corresponding CEA is periodic. In this case for any fixed s, the limit $\lim_{t\to+\infty} E^{[s,t]}$ does not exist in general, moreover its set of limit points (evolution algebras) can be a continuum set. We shall make this point clear as follows. Construct a time homogenous CEA which is periodic. Consider $n = 2$ take

$$a_{11}^{[t]} = a_{22}^{[t]} = a^{[t]}; \quad a_{12}^{[t]} = -b^{[t]}, \quad a_{21}^{[t]} = b^{[t]}.$$

Then Eq. (2.2) is equivalent to

$$a^{[t]} = a^{[\tau]}a^{[t-\tau]} - b^{[\tau]}b^{[t-\tau]};$$

$$b^{[t]} = a^{[\tau]}b^{[t-\tau]} + b^{[\tau]}a^{[t-\tau]}.$$

This system reminds the following identities

$$\cos t = \cos\tau\cos(t-\tau) - \sin\tau\sin(t-\tau);$$

$$\sin t = \cos\tau\sin(t-\tau) + \sin\tau\cos(t-\tau).$$

Consequently, one solution $\mathcal{M}^{[t]} = \left(a_{ij}^{[t]}\right)_{i,j=1,2}$ to Eq. (2.2) is

$$\mathcal{M}^{[t]} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{2.6}$$

Since that matrix is periodic with period $P = 2\pi$, the corresponding CEA $E^{[t]}$ is also periodic. Moreover this CEA is very interesting: for arbitrary 2-dimensional evolution algebra E_a^+, or E_a^-, $a \in [-1, 1]$ with structural constants matrix

$$\mathcal{M}_a^{\pm} = \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \mp\sqrt{1-a^2} & a \end{pmatrix}$$

respectively, there is a sequence $t_n = t_n(a)$ of times such that $\lim_{n\to\infty} E^{[t_n]} = E_a^+$ or E_a^-. We have $E_a^{\pm} \neq E_b^{\pm}$ if $a \neq b$. Moreover the following is true

Proposition 2.6. *(1) For any $a, b \in [-1, 1]$, $a \neq \pm b$, the algebras E_a^+ and E_b^+ are not isomorphic. The algebras E_a^+ and E_{-a}^+ are isomorphic.*

(2) For any $a, b \in [-1, 1]$, $a \neq \pm b$, the algebras E_a^- and E_b^- are not isomorphic. The algebras E_a^- and E_{-a}^- are isomorphic.

Proof. (1) Let $\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be an isomorphism of the evolution algebra E_a^+ to the evolution algebra E_b^+. Here $\det(\varphi) \neq 0$. By the multiplication table of the evolution algebras, we get the following relation between matrices $\mathcal{M}_a^+$ and $\mathcal{M}_b^+$:

$$\mathcal{M}_b^+ = \frac{1}{\det(\varphi)}\times$$

$$\begin{pmatrix} (a\delta - \sqrt{1-a^2}\gamma)\alpha^2 - (a\gamma + \sqrt{1-a^2}\delta)\beta^2 & (a\alpha + \sqrt{1-a^2}\beta)\beta^2 - (a\beta - \sqrt{1-a^2}\alpha)\alpha^2 \\ (a\delta - \sqrt{1-a^2}\gamma)\gamma^2 - (a\gamma + \sqrt{1-a^2}\delta)\delta^2 & (a\alpha + \sqrt{1-a^2}\beta)\delta^2 - (a\beta - \sqrt{1-a^2}\alpha)\gamma^2 \end{pmatrix}.$$

Since $\det(\mathcal{M}_a^+) = 1$, it is easy to see that there are two classes of isomorphisms:

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} : \alpha\delta \neq 0 \right\}, \quad \mathcal{C}_2 = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : \beta\gamma \neq 0 \right\}.$$

For the class $\mathcal{C}_1$ the matrix $\mathcal{M}_b^+$ must satisfy the following

$$\mathcal{M}_b^+ = \begin{pmatrix} b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & b \end{pmatrix} = \begin{pmatrix} a\alpha & \sqrt{1-a^2}\frac{\alpha^2}{\delta} \\ -\sqrt{1-a^2}\frac{\delta^2}{\alpha} & a\delta \end{pmatrix}.$$

From this equality we get $\alpha = \delta = \sqrt{\frac{1-b^2}{1-a^2}} = \frac{b}{a}$ if $a \neq 0, \pm 1$ which is satisfied iff $a = b$. Hence the isomorphisms from the class $\mathcal{C}_1$ can not give an isomorphism from $\mathcal{M}_a^+$ to $\mathcal{M}_b^+$. For $a = 0$ we get $b = 0$. One can take $\alpha = \delta = \mp 1$ if $a = \pm 1$ and $b = \mp 1$. Hence $E_{\pm 1}^+$ is isomorph to $E_{\mp 1}^+$.

For the class $\mathcal{C}_2$ the matrix $\mathcal{M}_b^+$ must satisfy the following

$$\mathcal{M}_b^+ = \begin{pmatrix} b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & b \end{pmatrix} = \begin{pmatrix} a\beta & -\sqrt{1-a^2}\frac{\beta^2}{\gamma} \\ \sqrt{1-a^2}\frac{\gamma^2}{\beta} & a\gamma \end{pmatrix}.$$

From this equality we get $\beta = \gamma = -\sqrt{\frac{1-b^2}{1-a^2}} = \frac{b}{a}$ if $a \neq 0, \pm 1$ which is satisfied iff $a = -b$. Hence the isomorphisms from the class $\mathcal{C}_2$ can only give an isomorphism from $\mathcal{M}_a^+$ to $\mathcal{M}_{-a}^+$.

