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On the $GI/M/1/N$ queue with multiple working vacations—analytic analysis and computation

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Abstract

We consider finite buffer single server $GI/M/1$ queue with exhaustive service discipline and multiple working vacations. Service times during a service period, service times during a vacation period and vacation times are exponentially distributed random variables. System size distributions at pre-arrival and arbitrary epoch with some important performance measures such as, probability of blocking, mean waiting time in the system etc. have been obtained. The model has potential application in the area of communication network, computer systems etc. where a single channel is allotted for more than one source.

Keywords: $GI/M/1$ queue; Finite buffer; Multiple working vacations

1. Introduction

Usefulness of vacation models in queueing theory have been well established as they are considered to be an effective instrument in modelling and analysis of communication networks, manufacturing and production systems in which single server is entitled to serve more than one queue. During the last two decades, queueing systems with vacations have been studied extensively. For more detail on this topic readers are referred to the survey paper by Doshi [1]. An extensive amount of literature is available on infinite- and finite-buffer $M/G/1$ type vacation models and can be found in Takagi [2,3]. However, a limited studies have been done on $GI/M/1$ type vacation models, e.g., Chatterjee and Mukherjee [4], Tian et al. [5], Tian [6,7] etc., whereas in [4] they have considered vacation time as generally distributed, in [5,7], vacation time is exponentially distributed. In [6] vacation time follows phase type distribution.

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In the study of vacation model generally it is assumed that server stops service during vacation period. However, there are numerous situations where the server will not completely remain inactive during the vacation period rather he will render service to the queue with a different rate. When a vacation ends and if there are customers in the queue, a service period begins and server serves the queue with his original service rate. Otherwise, on return from a vacation if there are no customers in the queue, the server goes for another vacation and continues to do so till on return from a vacation he finds at least one customer. Such type of vacation is called multiple working vacation and was introduced by Servi and Finn [8] whereby they studied $M/M/1$ queue assuming service times during service period, the service times during working vacation, and vacation times all are exponentially distributed with different rates. The model is denoted by $M/M/1/WV$. Later Wu and Takagi [9] investigated $M/G/1/WV$ model where the service times during service period, the service times during working vacation, and also vacation durations are generally distributed. Baba [10] considered similar model but assumed general independent arrival, i.e., $GI/M/1/WV$ queue where he assumed service times during service period, service times during the vacation period and vacation times are exponentially distributed. It may be remarked here that all the aforementioned studies assume availability of infinite buffer space in front of the server. However, finite buffer queues are more common in several practical applications. In such queues one of the main concerns of system designer is to provide sufficient buffer space so that the probability of customers rejection is small. Moreover, the analysis of finite buffer queue gives very different system behaviour than infinite buffer queue. In this direction Karaesmen and Gupta [11] studied finite buffer $GI/M/1$ queue with multiple vacations where they considered service and vacation times follow exponential distribution. They have also derived some results on bounds and approximation of blocking probability for some special cases of the model.

This paper analyzes $GI/M/1/N$ queue with multiple working vacation policy. The model was previously analyzed by Baba [10] for infinite buffer queue considering multiple working vacation policy. For the sake of notational convenience the model is denoted by $GI/M/1/N/MWV$, where MWV stands for 'multiple working vacation policy'. One final comment on the model is that by equating working vacation parameter (η) equal to zero one can get the results for $GI/M/1/N$ queue with multiple vacations.

The paper organizes as follows. In Section 2, we provide the model description and notations. Section 3 presents the analytic analysis of the model. Section 4 illustrates the numerical results where we provide a variety of tables for different values of the model parameters and also we have numerically verified our results in some special cases that exists in the literature. Some graphs are presented showing the effect of model parameters on some performance measures.

2. Description of the model

Let us consider a $GI/M/1/N$ queue where N is the capacity of the system including the one who is in service. The server is allowed to take working vacations whenever the system has been emptied. On return from a working vacation if the server finds the system nonempty he will serve the customers present in the queue, otherwise the server again goes for a working vacation and continues in this manner. During any working vacation the server will serve customers at a rate which is different from the rate of service during the service period.

