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Some Conditions for Boundedness of Solutions of Difference Volterra Equations

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1. INTRODUCTION

Boundedness of solutions of some difference Volterra equations is investigated, using some ideas from the books [1,2].

2. EQUATION WITH NONNEGATIVE COEFFICIENTS

Consider the scalar equation

$$x(i+1) = x(i) + \sum_{j=0}^{i} A(i,j)x(j), \qquad i \geq 0. \tag{2.1}$$

It is supposed that

$$A(i,j) \geq 0, \qquad i \geq j \geq 0. \tag{2.2}$$

THEOREM 2.1. *The necessary and sufficient condition for boundedness of the solution of equation (2.1) is*

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} A(i,j) < \infty. \tag{2.3}$$

PROOF. NECESSITY. Assume that all solutions of equation (2.1) are bounded, i.e.,

$$\sup_{i \geq 0} |x(i)| < \infty, \tag{2.4}$$

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and prove that inequality (2.3) is valid. Assume that $x(0) \geq 0$. Then from equation (2.1) for $i = 0$ and condition (2.2), it follows that $x(1) \geq x(0) \geq 0$. Note also that if $x(j+1) \geq x(j) \geq x(0)$ for $j = 0, \dots, i-1$, then from (2.1),(2.2) it follows that $x(i+1) \geq x(i) \geq x(0)$ for $i \geq 0$. Therefore, after summing equation (2.1) from $i = 0$ to k, we obtain

$$x(k+1) = x(0) + \sum_{i=0}^{k} \sum_{j=0}^{i} A(i,j)x(j) \geq x(0) \left[1 + \sum_{i=0}^{k} \sum_{j=0}^{i} A(i,j) \right].$$

From here and (2.4) follows (2.3).

SUFFICIENCY. Assume that inequality (2.3) is valid, but there exists unbounded solution $x(i) > 0$ of equation (2.1) such that $\Delta x(i) = x(i+1) - x(i) \geq 0$. Dividing equation (2.1) by $x(i)$ and summing both parts of the obtained equality from $i = 0$ to k, we obtain

$$\sum_{i=0}^{k} \frac{\Delta x(i)}{x(i)} = \sum_{i=0}^{k} \sum_{j=0}^{i} A(i,j) \frac{x(j)}{x(i)}. \tag{2.5}$$

Consider the continuous function $x(t) = x(i) + (t-i)\Delta x(i)$ for $t \in [i, i+1]$. Then $\dot{x}(t) = \Delta x(i)$ and $x(t) \geq x(i)$. Therefore,

$$\frac{\Delta x(i)}{x(i)} = \int_{i}^{i+1} \frac{\Delta x(i)}{x(i)}\, dt \geq \int_{i}^{i+1} \frac{\dot{x}(t)}{x(t)}\, dt = \ln x(i+1) - \ln x(i). \tag{2.6}$$

Substituting (2.6) into (2.5) and using (2.3) and $x(j) \leq x(i)$ for $j \leq i$, we obtain

$$\ln x(k+1) - \ln x(0) \leq \sum_{i=0}^{k} \sum_{j=0}^{i} A(i,j) \frac{x(j)}{x(i)} \leq \sum_{i=0}^{\infty} \sum_{j=0}^{i} A(i,j) < \infty.$$

So, as a result we obtain the absurdity of the assumption about unboundedness of $x(k)$. Theorem 2.1 is proven.

EXAMPLE 2.1. Consider the equation

$$x(i+1) = x(i) + \sum_{j=0}^{i} q^{i+j} x(j), \qquad q \in (0,1).$$

The solution of this equation is bounded since

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} q^{i+j} = \sum_{i=0}^{\infty} q^{i} \frac{1-q^{i+1}}{1-q} < \frac{1}{(1-q)^2} < \infty.$$

EXAMPLE 2.2. Consider the equation

$$x(i+1) = x(i) + \sum_{j=0}^{i} \frac{1}{(i+j+1)^{\alpha}} x(j).$$

If $\alpha > 2$, then the solution of this equation is bounded since

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{1}{(i+j+1)^{\alpha}} < \sum_{i=0}^{\infty} \frac{i+1}{(i+1)^{\alpha}} = \sum_{i=0}^{\infty} \frac{1}{(i+1)^{\alpha-1}} < \infty.$$

3. NONOSCILLATORY SOLUTIONS

Consider the Volterra difference equation in the form

$$x(i+1) = a_0(i)x(i) - \frac{a_1(i-1)}{a_1(i)} x(i-1) - \sum_{j=0}^{i-2} A(i,j)x(j), \qquad i \geq 0. \tag{3.1}$$

THEOREM 3.1. *Assume that equation (3.1) has nonoscillatory solutions, and the following inequalities are valid:*

$$\begin{aligned} a_0(i) - \frac{a_1(i-1)}{a_1(i)} \le 1, \quad a_1(i) > 0, \qquad i \ge 0, \\ A(i,j) \ge 0, \qquad i \ge 2, \quad j = 0, \dots, i-2, \end{aligned} \tag{3.2}$$

$$\sum_{i=0}^{\infty} \frac{1}{a_1(i)} < \infty. \tag{3.3}$$

Then all nonoscillatory solutions of equation (3.1) are bounded.