(2) The proof of (2) is similar to the proof of (1). □

Consider now discrete time n, $n \in \mathbb{N}$ and the CEA $\{E^{[n]}, n \in \mathbb{N}\}$ given by matrix (2.6).

Proposition 2.7. *The discrete time CEA $E^{[n]}$, $n \in \mathbb{N}$, is dense in the set $\{E_a^{\pm},\ a \in [-1, 1]\}$ of evolution algebras, i.e. for an arbitrary evolution algebra $E_a^{\pm}$ there exists a sequence $\{n_k\}_{k=1,2,\ldots}$ of natural numbers such that $\lim_{k\to\infty} E^{[n_k]} = E_a^+$ or E_a^-.*

Proof. It is known that the sequences $\{\sin n\}$ and $\{\cos n\}$, $n \in \mathbb{N}$, are dense in $[-1, 1]$ (see e.g.[5]). Hence for any $a \in [-1, 1]$ there is a sequence $\{n_k\}_{k=1,2,\ldots}$ of natural numbers such that $\lim_{k\to\infty} \cos(n_k) = a$. The same sequence can be used to get $\lim_{k\to\infty} E^{[n_k]} = E_a^+$ or E_a^-. □

3. A criterion for an evolution algebra to be baric

A *character* for an algebra A is a non-zero multiplicative linear form on A, that is, a non-zero algebra homomorphism from A to $\mathbb{R}$ [11]. Not every algebra admits a character. For example, an algebra with the zero multiplication has no character.

Definition 3.1. A pair (A, σ) consisting of an algebra A and a character σ on A is called a *baric algebra*. The homomorphism σ is called the weight (or baric) function of A and $\sigma(x)$ the weight (baric value) of x.

In [11] for the evolution algebra of a free population it is proven that there is a character $\sigma(x) = \sum_i x_i$, therefore that algebra is baric. But the evolution algebra E introduced in [15] is not baric, in general. The following theorem gives a criterion for an evolution algebra E to be baric.

Theorem 3.2. *An n-dimensional evolution algebra E, over field the $\mathbb{R}$, is baric if and only if there is a column $(a_{1i_0}, \ldots, a_{ni_0})^T$ of its structural constants matrix $\mathcal{M} = (a_{ij})_{i,j=1,\ldots,n}$, such that $a_{i_0i_0} \neq 0$ and $a_{ii_0} = 0$, for all $i \neq i_0$. Moreover, the corresponding weight function is $\sigma(x) = a_{i_0i_0}x_{i_0}$.*

Proof (*Necessity*). Take $x, y \in E$ with $x = \sum_{i=1}^n x_ie_i$, $y = \sum_{i=1}^n y_ie_i$. Assume $\sigma(x) = \sum_{i=1}^n \alpha_ix_i$, $x \in E$ is a character. We have

$$\sigma(xy) = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}\alpha_j\right) x_iy_i; \quad \sigma(x)\sigma(y) = \sum_{i=1}^n\sum_{j=1}^n \alpha_i\alpha_jx_iy_j.$$

From $\sigma(xy) = \sigma(x)\sigma(y)$ we get

$$\alpha_i\alpha_j = 0 \quad \text{for any} \quad i \neq j, \ i, j = 1, \ldots, n; \tag{3.1}$$

$$\sum_{j=1}^n a_{ij}\alpha_j = \alpha_i^2 \quad \text{for any} \quad i = 1, \ldots, n. \tag{3.2}$$

It is easy to see that the system (3.1) has a solution $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1^2 + \cdots + \alpha_n^2 > 0$ if and only if exactly one coordinate of α, say α_{i_0}, is not zero, and all others are zeros. Substituting this solution in (3.2) we get

$$a_{ii_0}\alpha_{i_0} = 0, \quad \text{if } i \neq i_0, \ i = 1, \ldots, n;$$

$$a_{i_0i_0}\alpha_{i_0} = \alpha_{i_0}^2, \quad \text{if } i = i_0.$$

From the last equations we get $a_{i_0i_0} \neq 0$, $a_{ii_0} = 0$, for all $i \neq i_0$ and $\alpha_{i_0} = a_{i_0i_0}$.
(Sufficiency). Assume there is a column $(a_{1i_0}, \ldots, a_{ni_0})^T$, such that $a_{i_0i_0} \neq 0$ and $a_{ii_0} = 0$, for all $i \neq i_0$. Then it is easy to see that $\sigma(x) = a_{i_0i_0}x_{i_0}$ is a weight function, therefore E is a baric evolution algebra. □

A baric algebra A may have several weight functions. As a corollary of Theorem 3.2 we have

Corollary 3.3. *If the matrix $\mathcal{M}$, mentioned in Theorem 3.2, has several columns $\left(a_{1i_j}, \ldots, a_{ni_j}\right)^T$, $j = i_1, \ldots, i_m$, $m \leqslant n$, which satisfy conditions of Theorem 3.2 then the evolution algebra E has exactly m weight functions $\sigma(x) = a_{i_ji_j}x_{i_j}$, $j = i_1, \ldots, i_m$.*

There are two types of trivial evolution algebras [15]: *zero evolution algebra*, which satisfies $e_ie_j = 0$ for all $i, j = 1, \ldots, n$; *non-zero trivial evolution algebra*, which satisfies $e_ie_j = 0$ for all $i \neq j$ and $e_i^2 = a_{ii}e_i$, where $a_{ii} \in \mathbb{R}$ is non-zero for some $i = 1, \ldots, n$. By Theorem 3.2 we conclude that the zero evolution algebra is not baric, but any non-zero trivial evolution algebra is a baric algebra. Moreover, there are baric evolution algebras which are not trivial.