Inter-arrival times are i.i.d.r.vs. Let $A(x)\{a(x)\}[A^*(\theta)]$ be the distribution function (DF) {probability density function (pdf)}[Laplace–Stieltjes transform (LST)] of the inter-arrival time A of customers. The mean inter-arrival time is $E(A) = -A^{*(1)}(0) = 1/\lambda$ (say), where λ is the mean arrival rate and $f^{*(j)}(\zeta)$ is the jth ($j \geqslant 1$) derivative of $f^*(\theta)$ at $\theta = \zeta$. Service times during service period, service times during a working vacation, and working vacation times all are assumed to be exponentially distributed with rate μ, η, and γ, respectively and they are independent of the arrival process. The traffic intensity is given by $\rho = \lambda/\mu$. The state of the system at time t is described by the following r.vs., namely

- $\xi(t) = \{1\}(0)$ if the server is {on service period}(on working vacation),
- $N_s(t) =$ number of customers present in the system including the one who is in service,
- $\tilde{A}(t) =$ remaining inter-arrival time of the customer who is going to enter into the system.

We define the joint probability densities of system length $N_s(t)$, state of the server $\xi(t)$ and the remaining inter-arrival time $\tilde{A}$, respectively, by

$$\pi_{n,1}(x;t)\Delta x = P\{N_s(t) = n,\ \xi(t) = 1,\ x < \tilde{A}(t) < x + \Delta x\}, \quad 1 \leqslant n \leqslant N,\ x \geqslant 0,$$
$$\pi_{n,0}(x;t)\Delta x = P\{N_s(t) = n,\ \xi(t) = 0,\ x < \tilde{A}(t) < x + \Delta x\}, \quad 0 \leqslant n \leqslant N,\ x \geqslant 0.$$

As we shall discuss the model in steady-state, i.e., when $t \to \infty$, the above probabilities will be denoted by $\pi_{n,1}(x)$ and $\pi_{n,0}(x)$, respectively.

3. Analysis of the model

In this section we will carry out the analytic analysis of $GI/M/1/N$ queue with multiple working vacation policy.

3.1. Steady-state distribution at pre-arrival epoch

Consider the system just before an arrival which are taken as embedded points. Let $t_0, t_1, t_2, \ldots$ be the time epochs at which successive arrivals occur and t_n^- denote the time epochs just before the arrival instant t_n. The inter-arrival times $T_{n+1} = t_{n+1} - t_n$, $n = 0, 1, 2, \ldots$ are i.i.d.r.vs. with common distribution function $A(x)$. The state of the system at t_i^- is defined as $\{N_s(t_i^-),\ \xi(t_i^-)\}$, where $N_s(t_i^-)$ is the number of customers in the system and $\xi(t_i^-)$ indicates whether the service period ($\xi(t_i^-) = 1$) is going on, or the server is on working vacation ($\xi(t_i^-) = 0$). $\{N_s(t_i^-),\ \xi(t_i^-)\}$ forms a bivariate Markov chain whose finite state space is equivalent to: $\{(0,0),((1,0),(1,1)),\ldots,((N,0),(N,1))\}$. In limiting case let us assume

$$\pi_{n,1}^- = \lim_{i\to\infty} P(N_s(t_i^-) = n,\ \xi(t_i^-) = 1), \quad 1 \leqslant n \leqslant N,$$
$$\pi_{n,0}^- = \lim_{i\to\infty} P(N_s(t_i^-) = n,\ \xi(t_i^-) = 0), \quad 0 \leqslant n \leqslant N,$$

where $\pi_{n,1}^-(\pi_{n,0}^-)$ represents the probability that there are n customers in the system just prior to an arrival epoch of a customer when the server is in the service period (on working vacation).

Let a_k and $b_k (k \geqslant 0)$ are the conditional probability that k customers have been served during an inter-arrival time when service period is going on, and working vacation continues, respectively. Similarly, c_k be the conditional probability that k customers have been served during an inter-arrival time given that working vacation terminates and service period is going on. Hence, for all $k \geqslant 0$, we have

$$a_k = \int_0^\infty \frac{(\mu t)^k}{k!} \mathrm{e}^{-\mu t}\, \mathrm{d}A(t), \quad b_k = \int_0^\infty \mathrm{e}^{-\gamma t} \frac{(\eta t)^k}{k!} \mathrm{e}^{-\eta t}\, \mathrm{d}A(t),$$

and

$$c_k = \int_0^\infty \sum_{j=0}^{k} \left\{ \int_0^t \gamma \mathrm{e}^{-\gamma x} \frac{(\eta x)^j}{j!} \mathrm{e}^{-\eta x} \times \frac{[\mu(t-x)]^{k-j}}{(k-j)!} \mathrm{e}^{-\mu(t-x)}\, \mathrm{d}x \right\} \mathrm{d}A(t).$$