PROOF. Without loss of generality we can assume that the solution $x(i)$ of equation (3.1) is greater than 0 for all $i \ge -1$ and nonoscillatory. Hence, from equation (3.1) and inequalities (3.2) it follows that

$$\begin{aligned} x(i+1) &\le \left(a_0(i) - \frac{a_1(i-1)}{a_1(i)} \right) x(i) + \frac{a_1(i-1)}{a_1(i)} \Delta x(i-1) \\ &\le x(i) + \frac{a_1(i-1)}{a_1(i)} \Delta x(i-1) \end{aligned}$$

or

$$a_1(i) \Delta x(i) \le a_1(i-1) \Delta x(i-1). \tag{3.4}$$

Continuing inequality (3.4), we have

$$a_1(i) \Delta x(i) \le a_1(-1) \Delta x(-1)$$

or

$$x(i+1) \le x(i) + \frac{a_1(-1)}{a_1(i)} \Delta x(-1). \tag{3.5}$$

Summing both parts of inequality (3.5), we obtain

$$x(i+1) \le x(0) + a_1(-1) \Delta x(-1) \sum_{j=0}^{i} \frac{1}{a_1(j)}.$$

From here and (3.3), it follows that the nonoscillating solutions of Volterra equation (3.1) are bounded for all values of $i \ge 0$. Theorem 3.1 is proven.

EXAMPLE 3.1. Consider the difference equation

$$a_1(i) \Delta x(i) = a_1(i-1) \Delta x(i-1) - a_1^{-1}(i) x(i), \tag{3.6}$$

where $a_1(i) > 0$ for $i \ge 0$ and satisfies the condition $\sum_{i=0}^{\infty} a_1^{-1}(i) < \infty$.

Equation (3.6) can be rewritten in form (3.1) with $A(i,j) = 0$ for $0 \le j \le i-2$ and

$$a_0(i) = 1 + \frac{a_1(i-1)}{a_1(i)} - \frac{1}{a_1^2(i)}.$$

From Theorem 3.1, it follows that all nonoscillatory solutions of equation (3.6) are bounded.

4. BEHAVIOR OF SOLUTION INCREMENT

Consider Volterra difference equation in the form

$$x(i+1) = a_0(i) x(i) - \frac{a_1(i-1)}{a_1(i)} x(i-1) + \frac{a_2(i)}{a_1(i)} \sum_{j=0}^{i-2} A(i,j) x(j), \qquad i \ge 0, \tag{4.1}$$

where

$$a_2(i) = a_1(i) \left(a_0(i) - \frac{a_1(i-1)}{a_1(i)} - 1 \right). \tag{4.2}$$

THEOREM 4.1. *Let*

$$\begin{aligned} a_1(i) > 0, \quad a_2(i) \geq 0, \qquad & i \geq 0, \\ A(i,j) \geq 0, \qquad i \geq 2, \quad & j = 0, \dots, i-2. \end{aligned} \tag{4.3}$$

Then the necessary condition for every function $a_1(i)\Delta x(i)$ of equation (4.1) to be bounded are

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{a_2(i+1)}{a_1(j)} < \infty. \tag{4.4}$$

PROOF. It is easy to see that equation (4.1) can be rewritten as Volterra difference equation in the following form:

$$\Delta[a_1(i)\Delta x(i)] = a_2(i+1) \left(x(i+1) + \sum_{j=0}^{i-1} A(i+1,j)x(j) \right). \tag{4.5}$$

Choose the initial condition of equation (4.1) such that $x(0) > x(-1) > 0$ and assume that the function $a_1(i)\Delta x(i)$ is bounded. Then the solution of equation (4.1) satisfies the conditions

$$x(i) > 0, \quad \Delta x(i) > 0, \qquad i \geq 0. \tag{4.6}$$

From (4.5), it follows that

$$\Delta[a_1(i)\Delta x(i)] \geq a_2(i+1)x(i+1) \geq 0, \qquad i \geq 0. \tag{4.7}$$