4. Property transition

If a system has parameters (as usually like: temperature, time, interaction, etc.) then a property of the system can variate by a parameter. For example, the behavior of phases (states) of a system in physics, depends on temperature $T > 0$, if for some values of T there is a unique phase and for other values there are several phases, then the physical system has a phase transition [8]. Similar transitions of a property can be seen for systems of biology, chemistry, etc. Here we shall define a notion of property transition for CEA.

Definition 4.1. Assume a CEA, $E^{[s,t]}$, has a property, say P, at pair of times (s_0, t_0); we say that the CEA has P property transition if there is a pair $(s, t) \neq (s_0, t_0)$ at which the CEA has no the property P.

Denote

$$\mathcal{T} = \{(s,t) : 0 \leqslant s \leqslant t\};$$
$$\mathcal{T}_P = \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has property } P\}$$
$$\mathcal{T}_P^0 = \mathcal{T} \setminus \mathcal{T}_P = \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has no property } P\}.$$

Definition 4.2. We call the set
$\mathcal{T}_P$-the duration of the property P;
$\mathcal{T}_P^0$-the lost duration of the property P;
The partition $\{\mathcal{T}_P, \mathcal{T}_P^0\}$ of the set $\mathcal{T}$ is called P property diagram.

For example, if $P =$ commutativity then since any evolution algebra is commutative, we conclude that any CEA has not commutativity property transition.

4.3. Baric property transition

Since a CEA is not a baric algebra, in general, using Theorem 3.2 we can give baric property diagram. Let us do this for the above given Examples 1–4.

Example 1′. For the case of Example 1, by Theorem 3.2 we have that $E^{[t]}$ is baric iff

$$a_{ii}^{[t]} = \frac{2}{3}e^{-\frac{3}{2}At}\cos(\alpha t) + \frac{1}{3} = 1.$$

This has unique solution $t = 0$. Consequently, $\mathcal{T}_{\text{baric}} = \{0\}$, $\mathcal{T}_{\text{baric}}^0 = \{t : t > 0\}$. Thus the CEA $E^{[t]}$ is baric (even non-zero trivial) evolution algebra only at initial time, and it loses baricity as soon as the time turned on.

Example 2′. In Example 2, using Theorem 3.2 we obtain that

$$\mathcal{T}_{\text{baric}} = \begin{cases} \{0\} & \text{if } \lambda \neq \mu; \\ \mathcal{T} & \text{if } \lambda = \mu. \end{cases}$$

Thus the CEA $E^{[t]}$ has not baric property transition if $\lambda = \mu$, and it has a baric property transition, as in Example 1', if $\lambda \neq \mu$.

Example 3′. *Baric property transition for a two-state evolution.* Since in case of Example 3, we have a rich class of CEA here we shall give a special theory of the baric property transition. Using Theorem 3.2 we obtain that $\mathcal{T}_{\text{baric}}$ is the set of (s, t) such that

$$1 + \Phi(t)(\Psi(t) - \Psi(s)) - \frac{\Phi(t)}{\Phi(s)} = 0 \text{ or } 1 - \Phi(t)(\Psi(t) - \Psi(s)) - \frac{\Phi(t)}{\Phi(s)} = 0.$$

These equations can be rewritten as

$$\theta(t) = \theta(s), \;\; \theta^{-}(t) = \theta^{-}(s),$$

where

$$\theta(t) = \frac{1}{\Phi(t)} + \Psi(t), \; \theta^{-}(t) = \frac{1}{\Phi(t)} - \Psi(t). \tag{4.1}$$

Thus

$$\mathcal{T}_{\text{baric}} = \mathcal{T}_{\text{baric}}(\theta) \; \cup \; \mathcal{T}_{\text{baric}}(\theta^{-}),$$

here $\mathcal{T}_{\text{baric}}(\theta) = \{(s, t) \in \mathcal{T} : \theta(t) = \theta(s)\}$.

Remark 4.4. To describe the set $\mathcal{T}_{\text{baric}}$ one has to describe the sets $\mathcal{T}_{\text{baric}}(\theta)$ and $\mathcal{T}_{\text{baric}}(\theta^{-})$, both of which are defined by the parameter functions Φ and Ψ. Note that if we replace Φ with $-\Phi$ or Ψ with $-\Psi$ then these sets transfer to each other. Since Φ and Ψ are arbitrary functions, it will be enough to describe only $\mathcal{T}_{\text{baric}}(\theta)$ for arbitrary θ. Thus in the sequel of this subsection we shall deal with description of $\mathcal{T}_{\text{baric}}(\theta)$.

The function $\theta(t)$ is called *baric property controller* of the CEA. Because, it really controls the baric duration set, for example, if θ is a strong monotone function then the duration is “minimal”, i.e. the line $s = t$, but if θ is a constant function then the baric duration set is “maximal”, i.e. it is $\mathcal{T}$. Since Φ and Ψ are arbitrary functions, we have a rich class of controller functions, therefore we have a “powerful” control on the property to be baric.

For a special choose of θ we have

Proposition 4.5. *If $\Phi(t) = \lambda^{t}$, $\lambda > 0$ and $\Psi(t) = ct$, $c \in \mathbb{R}$. Then*

$$\mathcal{T}_{\text{baric}}(\theta) = \mathcal{T}_{\text{baric}}(\lambda, c) = \{(s, t) : s = t\} \cup$$

$$\begin{cases} \emptyset & \text{if } \; 0 < \lambda \leqslant 1, \; c \geqslant \ln \lambda; \;\; \text{or} \\ & \lambda > 1, \; c \in (-\infty, 0] \; \cup \; [\ln \lambda, +\infty), \\ \{(s, t) : 0 \leqslant s \leqslant t_{\mathrm{c}}, \; t_{c} \leqslant t \leqslant t_{\mathrm{c}}', \; \theta(s) = \theta(t)\} & \text{if } \; 0 < \lambda \leqslant 1, \; c < \ln \lambda; \;\; \text{or} \\ & \lambda > 1, \; c \in (0, \ln \lambda), \end{cases}$$

where t_{c} and t_{c}' serve as critical times, which defined by $t_{\mathrm{c}} = \frac{1}{\ln \lambda} \ln \left(\frac{\ln \lambda}{c}\right)$ and $t_{\mathrm{c}}' > 0$ is a unique solution to $\theta(t_{\mathrm{c}}') = 1$.