The p.g.f. of a_k, b_k and c_k are given by

$$\bar{A}(z) = \sum_{i=0}^{\infty} a_i z^i = A^*(\mu - \mu z), \quad \bar{B}(z) = \sum_{i=0}^{\infty} b_i z^i = A^*(\gamma + \eta - \eta z),$$

and

$$\overline{C}(z) = \sum_{i=0}^{\infty} c_i z^i = \frac{\gamma\{A^*(\mu - \mu z) - A^*(\gamma + \eta - \eta z)\}}{\gamma - (\mu - \eta)(1 - z)}.$$

Observing the state of the system at two consecutive embedded points, we have the one step transition probability matrix (TPM) $\mathscr{P}$ as follows:

$$\mathscr{P} = \begin{pmatrix} \mathbf{B}_{0,0} & \mathbf{A}_{0,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_{1,0} & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_{2,0} & \mathbf{A}_2 & \mathbf{B}_1 & \mathbf{A}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{B}_{3,0} & \mathbf{A}_3 & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{B}_{N-1,0} & \mathbf{A}_{N-1} & \mathbf{A}_{N-2} & \mathbf{A}_{N-3} & \mathbf{A}_{N-4} & \cdots & \mathbf{A}_0 \\ \mathbf{B}_{N-1,0} & \mathbf{A}_{N-1} & \mathbf{A}_{N-2} & \mathbf{A}_{N-3} & \mathbf{A}_{N-4} & \cdots & \mathbf{A}_0 \end{pmatrix}_{(2N+1)\times(2N+1)},$$

where

$$\mathbf{B}_{0,0} = 1 - b_0 - c_0, \quad \mathbf{A}_{0,1} = (b_0, c_0),$$
$$\mathbf{A}_k = \begin{pmatrix} b_k & c_k \\ 0 & a_k \end{pmatrix}, \quad 0 \leqslant k \leqslant N-1,$$
$$\mathbf{B}_{k,0} = \begin{pmatrix} 1 - \sum_{i=0}^{k}(b_i + c_i) \\ 1 - \sum_{i=0}^{k} a_i \end{pmatrix}, \quad 1 \leqslant k \leqslant N-1.$$

The pre-arrival epoch probabilities $\pi^-_{n,0}(0 \leqslant n \leqslant N)$ and $\pi^-_{n,1}(1 \leqslant n \leqslant N)$ can be obtained by solving the system of equations: $\boldsymbol{\pi}^- = \boldsymbol{\pi}^-\mathscr{P}$, where $\boldsymbol{\pi}^- = (\pi^-_{0,0}, \pi^-_{1,0}, \pi^-_{1,1}, \pi^-_{2,0}, \pi^-_{2,1}, \ldots, \pi^-_{N,0}, \pi^-_{N,1})$. We have used GTH (Grassmann, Taksar and Heyman) algorithm given in Latouche and Ramaswami [12, p. 123] for solving the system of equations as it works very well even for large number of states.

3.2. Steady-state distribution at arbitrary epoch

To obtain the system length distribution at arbitrary epoch we develop relations between distributions of number of customers in the system at pre-arrival and arbitrary epochs. For this we use supplementary variable method and relate the state of the system at two consecutive time epochs t and $t + \Delta t$. Using probabilistic arguments, we get a set of partial differential equations. Taking limit as $t \to \infty$, those equations can be written as

$$-\frac{\mathrm{d}}{\mathrm{d}x}\pi_{1,1}(x) = -\mu\pi_{1,1}(x) + \mu\pi_{2,1}(x) + \gamma\pi_{1,0}(x), \tag{1}$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\pi_{n,1}(x) = -\mu\pi_{n,1}(x) + \mu\pi_{n+1,1}(x) + \pi_{n-1,1}(0)a(x) + \gamma\pi_{n,0}(x), \quad 2 \leqslant n \leqslant N-1, \tag{2}$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\pi_{N,1}(x) = -\mu\pi_{N,1}(x) + (\pi_{N-1,1}(0) + \pi_{N,1}(0))a(x) + \gamma\pi_{N,0}(x), \tag{3}$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\pi_{0,0}(x) = \mu\pi_{1,1}(x) + \eta\pi_{1,0}(x), \tag{4}$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\pi_{n,0}(x) = -(\gamma + \eta)\pi_{n,0}(x) + \pi_{n-1,0}(0)a(x) + \eta\pi_{n+1,0}(x), \quad 1 \leqslant n \leqslant N-1, \tag{5}$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}\pi_{N,0}(x) = -(\gamma + \eta)\pi_{N,0}(x) + (\pi_{N-1,0}(0) + \pi_{N,0}(0))a(x), \tag{6}$$

where $\pi_{n,1}(0)$ and $\pi_{n,0}(0)$ are the respective probabilities with remaining inter-arrival time is zero, i.e., an arrival is about to occur. Let us define the Laplace transform of $\pi_{n,1}(x)$ and $\pi_{n,0}(x)$ as

$$\pi^*_{n,1}(\theta) = \int_0^\infty \mathrm{e}^{-\theta x}\pi_{n,1}(x)\mathrm{d}x, \quad \pi^*_{n,0}(\theta) = \int_0^\infty \mathrm{e}^{-\theta x}\pi_{n,0}(x)\,\mathrm{d}x, \quad \operatorname{Re}\theta \geqslant 0.$$