Summation of inequality (4.7) gives

$$a_1(i)\Delta x(i) \geq a_1(0)\Delta x(0). \tag{4.8}$$

Dividing both parts of inequality (4.8) on $a_1(i)$ and summing it, we obtain

$$x(i+1) \geq x(0) + a_1(0)\Delta x(0) \sum_{j=0}^{i} \frac{1}{a_1(j)}. \tag{4.9}$$

From (4.7),(4.9) it follows

$$\Delta[a_1(i)\Delta x(i)] \geq a_2(i+1) \left(x(0) + a_1(0)\Delta x(0) \sum_{j=0}^{i} \frac{1}{a_1(j)} \right). \tag{4.10}$$

Summation of inequality (4.10) from $i = 0$ to k gives

$$a_1(k+1)\Delta x(k+1) \geq x(0) \sum_{i=0}^{k} a_2(i+1) + a_1(0)\Delta x(0) \left(1 + \sum_{i=0}^{k} \sum_{j=0}^{i} \frac{a_2(i+1)}{a_1(j)} \right) \geq 0.$$

Consequently, because of boundedness of the function $a_1(k)\Delta x(k)$, we can conclude that inequality (4.4) is valid. Theorem 4.1 is proven.

THEOREM 4.2. *Assume that conditions (4.3), (4.4), and*

$$\hat{A} = \sup_{i \geq 2} \sum_{j=0}^{i-2} A(i,j) < \infty \tag{4.11}$$

are satisfied. Then for all solutions of equation (4.1), the function $a_1(k)\Delta x(k)$ is bounded.

PROOF. Without loss of generality, we can assume that the solution of equation (4.1) is positive and increasing, i.e., inequalities (4.6) are valid. Using inequalities (4.6) and $a_1(i) > 0$, let us find the first difference of the fraction $x(i+1)/a_1(i)\Delta x(i)$. We have

$$\Delta\left[\frac{x(i+1)}{a_1(i)\Delta x(i)}\right] = \frac{x(i+2)}{a_1(i+1)\Delta x(i+1)} - \frac{x(i+1)}{a_1(i)\Delta x(i)} = \frac{a_1(i)x(i+2)\Delta x(i) - a_1(i+1)x(i+1)\Delta x(i+1)}{a_1(i)a_1(i+1)\Delta x(i)\Delta x(i+1)}. \tag{4.12}$$

In the first of the right-hand side of equality (4.12), we have

$$a_1(i)x(i+2)\Delta x(i) = a_1(i)\Delta x(i+1)\Delta x(i) + a_1(i)x(i+1)\Delta x(i). \tag{4.13}$$

From (4.12), (4.13), (4.7) it follows that

$$\Delta\left[\frac{x(i+1)}{a_1(i)\Delta x(i)}\right] = \frac{1}{a_1(i+1)} - \frac{x(i+1)\Delta[a_1(i)\Delta x(i)]}{a_1(i)a_1(i+1)\Delta x(i)\Delta x(i+1)} \le \frac{1}{a_1(i+1)}.$$

Consequently, summing this inequality we obtain

$$\frac{x(i+1)}{a_1(i)\Delta x(i)} \le \frac{x(1)}{a_1(0)\Delta x(0)} + \sum_{j=0}^{i}\frac{1}{a_1(j)}. \tag{4.14}$$

Dividing both parts of equation (4.5) on $a_1(i)\Delta x(i)$ and using inequality (4.14), we conclude that

$$\frac{\Delta[a_1(i)\Delta x(i)]}{a_1(i)\Delta x(i)} = a_2(i+1)\left(\frac{x(i+1)}{a_1(i)\Delta x(i)} + \frac{S(i+1)}{a_1(i)\Delta x(i)}\right) \le a_2(i+1)\left(\frac{x(1)}{a_1(0)\Delta x(0)} + \sum_{j=0}^{i}\frac{1}{a_1(j)} + \frac{S(i+1)}{a_1(i)\Delta x(i)}\right),$$

where

$$S(i) = \sum_{j=0}^{i-2} A(i,j)x(j).$$

Summing the above inequality over i, we have

$$\sum_{i=0}^{k}\frac{\Delta[a_1(i)\Delta x(i)]}{a_1(i)\Delta x(i)} \le \sum_{i=0}^{k}\sum_{j=1}^{i}\frac{a_2(i+1)}{a_1(j)} + \frac{x(1)}{a_1(0)\Delta x(0)}\sum_{i=0}^{k}a_2(i+1) + \sum_{i=0}^{k}\frac{a_2(i+1)S(i+1)}{a_1(i)\Delta x(i)}. \tag{4.15}$$