Proof. Under the conditions of the proposition we have $\theta(t) = \lambda^{-t} + ct$, and the simple analysis of the equation $\theta(s) = \theta(t)$ for this θ gives the full set $\mathcal{T}_{\text{baric}}(\lambda, c)$. $\square$

In Fig. 1, the baric property diagram is given.

As a corollary of Proposition 4.5 we have

Corollary 4.6. *(1) For any fixed s, with $0 \leqslant s < t_c$ (resp. $t_c \leqslant s \leqslant t$), the time t has two (resp. one) critical values: $t_c^{(1)} = s$ (resp. s) and $t_c^{(2)}$ which is a unique solution of $\theta(t_c^{(2)}) = \theta(s)$.*

(2) For any fixed t, with $0 \leqslant s \leqslant t \leqslant t_c$ or $t_c' < t$ (resp. $t_c < t \leqslant t_c'$), the time s has one (resp. two) critical values: $s_c^{(1)} = t$ (resp. $s_c^{(1)} = t$ and $s_c^{(2)}$ which is a unique solution of $\theta(t_c^{(2)}) = \theta(s)$).

Let us discuss some more examples of the controller θ. If $\theta(t) = \tan(t)$ then $\tan(s) = \tan(t)$ has solution $t = s + \pi k$, $k \in \mathbb{Z}$. The intersection of this family of lines with $\mathcal{T}$ gives the family of half lines,

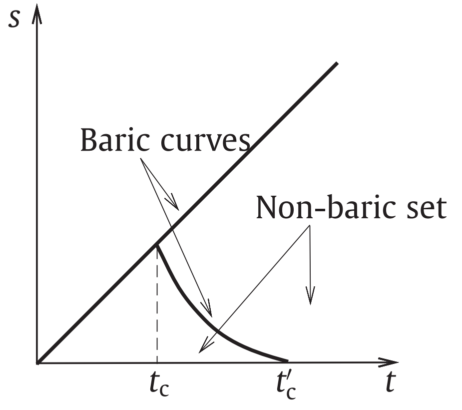

Fig. 1. The baric property diagram for $\theta(t) = \lambda^{-t} + ct$.

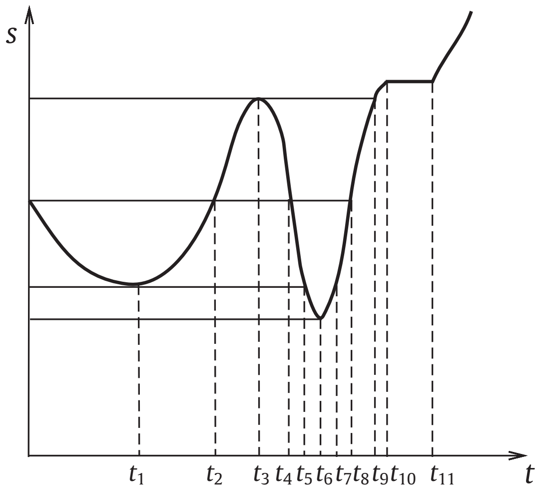

Fig. 2. An example of controller θ.

i.e.

$$\mathcal{T}_{baric}(\tan(t)) = \bigcup_{k=0,1,2,\ldots} \{(s,t) \in \mathcal{T} : s = t - \pi k\}.$$

If $\theta(t) = \sin(t)$ then $\sin(s) = \sin(t)$ has two family of solutions: $s = t + 2\pi k$, $k \in \mathbb{Z}$ and $s = -t + (2k+1)\pi$, $k \in \mathbb{Z}$. The intersection of these families of lines with $\mathcal{T}$ is

$$\mathcal{T}_{baric}(\sin(t)) = \bigcup_{k=0,1,2,\ldots} \{(s,t) \in \mathcal{T} : t = s + 2\pi k \text{ or } t = -s + (2k+1)\pi\}.$$

In all above considered examples we obtained a set $\mathcal{T}_{baric}(\theta)$ which has zero Lebesgue measure. But there is controllers for which this set has non-zero Lebesgue measure, for example, if $\theta(t)$ is a controller function with the graph as shown in Fig. 2, then the corresponding baric property diagram is as shown in Fig. 3. Thus any "constant part" of the graph of the controller gives a full triangle in the diagram, moreover, any "non-constant part" gives several curves. In this case the set $\mathcal{T}_{baric}(\theta)$ has a non-zero Lebesgue measure.

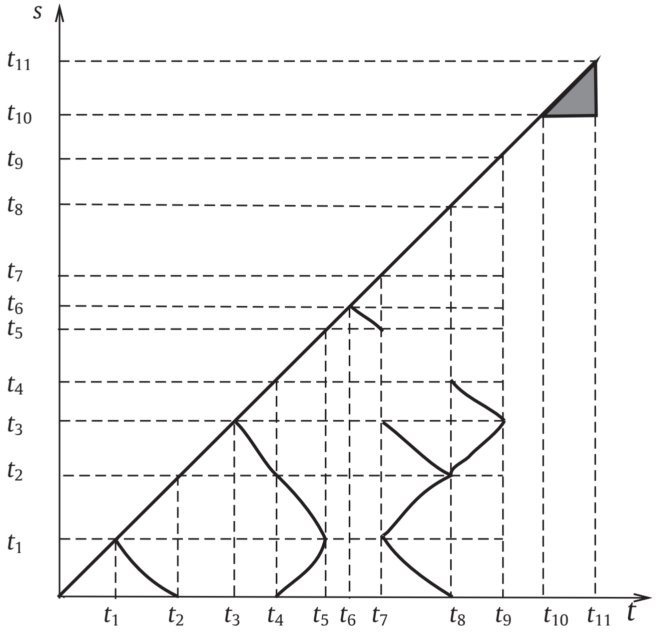

Fig. 3. The baric property diagram for the controller θ with graph as in Fig. 2.