So that

$$\pi_{n,1} \equiv \pi^*_{n,1}(0) = \int_0^\infty \pi_{n,1}(x)\,\mathrm{d}x, \quad \pi_{n,0} \equiv \pi^*_{n,0}(0) = \int_0^\infty \pi_{n,0}(x)\,\mathrm{d}x, \tag{7}$$

where $\pi_{n,1}$ is the joint probability that there are n customers in the system while service period is going on. Similarly, $\pi_{n,0}$ is the joint probability that there are n customers in the system and the server is on working vacation. Multiplying Eqs. (1)–(6) by $\mathrm{e}^{-\theta x}$ and integrating w.r.t. x over 0–∞, we obtain

$$(\mu-\theta)\pi^*_{1,1}(\theta) = \mu\pi^*_{2,1}(\theta) + \gamma\pi^*_{1,0}(\theta) - \pi_{1,1}(0), \tag{8}$$

$$(\mu-\theta)\pi^*_{n,1}(\theta) = \mu\pi^*_{n+1,1}(\theta) + \pi_{n-1,1}(0)A^*(\theta) + \gamma\pi^*_{n,0}(\theta) - \pi_{n,1}(0), \quad 2 \leqslant n \leqslant N-1, \tag{9}$$

$$(\mu-\theta)\pi^*_{N,1}(\theta) = (\pi_{N-1,1}(0) + \pi_{N,1}(0))A^*(\theta) + \gamma\pi^*_{N,0}(\theta) - \pi_{N,1}(0), \tag{10}$$

$$-\theta\pi^*_{0,0}(\theta) = \mu\pi^*_{1,1}(\theta) + \eta\pi^*_{1,0}(\theta) - \pi_{0,0}(0), \tag{11}$$

$$(\gamma+\eta-\theta)\pi^*_{n,0}(\theta) = \eta\pi^*_{n+1,0}(\theta) + \pi_{n-1,0}(0)A^*(\theta) - \pi_{n,0}(0), \quad 1 \leqslant n \leqslant N-1, \tag{12}$$

$$(\gamma+\eta-\theta)\pi^*_{N,0}(\theta) = (\pi_{N-1,0}(0) + \pi_{N,0}(0))A^*(\theta) - \pi_{N,0}(0), \tag{13}$$

One important result listed below in the form of a lemma using Eqs. (8)–(13).

Lemma 1

$$\sum_{n=1}^{N}\pi_{n,1}(0) + \sum_{n=0}^{N}\pi_{n,0}(0) = \lambda. \tag{14}$$

The left hand side denote mean number of entrances into the system per unit time and is obviously equal to the mean arrival rate λ.

Proof. Adding (8)–(13),

$$\sum_{n=1}^{N}\pi^*_{n,1}(\theta) + \sum_{n=0}^{N}\pi^*_{n,0}(\theta) = \frac{1-A^*(\theta)}{\theta}\left(\sum_{n=1}^{N}\pi_{n,1}(0) + \sum_{n=0}^{N}\pi_{n,0}(0)\right). \tag{15}$$

Taking limit $\theta \to 0$ we obtain the desired result. □

3.2.1. Relation between steady-state distribution at arbitrary and pre-arrival epochs

The relation between pre-arrival epoch probabilities $\pi^-_{n,1}(\pi^-_{n,0})$ and $\pi_{n,1}(0)(\pi_{n,0}(0))$ are given by

$$\pi^-_{n,1} = \frac{1}{\lambda}\pi_{n,1}(0), \quad 1 \leqslant n \leqslant N, \tag{16}$$

and

$$\pi^-_{n,0} = \frac{1}{\lambda}\pi_{n,0}(0), \quad 0 \leqslant n \leqslant N, \tag{17}$$

where λ is given in Lemma 1. Now we are in a position to express arbitrary epoch probabilities in terms of pre-arrival epoch probabilities. Setting $\theta = 0$ in the Eqs. (8)–(13) and using (16) and (17), we obtain after simplification