Now let us use inequality (4.14) in the following form:

$$\frac{1}{a_1(i)\Delta x(i)} \le \frac{1}{x(i+1)}\left(\frac{x(1)}{a_1(0)\Delta x(0)} + \sum_{j=0}^{i}\frac{1}{a_1(j)}\right). \tag{4.16}$$

As a result from inequalities (4.15),(4.16) it follows that

$$\sum_{i=0}^{k}\frac{\Delta[a_1(i)\Delta x(i)]}{a_1(i)\Delta x(i)} \le \sum_{i=0}^{k}\sum_{j=0}^{i}\frac{a_2(i+1)}{a_1(j)} + \frac{x(1)}{a_1(0)\Delta x(0)}\sum_{i=0}^{k}a_2(i+1) + \sum_{i=0}^{k}\frac{a_2(i+1)S(i+1)}{x(i+1)}\left(\frac{x(1)}{a_1(0)\Delta x(0)} + \sum_{j=0}^{i}\frac{1}{a_1(j)}\right). \tag{4.17}$$

At the left-hand side of inequality (4.17) we can replace sum by the integral using the functions

$$x(t) = a_1(i)\Delta x(i) + (t-i)\Delta[a_1(i)\Delta x(i)], \qquad i \le t \le i+1,$$
$$\dot{x}(t) = \Delta[a_1(i)\Delta x(i)], \qquad x(i) = a_1(i)\Delta x(i).$$

As a result, we have

$$\begin{aligned}\sum_{i=0}^{k} \frac{\Delta[a_1(i)\Delta x(i)]}{a_1(i)\Delta x(i)} \ge \sum_{i=0}^{k} \int_i^{i+1} \frac{\dot{x}(t)}{x(t)}\,dt = \sum_{i=0}^{k} \ln[a_1(i+1)\Delta x(i+1)] \\ - \ln[a_1(i)\Delta x(i)] = \ln[a_1(k+1)\Delta x(k+1)] - \ln[a_1(0)\Delta x(0)].\end{aligned} \tag{4.18}$$

Now let us estimate the right part of (4.17). Because the sequence $x(i)$ is increasing and (4.11), we have

$$\frac{S(i+1)}{x(i+1)} = \sum_{j=0}^{i-1} A(i+1,j)\frac{x(j)}{x(i+1)} \le \sum_{j=0}^{i-1} A(i+1,j) \le \hat{A}. \tag{4.19}$$

From (4.17)–(4.19), it follows that

$$\begin{aligned}&\ln[a_1(k+1)\Delta x(k+1)] - \ln[a_1(0)\Delta x(0)] \\ &\le \left(1+\hat{A}\right)\left(\sum_{i=0}^{k}\sum_{j=0}^{i} \frac{a_2(i+1)}{a_1(j)} + \frac{x(1)}{a_1(0)\Delta x(0)} \sum_{i=0}^{k} a_2(i+1)\right).\end{aligned} \tag{4.20}$$

Using (4.4), note that

$$\infty > \sum_{i=0}^{\infty}\sum_{j=0}^{i} \frac{a_2(i+1)}{a_1(j)} > \frac{1}{a_1(0)} \sum_{i=0}^{\infty} a_2(i+1).$$

From here and (4.4), it follows that the right part of (4.20) is bounded. Hence, the function $a_1(k)\Delta x(k)$ is bounded. Theorem 4.2 is proven.

EXAMPLE 4.1. Consider the difference equation

$$a_1(i)\Delta x(i) = a_1(i-1)\Delta x(i-1) + a_1^{-1}(i)x(i), \tag{4.21}$$

where $a_1(i) > 0$ for $i \ge 0$ and satisfies the condition $\sum_{i=0}^{\infty} a_1^{-1}(i) < \infty$.

Equation (4.21) can be rewritten in form (4.1) with $A(i,j) = 0$ for $0 \le j \le i-2$ and

$$a_0(i) = 1 + \frac{a_1(i-1)}{a_1(i)} + \frac{1}{a_1^2(i)}.$$

As follows from (4.2), in this case $a_2(i) = a_1^{-1}(i)$, and condition (4.4) takes the form

$$\sum_{i=0}^{\infty} \frac{1}{a_1(i+1)} \sum_{j=0}^{i} \frac{1}{a_1(j)} < \left(\sum_{i=0}^{\infty} \frac{1}{a_1(i)}\right)^2 < \infty.$$

So, all conditions of Theorem 4.2 are valid. Therefore, for each solution $x(i)$ of equation (4.21) the function $a_1(i)\Delta x(i)$ is bounded.

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