Let $\theta(t) = D(t)$ be the Dirichlet function defined by

$$D(t) = \begin{cases} 1 & \text{if } t \text{ rational;} \\ 0 & \text{if } t \text{ irrational.} \end{cases}$$

In this case we have a rich set of baric property duration, i.e.

$$\mathcal{T}_{baric}(D(t)) = \{(s,t) \in \mathcal{T} : t \text{ and } s \text{ rational}\} \cup \{(s,t) \in \mathcal{T} : t \text{ and } s \text{ irrational}\}.$$

Definition 4.7. A function θ defined on $\mathbb{R}$ is called a function of *countable variation* if it has the following properties:

1. it is continuous except at most on a countable set, (which is denoted by $X_c = \{x_1, x_2, \dots\}$), it has only jump-type discontinuities (denote the one-sided limit from the negative direction by $\theta(x_i^-)$ and from the positive direction by $\theta(x_i^+)$, $i = 1, 2, \dots$);

2. it has at most a countable set of singular (extremum) points (which is denoted by $X_e = \{y_1, y_2, \dots\}$).

Note that any function of countable variation has not "constant parts" in its graph.

The following theorem gives a characteristics of the baric property duration set.

Theorem 4.8. *If the controller θ (see Eq. (4.1)) is a function of countable variation, then the baric duration set $\mathcal{T}_{\text{baric}}(\theta)$ has zero Lebesgue measure, that is the corresponding CEA is not baric almost surely.*

Proof. Using the (finite or infinite) sequences X_c and X_e we construct the sequences $\{t_{i,k}^-\}_{k=1,2,\dots}$, $i = 1, 2, \dots$ with $\theta(t_{i,k}^-) = \theta(x_i^-)$ for all k; $\{t_{i,q}^+\}_{q=1,2,\dots}$, $i = 1, 2, \dots$ with $\theta(t_{i,q}^+) = \theta(x_i^+)$ and $\{t_{j,l}^e\}_{l=1,2,\dots}$, $j = 1, 2, \dots$, where $\theta(t_{j,l}^e) = \theta(y_j)$ for all l. Now define a sequence $\{t_i\}_{i=1,2,\dots}$, with

$t_1 < t_2 < t_3 < \ldots$ as follows

$$\{t_i\}_{i=1,2,\ldots} = X_c \cup X_e \bigcup_i \left(\{t^-_{i,k}\}_{k=1,2,\ldots} \cup \{t^+_{i,q}\}_{q=1,2,\ldots}\right) \bigcup_j \{t^e_{j,l}\}_{l=1,2,\ldots}.$$

Since θ is a function of countable variation, the sequence $\{t_i\}_{i=1,2,\ldots}$ is at most countable. In a case, if it is a bounded sequence (in particular, a finite sequence), then we add the last term to be $+\infty$. Consider rectangles

$$\mathcal{T}_{ij} = \left\{(s,t) \in \mathbb{R}^2 : t_i \leqslant s \leqslant t_{i+1},\ t_j \leqslant t \leqslant t_{j+1}\right\}.$$

Denote $G(\theta) = \{(t, y) : y = \theta(t)\}$.

By the construction, the rectangles have the following properties:

– The set of all rectangles is at most a countable set;

– The intersection $G(\theta) \cap \mathcal{T}_{ij}$ is empty or contains a monotone part of the graph $G(\theta)$.

If $G(\theta) \cap \mathcal{T}_{ij}$ is empty then we say $\mathcal{T}_{ij}$ is empty.

Now we shall construct the set $\mathcal{T}_{\text{baric}}(\theta)$. Fix i, j such that the rectangle $\mathcal{T}_{ij}$ is not-empty (an empty rectangle does not give any contribution to the set $\mathcal{T}_{\text{baric}}(\theta)$), then we have

$$\mathcal{T}_{\text{baric}}(\theta) \cap \mathcal{T}_{kj} = \text{a curve giving an one-to-one corespondence between } [t_j, t_{j+1}]$$
$$\text{and } [t_k, t_{k+1}] \text{ if } \mathcal{T}_{ik} \neq \emptyset,\ k = 1, \ldots, j-1.$$

Thus we have

$$\mathcal{T}_{\text{baric}}(\theta) = \bigcup_{kj} \left(\mathcal{T}_{\text{baric}}(\theta) \cap \mathcal{T}_{kj}\right).$$

Since there are a countable set of rectangles and in each rectangle we may have at most a curve which has Lebesgue measure zero (because, these curves give one-to-one correspondences), we conclude that the set $\mathcal{T}_{\text{baric}}(\theta)$ also has zero Lebesgue measure. □

Example 4′. Consider the CEA $E^{[s,t]}$ constructed in Example 4 by a family of invertible lower (or upper) triangular matrices $A^{[t]}$, $t \geqslant 0$.