$$\pi_{N,0} = \frac{\lambda}{\gamma+\eta}\pi^-_{N-1,0}, \tag{18}$$

$$\pi_{n,0} = \frac{\lambda}{\gamma+\eta}(\pi^-_{n-1,0} - \pi^-_{n,0}) + \frac{\eta}{\gamma+\eta}\pi_{n+1,0}, \quad n = N-1, N-2, \ldots, 1, \tag{19}$$

$$\pi_{N,1} = \frac{\lambda}{\mu}\pi^-_{N-1,1} + \frac{\gamma}{\mu}\pi_{N,0}, \tag{20}$$

$$\pi_{n,1} = \pi_{n+1,1} + \frac{\lambda}{\mu}(\pi^-_{n-1,1} - \pi^-_{n,1}) + \frac{\gamma}{\mu}\pi_{n,0}, \quad n = N-1, N-2, \ldots, 2, \tag{21}$$

$$\pi_{1,1} = \pi_{2,1} - \frac{\lambda}{\mu}\pi^-_{1,1} + \frac{\gamma}{\mu}\pi_{1,0}, \tag{22}$$

It may be noted here that we do not have explicit expressions for $\pi_{0,0}$. However, it can be computed by using the normalization condition, that is, $\pi_{0,0} = 1 - \sum_{n=1}^{N}(\pi_{n,0} + \pi_{n,1})$.

3.3. Performance measures

As the steady-state probabilities at various epochs are known, performance measures of the queue can easily be obtained and are given as: the average number of customers in the system $(L) = \sum_{i=1}^{N} i(\pi_{i,1} + \pi_{i,0})$, the average number of customers in the system when the server is in the service period $(L_1) = \sum_{i=1}^{N} i\pi_{i,1}$, the average number of customers in the system when the server is on working vacation $(L_2) = \sum_{i=0}^{N} i\pi_{i,0}$. The probability of loss or blocking $(P_{\text{loss}}) = \pi^{-}_{N,0} + \pi^{-}_{N,1}$. Finally, mean waiting time in the system $(w) = L/\lambda'$, where $\lambda' = \lambda(1 - P_{\text{loss}})$ is the effective arrival rate.

3.3.1. Waiting time analysis

In this section we obtain the LST of waiting time distribution of a customer who is accepted in the system. If $W(x)$ be the actual waiting time distribution (in the system) of a customer who is accepted in the system and let $W^*(\theta)$ be its LST then considering various possible cases, we have

$$W^*(\theta) = \frac{1}{1 - P_{\text{loss}}}\left\{\pi^{-}_{0,0}\left(\frac{\gamma\mu}{(\theta+\mu)(\theta+\gamma+\eta)} + \frac{\eta}{\theta+\gamma+\eta} + \sum_{n=1}^{N-1}\pi^{-}_{n,1}\left(\frac{\mu}{\theta+\mu}\right)^{n+1}\right)\right.$$
$$\left. + \sum_{n=1}^{N-1}\pi^{-}_{n,0}\sum_{k=0}^{n}\left(\frac{\eta}{\theta+\gamma+\eta}\right)^{k}\left(\frac{\gamma}{\theta+\gamma+\eta}\right)\left(\frac{\mu}{\theta+\mu}\right)^{n+1-k} + \sum_{n=1}^{N-1}\pi^{-}_{n,0}\left(\frac{\eta}{\theta+\gamma+\eta}\right)^{n+1}\right\}.$$

From this expression one can easily obtain mean waiting time in the system which is given by

$$w = -W^{*(1)}(0) = \frac{1}{1 - P_{\text{loss}}}\left\{\pi^{-}_{0,0}\left(\frac{\gamma+\mu}{\mu(\gamma+\eta)}\right) + \sum_{n=1}^{N-1}\pi^{-}_{n,1}\left(\frac{n+1}{\mu}\right)\right.$$
$$\left. + \sum_{n=1}^{N-1}\pi^{-}_{n,0}\sum_{k=0}^{n}\left(\frac{k\eta^{k}\gamma}{(\gamma+\eta)^{k+2}} + \frac{\eta^{k}\gamma}{(\gamma+\eta)^{k+2}} + \frac{\eta^{k}\gamma(n+1-k)}{(\gamma+\eta)^{k+1}\mu}\right) + \sum_{n=1}^{N-1}\pi^{-}_{n,0}\frac{(n+1)\eta^{n+1}}{(\gamma+\eta)^{n+2}}\right\}.$$

4. Numerical result and discussion

To demonstrate the applicability of the results obtained in previous sections, a variety of numerical results have been presented in self explanatory tables (Tables 1–3) and graphs. In the bottom of the tables various performance measures are given. To compare our results with those of Baba [10], we have used the formulae given in [10] to evaluate pre-arrival and arbitrary epoch probabilities of $E_2/M/1$ queue. Using our method we obtained the results for $E_2/M/1/25$ queue with mean inter-arrival time equal to 1.6667, and the service time during service period, vacation time and service time during vacation period follow exponential distribution with respective rates equal to 1.5, 0.5 and 0.8. Since offered load $(\rho = 0.4) < 1$ and buffer space (=25) is high, our model behaves as an infinite buffer queue. It is found that the pre-arrival and arbitrary epoch probabilities of $E_2/M/1/25$ queue match exactly with those of $E_2/M/1$ queue. Moreover, in our numerical computation, mean waiting time obtained through transform exactly matches with the one obtained from Little's rule. Further, we have also compared our result with Karaesmen and Gupta [11] for non-working vacation models $(\eta = 0)$ and found that our P_{loss} matches (see Table 3) exactly with the one obtained by them.

In Table 3 we provide the results for $HE_2/M/1/10$ queue with multiple vacations by setting the working vacation parameter $(\eta = 0.0)$ equal to zero. Other parameters are same as those given in Table 2.

Table 1
Distribution of number of customers in the system at various epochs for $E_2/M/1/10/MWV$ queue with parameters: $\lambda = 1.0$, $\mu = 1.2$, $\gamma = 0.6$ and $\eta = 1.4$

n	$\pi^-_{n,0}$	$\pi^-_{n,1}$	$\pi_{n,0}$	$\pi_{n,1}$
0	0.318654	–	0.171164	–
1	0.075534	0.126057	0.145743	0.095512
2	0.017905	0.122310	0.034547	0.127687
3	0.004244	0.096766	0.008189	0.107292
4	0.001006	0.072632	0.001941	0.081910
5	0.000238	0.053653	0.000460	0.060828
6	0.000057	0.039427	0.000109	0.044783
7	0.000013	0.028878	0.000026	0.032872
8	0.000003	0.020918	0.000006	0.024069
9	0.000001	0.014300	0.000001	0.017432
10	0.000000	0.007403	0.000000	0.011917
Sum	0.417656	0.582344	0.395697	0.604303

$L_1 = 2.271963$, $L_2 = 0.250372$, $L = 2.522335$, $P_{\text{loss}} = 0.007403$.

It is to be noted here that the blocking probability in this case exactly match with the respective blocking probability obtained in [11, p. 822, Table 1].

In Fig. 1, we have plotted loss probability against system capacity. Three types of inter-arrival time distributions are assumed viz. (i) deterministic, (ii) E_2, and (iii) HE_2. They all have equal mean ($=1/\lambda$) where $\lambda = 1.25$. Other parameters are taken as $\mu = 2.0$, $\gamma = 1.0$, $\eta = 0.6$. For case (ii), we have considered two coefficient of variation (CV = 0.71, 0.95). Similarly, for case (iii) we have considered three CVs (=1.29, 1.71, 3.09).

From Fig. 1 it can be seen that for any inter-arrival time as system capacity increases, P_{loss} asymptotically approaches to zero except for case (iii) with a very high CV(=3.09). This study reveals that not only the service process and duration of vacation time play major role in the performance of queueing system but also the arrival process plays very important role. Implying that the proper determination of the arrival process is very much necessary for the allocation of buffer space. Similarly, in Fig. 1(a), we have conducted the same experiment as we did in Fig. 1 except that here we have considered a very high arrival rate ($\lambda = 2.17391$) so that traffic intensity will be very high ($\rho = \lambda/\mu = 1.08696 > 1$). All other parameters remain the same as we have taken for Fig. 1. From this figure it can be seen that as buffer size increases loss probability reduces very little and asymptotically converges to its minimum value. After certain level though system size ($N = 10$) increases but the rejection rate remains almost unchanged.