Theorem 4.9. *For any pair of time (s, t) the n-dimensional evolution algebra $E^{[s,t]}$, constructed by a family of (lower or upper) triangular invertible matrices is baric. Moreover, $E^{[s,t]}$ has a weight function $\sigma(x) = \mathcal{M}^{[s,t]}_{nn} x_n$, where $\mathcal{M}^{[s,t]}_{ii}$, $i = 1, \ldots, n$ are diagonal entries of $\mathcal{M}^{[s,t]} = A^{[s]}(A^{[t]})^{-1}$.*

Proof. It is known that the standard operations on triangular matrices conveniently preserve the triangular form: the sum and product of two lower triangular matrices is again lower triangular. The inverse of a lower triangular matrix is also lower triangular, and of course we can multiply a lower triangular matrix by a constant and it will still be lower triangular. This means that the lower triangular matrices form a subalgebra of the ring of square matrices for any given size. The analogous result holds for upper triangular matrices. Using these properties we get that $\mathcal{M}^{[s,t]}$ is also a triangular matrix. Moreover, since $A^{[t]}$ is invertible, its determinant is non-zero for all t. Thus

$$\det(\mathcal{M}^{[s,t]}) = \prod_{i=1}^{n} \mathcal{M}^{[s,t]}_{ii} = \det(A^{[s]}) \det((A^{[t]})^{-1}) \neq 0.$$

Consequently, all diagonal entries of the matrix are non-zero. In particular, $\mathcal{M}^{[s,t]}_{nn} \neq 0$, and Theorem 3.2 completes the proof. □

Corollary 4.10. *The CEA $E^{[s,t]}$ constructed by triangular invertible matrices has not baric property transition.*

4.11. Absolute nilpotent elements transition

The element x of an algebra A is called an *absolute nilpotent* if $x^2 = 0$.

Let $E = \mathbb{R}^n$ be an evolution algebra over the field $\mathbb{R}$ with structural constant coefficients matrix $\mathcal{M} = (a_{ij})$, then for arbitrary $x = \sum_i x_i e_i$ and $y = \sum_i y_i e_i \in \mathbb{R}^n$ we have

$$xy = \sum_j \left(\sum_i a_{ij} x_i y_i \right) e_j, \quad x^2 = \sum_j \left(\sum_i a_{ij} x_i^2 \right) e_j.$$

For a n-dimensional evolution algebra $\mathbb{R}^n$ consider operator $V: \mathbb{R}^n \to \mathbb{R}^n, x \mapsto V(x) = x'$ defined as

$$x'_j = \sum_{i=1}^{n} a_{ij} x_i^2, \quad j = 1, \dots, n. \tag{4.2}$$

This operator is called *evolution operator* [11].

We have $V(x) = x^2$, hence the equation $V(x) = x^2 = 0$ is given by the following system

$$\sum_i a_{ij} x_i^2 = 0, \quad j = 1, \dots, n. \tag{4.3}$$

If $\det(\mathcal{M}) \neq 0$ then the system (4.3) has unique solution $(0, \dots, 0)$. If $\det(\mathcal{M}) = 0$ and $\mathrm{rank}(\mathcal{M}) = r$ then we can assume that the first r rows of $\mathcal{M}$ are linearly independent, consequently, the system of Eq. (4.3) can be written as

$$x_i^2 = -\sum_{j=r+1}^{n} d_{ij} x_j^2, \quad i = 1, \dots, r, \tag{4.4}$$

where $d_{ij} = \frac{\det(\mathcal{M}_{ij})}{\det(M_r)}$ with $\mathcal{M}_r = (a_{ij})_{i,j=1,\dots,r}$,

$$\mathcal{M}_{ij} = \begin{pmatrix} a_{11} & \dots & a_{i-1,1} & a_{j1} & a_{i+1,1} & \dots & a_{r1} \\ a_{12} & \dots & a_{i-1,2} & a_{j2} & a_{i+1,2} & \dots & a_{r2} \\ & \dots & & \dots & & \dots & \\ a_{1r} & \dots & a_{i-1,r} & a_{jr} & a_{i+1,r} & \dots & a_{rr} \end{pmatrix}.$$

An interesting problem is to find a necessary and sufficient condition on matrix $D = (d_{ij})_{\substack{i=1,\dots,r \\ j=r+1,\dots,n}}$ under which the system (4.4) has unique solution. The difficulty of the problem depends on rank r, here we shall consider the case $r = n - 1$.

Proposition 4.12. *(1) If* $\det(\mathcal{M}) \neq 0$ *then the finite dimensional evolution algebra* $\mathbb{R}^n$ *has unique absolute nilpotent* $(0, \dots, 0)$.

(2) If $\det(\mathcal{M}) = 0$ *and rank*$(\mathcal{M}) = n - 1$ *then the evolution algebra* R^n *has unique absolute nilpotent* $(0, \dots, 0)$ *if and only if*

$$\det(\mathcal{M}_{i_0 n}) \cdot \det(\mathcal{M}_{n-1}) > 0, \tag{4.5}$$

for some $i_0 \in \{1, \dots, n-1\}$.

Proof. (1) Straightforward.
(2) If $\operatorname{rank}(\mathcal{M}) = n-1$ then from (4.4) we get

$$x_i^2 = -\frac{\det(\mathcal{M}_{in})}{\det(M_{n-1})} x_n^2, \quad i = 1, \dots, n-1. \tag{4.6}$$

From (4.6) it follows that the condition (4.5) is necessary and sufficient to have unique solution $(0, \dots, 0)$. □

For a CEA $E^{[s,t]}$ with matrix $\mathcal{M}^{[s,t]}$ denote

$$\mathcal{T}_{nil} = \{(s,t) \in \mathcal{T} : E^{[s,t]} \text{ has unique absolute nilpotent}\}, \quad \mathcal{T}_{nil}^0 = \mathcal{T} \setminus \mathcal{T}_{nil}.$$

The following theorem gives an answer on problem of existence of "uniqueness of absolute nilpotent element" property transition.