Table 2
Distribution of number of customers in the system at various epochs for $HE_2/M/1/10/MWV$ queue with parameters: $\sigma_1 = 0.149883$, $\sigma_2 = 0.850117$, $\lambda_1 = 0.419671$, $\lambda_2 = 2.38033$, $\lambda = 1.4$ (CV = 1.71), $\mu = 1.0$, $\gamma = 0.25$ and $\eta = 0.5$

n	$\pi^-_{n,0}$	$\pi^-_{n,1}$	$\pi_{n,0}$	$\pi_{n,1}$
0	0.016294	–	0.042370	–
1	0.013496	0.008643	0.011667	0.016979
2	0.011180	0.017356	0.009666	0.026163
3	0.009262	0.026455	0.008012	0.035944
4	0.007678	0.036355	0.006649	0.046681
5	0.006375	0.047722	0.005538	0.058877
6	0.005319	0.061886	0.004658	0.073407
7	0.004517	0.081888	0.004031	0.092072
8	0.004121	0.115224	0.003800	0.119066
9	0.004984	0.181008	0.004591	0.164788
10	0.011642	0.328593	0.009304	0.255738
Sum	0.094869	0.905131	0.110286	0.889714

$L_1 = 6.736186$, $L_2 = 0.330244$, $L = 7.066430$, $P_{\text{loss}} = 0.340234$.

Table 3
Distribution of number of customers in the system at various epochs for $HE_2/M/1/10$ queue with multiple vacations

n	$\pi^-_{n,0}$	$\pi^-_{n,1}$	$\pi_{n,0}$	$\pi_{n,1}$
0	0.011626	–	0.034355	–
1	0.010036	0.008040	0.008903	0.016276
2	0.008664	0.016428	0.007686	0.025306
3	0.007479	0.025428	0.006635	0.035128
4	0.006456	0.035416	0.005727	0.046070
5	0.005573	0.047041	0.004944	0.058621
6	0.004811	0.061637	0.004268	0.073659
7	0.004153	0.082312	0.003684	0.093028
8	0.003585	0.116766	0.003181	0.121052
9	0.003095	0.184697	0.002746	0.168491
10	0.019536	0.337221	0.017331	0.262908
Sum	0.085014	0.914986	0.099460	0.900540

$L_1 = 6.856726$, $L_2 = 0.366677$, $L = 7.223403$, $P_{\text{loss}} = 0.356757$.

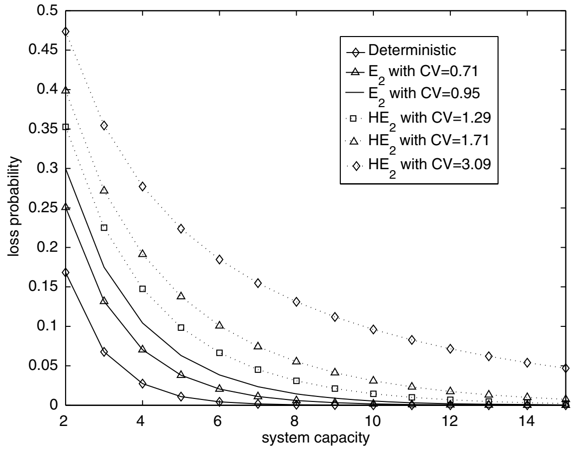

Fig. 1. N versus P_{loss}.

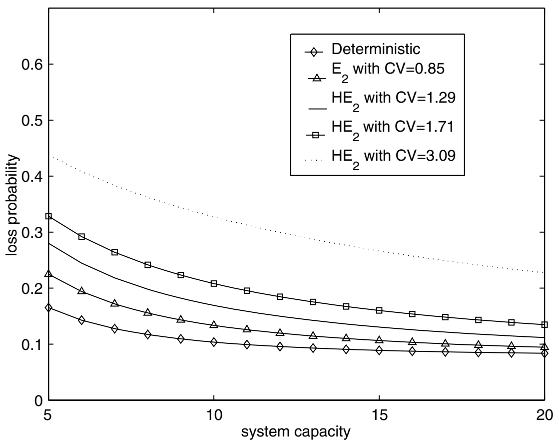

Fig. 1(a). N versus P_{loss}.

In Fig. 2 and 2(a), we examine the behaviour of working vacation model compared to non-working vacation model in an $E_2/M/1/N$ queue with buffer space (N) varying from 2 to 20. For this we have chosen the

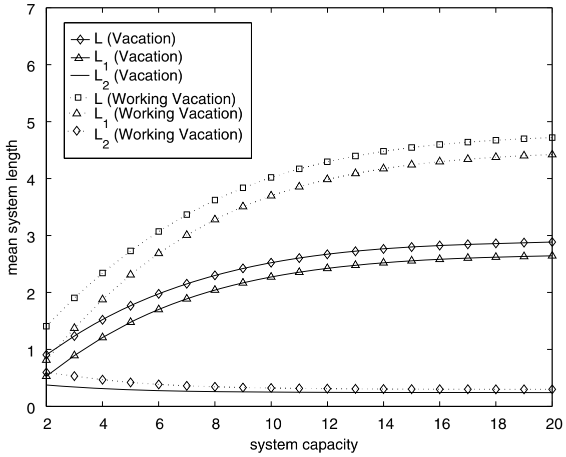

Fig. 2. N versus mean system length.