Theorem 4.13. *(1) There are CEAs which have not "uniqueness of absolute nilpotent element" property transition.*
(2) There is CEA which has "uniqueness of absolute nilpotent element" property transition.

Proof. Denote $d(s,t) = \det(\mathcal{M}^{[s,t]})$. By Eq. (2.2) we get

$$d(s,t) = d(s,\tau)d(\tau,t), \quad \text{for all} \quad \tau,\ s < \tau < t. \tag{4.7}$$

As it was mentioned above, the Eq. (4.7) is known as Cantor's second equation.

(1) The Eq. (4.7) has solutions $d(s,t) = \frac{\Phi(s)}{\Phi(t)}$, where $\Phi(t) \neq 0$ is an arbitrary function. Thus for such solutions we conclude that if $d(s_0,t_0) \neq 0$ for some (s_0,t_0) then $d(s,t) \neq 0$ for any (s,t). Consequently, corresponding CEAs have not "uniqueness of absolute nilpotent element" property transition.

(2) Note that the Eq. (4.7) has solution $d(s,t) = f(t)$, where $f(t) = 1$ for $t < 1$ and $f(t) = 0$ otherwise. For this solution we have $d(s,t) = 1, s < t < 1$ and $d(s,t) = 0, t \geqslant 1$. For some $t \geqslant 1$ one can construct a matrix $\mathcal{M}^{[s,t]}$ which does not satisfy uniqueness condition mentioned in part 2) of Proposition 4.12. Indeed let us consider the matrix $\mathcal{M}^{[s,t]} = \left(a_{ij}^{[s,t]}\right)_{i,j=1,2}$ with entries as in (2.4). The second equation of the system (2.5) has a solution:

$$\beta(s,t) = \begin{cases} 1, & \text{if } s < t < 1 \\ 0, & \text{if } t \geqslant 1. \end{cases}$$

Substituting this solution in the first equation of (2.5) we obtain

$$\alpha(s,t) = \begin{cases} \psi(t) - \psi(s), & \text{if } s < t < 1 \\ g(t), & \text{if } t \geqslant 1, \end{cases}$$

where ψ and g are arbitrary functions. The corresponding matrix has the following form

$$\mathcal{M}^{[s,t]} = \frac{1}{2}\begin{pmatrix} 2 + \psi(t) - \psi(s) & -\psi(t) + \psi(s) \\ \psi(t) - \psi(s) & 2 - \psi(t) + \psi(s) \end{pmatrix}, \quad \text{if } s < t < 1,$$

and

$$\mathcal{M}^{[s,t]} = \frac{1}{2}\begin{pmatrix} 1+g(t) & 1-g(t) \\ 1+g(t) & 1-g(t) \end{pmatrix}, \quad \text{if } t \geqslant 1. \tag{4.8}$$

We have

$$d(s,t) = \det(\mathcal{M}^{[s,t]}) = \begin{cases} 1, & \text{if } s < t < 1 \\ 0, & \text{if } t \geqslant 1. \end{cases}$$

Assume $g(t) \neq -1$ then for (4.8) the Eq. (4.6) has the form

$$x_1^2 = -\frac{1-g(t)}{1+g(t)}x_2^2.$$

This equation has infinitely many solutions if $|g(t)| > 1$ for some $t \geqslant 1$. Thus corresponding CEA has "uniqueness of absolute nilpotent element" property transition. □

Now let us construct the set $\mathcal{T}_{nil}$ for Examples 1–5: It is easy to see that

$$\det(\mathcal{M}^{[s,t]}) = \begin{cases} e^{-3At}, & \text{for Example 1;} \\ (\lambda\mu)^t, & \text{for Example 2;} \\ \frac{\Phi(t)}{\Phi(s)}, & \text{for Example 3;} \\ \prod_{i=1}^n \mathcal{M}_{ii}^{[s,t]}, & \text{for Example 4;} \\ 1, & \text{for Example 5.} \end{cases}$$

Thus in each one of the considered examples we have $\det(\mathcal{M}^{[s,t]}) \neq 0$, consequently, $\mathcal{T}_{nil} = \mathcal{T}$, i.e. the CEAs constructed in Examples 1–5 have not "uniqueness of the absolute nilpotent element" property transition.

There are CEAs which have infinitely many absolute nilpotent elements independently on time. For example, take $\mathcal{M}^{[s,t]}$ with identical rows $(\frac{\Phi(s)}{\Phi(t)}, 0, 0, \ldots, 0)$, where Φ is an arbitrary function with $\Phi(t) \neq 0$ for all t. It is easy to see that this matrix satisfies the Eq. (2.2), hence it determines a CEA, $E^{[s,t]}$, which has infinitely many absolute nilpotent elements: $(0, x_2, \ldots, x_n)$, where $x_2, \ldots, x_n \in \mathbb{R}$ are arbitrary numbers. Thus for this example we have $\mathcal{T}_{nil} = \emptyset$, $\mathcal{T}_{nil}^0 = \mathcal{T}$. In other words the CEA has not "non-uniqueness of absolute nilpotent element" property transition.

Remark 4.14. These examples (Examples 1–4) of "uniqueness of nilpotent element" property transition of CEAs with time-parameter are similar to the "uniqueness of Gibbs phase" property transition, i.e. phase transition of physical systems with respect to temperature-parameter, $T > 0$. Usually there is a phase transition if the temperature is very low ($T \sim 0$) or if it is very high ($T \sim +\infty$) (see [8]). Example 5 is an analogue of a physical system which has unique (Gibbs) phase for any temperature. There a lot of examples of such physical systems (see e.g. [8]).

4.15. Idempotent elements transition

A element x of an algebra $\mathcal{A}$ is called *idempotent* if $x^2 = x$; such points of an evolution algebra are especially important, because they are the fixed points (i.e. $V(x) = x$) of the evolution operator V, (4.2). We denote by $\mathcal{I}d(E)$ the idempotent elements of an algebra E. Using (4.2) the equation $x^2 = x$

can be written as

$$x_j = \sum_{i=1}^{n} a_{ij} x_i^2, \quad j = 1, \ldots, n. \tag{4.9}$$

The general analysis of the solutions of the system (4.9) is very difficult. We shall solve this problem for the CEA $E^{[t]}$, $t \geqslant 0$, corresponding to the Example 2. In case of Example 2 the system (4.9) has the following form

$$\begin{cases} 2x = (\lambda^t + \mu^t)x^2 + (\lambda^t - \mu^t)y^2; \\ 2y = (\lambda^t - \mu^t)x^2 + (\lambda^t + \mu^t)y^2, \end{cases} \tag{4.10}$$

where $\lambda > 0, \quad \mu > 0$ and $t \geqslant 0$.

Case $\lambda = \mu$. It is easy to see that if $\lambda = \mu$ then the system (4.10) has only four solutions $0 = (0, 0), z_1 = z_1(t) = (0, \lambda^{-t}), z_2 = z_2(t) = (\lambda^{-t}, 0), z_3 = z_3(t) = (\lambda^{-t}, \lambda^{-t})$.

Case $\lambda \neq \mu$. For $\lambda \neq \mu$ the solutions 0 and z_3 still exist. If $x = 0$ or $y = 0$ there is no any new solution. Thus we consider the case $xy \neq 0$. Denote

$$u = \frac{x}{y}, \quad \gamma(t) = \frac{\lambda^t - \mu^t}{\lambda^t + \mu^t} = \frac{(\lambda/\mu)^t - 1}{(\lambda/\mu)^t + 1}.$$

For $t > 0$ it is easy to see that if $\lambda < \mu$ then $-1 < \gamma(t) < 0$ and if $\lambda > \mu$ then $0 < \gamma(t) < 1$. Note that for $t = 0$ there is no any new solution. From system (4.10) we get

$$\gamma(t)u^3 - u^2 + u - \gamma(t) = (u - 1)\left(\gamma(t)u^2 + (\gamma(t) - 1)u + \gamma(t)\right) = 0. \tag{4.11}$$

Subcase $\lambda < \mu$. In this case for any $t > 0$ the Eq. (4.11) has three solutions

$$u_1 = 1, \quad u_\pm = \frac{1 - \gamma(t) \pm \sqrt{1 - 2\gamma(t) - 3\gamma^2(t)}}{2\gamma(t)}. \tag{4.12}$$

Subcase $\lambda > \mu$. In this case the number of solutions to the Eq. (4.11) varies by γ, i.e.

$$\text{solutions to (4.11)} = \begin{cases} 1, & \text{if } \frac{1}{3} \leqslant \gamma(t) < 1; \\ 1, u_-, u_+ & \text{if } 0 < \gamma(t) < \frac{1}{3}, \end{cases} \tag{4.13}$$

where $u_\pm$ are defined in (4.12).

Now we shall describe x, y corresponding to the solutions of (4.11). The case $u = 1$, i.e. $x = y$ does not give any new solution. For $u = u_\pm$ we have $x = u_\pm y$, substituting this in the second equation of (4.10) after simple calculations we get the following two non-zero solutions to (4.10):

$$x_\pm = \frac{\mu^t \pm \sqrt{\lambda^t(2\mu^t - \lambda^t)}}{\mu^t(\lambda^t \pm \sqrt{\lambda^t(2\mu^t - \lambda^t)})}, \quad y_\pm = \frac{\lambda^t - \mu^t}{\mu^t(\lambda^t \pm \sqrt{\lambda^t(2\mu^t - \lambda^t)})}.$$

Note that $x_\pm, \quad y_\pm$ are well defined for any $\lambda \neq \mu$. For $\lambda > \mu$ we have critical time

$$t_c = \frac{\ln 2}{\ln \lambda - \ln \mu}, \tag{4.14}$$

which is the unique solution to the equation $\gamma(t) = \frac{1}{3}$.

Thus we have proved the following

Proposition 4.16. *We have*

$$\mathcal{I}d(E^{[t]}) = \begin{cases} \{0, z_1, z_2, z_3\}, & \text{if } \lambda = \mu; \\ \{0, z_3, (x_-, y_-), (x_+, y_+)\}, & \text{if } \lambda < \mu; \\ \{0, z_3\}, & \text{if } \lambda > \mu;\ t \geqslant t_c \\ \{0, z_3, (x_-, y_-), (x_+, y_+)\}, & \text{if } \lambda > \mu,\ t < t_c. \end{cases}$$

This proposition gives a very interesting "a fixed set of idempotent elements" property transition, i.e. we have

Corollary 4.17. *The CEA $E^{[t]}$ constructed in Example 2 has not "a fixed set of idempotent elements" property transition if $\lambda \leqslant \mu$; it has such property transition if $\lambda > \mu$. Moreover the transition point (the critical time) is $t = t_c$ defined by formula (4.14).*

Remark 4.18. There are exactly solvable models in statistical mechanics, here an imprecise notion of "exactly solvable" as meaning: "The solutions can be expressed explicitly in terms of some previously known functions" is also sometimes used [1]. In such models, for example, the critical temperature can be expressed explicitly. Comparing this with our examples of a property transition we also can say that a property transition of a CEA is exactly solvable if the critical time can be found exactly. Thus our Example 2 is exactly solvable for investigation of properties of idempotent elements.

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