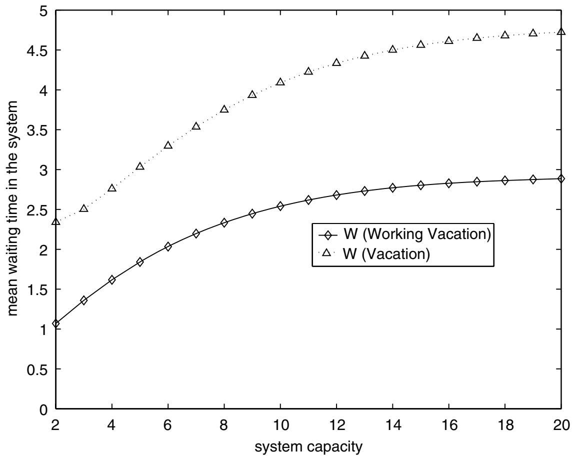

Fig. 2(a). N versus w.

following parameters: $\lambda = 1.00$, $\mu = 1.2$, $\gamma = 0.6$ and $\eta = 1.4$ ($\eta = 0.0$ for non-working vacation model). Fig. 2 shows the effect of buffer space on mean system length and it is observed that mean system lengths are higher for working vacation models. Moreover, there is very little influence of buffer size on mean system lengths when server is on vacation. Also for vacation and working vacation models they are almost same. After certain level (say, $N = 14$) the mean system lengths are parallel to x-axis, i.e., they asymptotically approach to their maximum value. Fig. 2(a) plots the effect of buffer space on mean waiting time in the system. Quite naturally we observe opposite behaviour than Fig. 2, i.e., here mean waiting time is less for working vacation model in comparison of non-working vacation.

In Fig. 3 and 3(a), we have plotted mean system lengths and probability of blocking against the service rate during working vacation, respectively. For this we have taken $E_2/M/1/7$ queue with parameters: $\lambda = 1.00$, $\mu = 1.2$, $\gamma = 0.6$ and η varies from 0 to 4.0. Fig. 3 shows that as η increases, mean system lengths asymptotically approaches to their minimum value. Moreover, η has very little influence on L_2, mean system length when server is on vacation. It can be seen from Fig. 3(a) that initially P_{loss} linearly decreases as η increases up to 1.3, afterwards it asymptotically approaches to its minimum value. So the blocking of customers is very much affected by the rate of service during a working vacation.

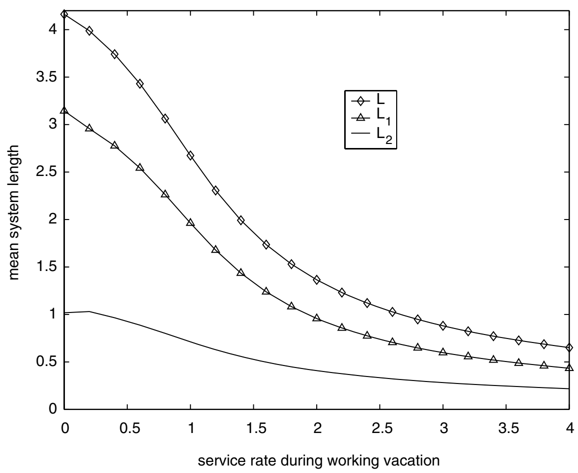

Fig. 3. η versus mean system length.

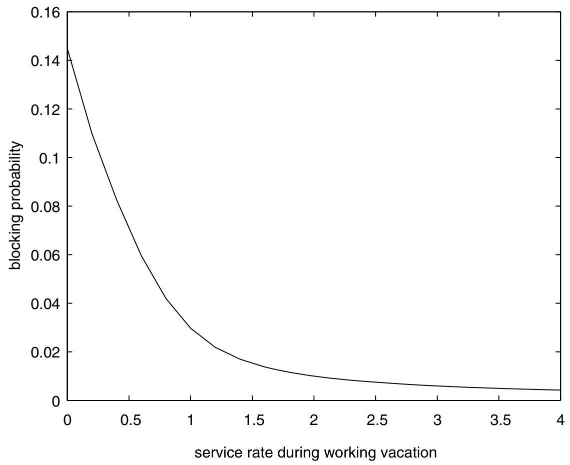

Fig. 3(a). η versus P_{loss}.

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