Bohr Hamiltonian with deformation-dependent mass term

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A R T I C L E   I N F O

A B S T R A C T

The Bohr Hamiltonian describing the collective motion of atomic nuclei is modified by allowing the mass to depend on the nuclear deformation. Exact analytical expressions are derived for spectra and wave functions in the case of a γ-unstable Davidson potential, using techniques of supersymmetric quantum mechanics. Numerical results in the Xe–Ba region are discussed.

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1. Introduction

The Bohr Hamiltonian [1] and its extension, the geometrical collective model [2,3], have provided for several decades a sound framework for understanding the collective behaviour of atomic nuclei. A puzzle remaining unsolved since the early days of the Bohr Hamiltonian regards the behaviour of the moments of inertia of atomic nuclei [4]. They are predicted to increase proportionally to β^2, where β is the collective variable corresponding to nuclear deformation, while experimentally a much more moderate increase is observed, especially for well deformed nuclei. In addition, the use of a constant mass has been recently questioned [5], pointing out that the mass tensor of the collective Hamiltonian cannot be considered as a constant and should be taken as a function of the collective coordinates.

On the other hand, the algebraic framework of the Interacting Boson Model (IBM) [6] has been very useful in the systematic study of nuclei corresponding to its three limiting symmetries [vibrational U(5), axially symmetric deformed SU(3), γ-unstable O(6)], as well as to intermediate cases. In the geometrical limit of the IBM [6], obtained through the use of coherent states [6], it is worth remembering that in addition to the usual term of the kinetic energy, π^2, terms of the form $\beta^2\pi^2$ appear in the O(6) limiting symmetry and in the U(5)–O(6) transition region, while more complicated terms appear in the SU(3) limiting symmetry, as well as in the U(5)–SU(3) and SU(3)–O(6) transition regions [7]. These terms indicate that it might be appropriate to search for a modified form of the Bohr Hamiltonian, in which the kinetic energy term will be modified by terms containing β^2, and even by more involved terms.

The above reasoning leads to the consideration of a Bohr Hamiltonian with a mass depending on the collective variable β. Position-dependent effective masses have been considered recently in a general framework [8], demonstrating the equivalence of this approach to the consideration of deformed canonical commutation relations, as well as to the consideration of curved spaces. Furthermore, several Hamiltonians known to be soluble through techniques of supersymmetric quantum mechanics (SUSYQM) [9,10], have been appropriately generalized [11] to include position-dependent effective masses, the 3-dimensional harmonic oscillator being among them [11].

In the present work we are going to show that the Bohr Hamiltonian with a harmonic oscillator potential in β (to which a term proportional to $1/\beta^2$ can be added at no cost) can be generalized in order to include a mass depending on β, $B = B_0/(1 + a\beta^2)^2$, where B_0 and a are constants. Exact solutions will be constructed using techniques of SUSYQM. In order to achieve exact separation of variables, we are going to limit ourselves to potentials independent of the collective variable γ, called γ-unstable potentials. Numerical results for the Xe–Ba isotopes, well-known examples of γ-unstable behaviour [12], will also be shown.

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2. Formalism of position-dependent effective masses

When the mass $m(\mathbf{x})$ is position dependent [8], it does not commute with the momentum $\mathbf{p} = -i\hbar\nabla$. As a consequence, there are many ways to generalize the usual form of the kinetic energy, $\mathbf{p}^2/(2m_0)$, where m_0 is a constant mass, in order to obtain a Hermitian operator. In order to avoid any specific choices, one can use the general two-parameter form proposed by von Roos [13], with a Hamiltonian

$$H = -\frac{\hbar^2}{4}\left[m^{\delta'}(\mathbf{x})\nabla m^{\kappa'}(\mathbf{x})\nabla m^{\lambda'}(\mathbf{x}) + m^{\lambda'}(\mathbf{x})\nabla m^{\kappa'}(\mathbf{x})\nabla m^{\delta'}(\mathbf{x})\right] + V(\mathbf{x}), \tag{1}$$

where V is the relevant potential and the parameters δ', κ', λ' are constrained by the condition $\delta' + \kappa' + \lambda' = -1$. Assuming a position dependent mass of the form

$$m(\mathbf{x}) = m_0 M(\mathbf{x}), \quad M(\mathbf{x}) = \frac{1}{(f(\mathbf{x}))^2}, \quad f(\mathbf{x}) = 1 + g(\mathbf{x}), \tag{2}$$

where m_0 is a constant mass and $M(\mathbf{x})$ is a dimensionless position-dependent mass, the Hamiltonian becomes

$$H = -\frac{\hbar^2}{4m_0}\left[f^{\delta}(\mathbf{x})\nabla f^{\kappa}(\mathbf{x})\nabla f^{\lambda}(\mathbf{x}) + f^{\lambda}(\mathbf{x})\nabla f^{\kappa}(\mathbf{x})\nabla f^{\delta}(\mathbf{x})\right] + V(\mathbf{x}), \tag{3}$$

with $\delta + \kappa + \lambda = 2$. It is known [8] that this Hamiltonian can be put into the form

$$H = -\frac{\hbar^2}{2m_0}\sqrt{f(\mathbf{x})}\nabla f(\mathbf{x})\nabla\sqrt{f(\mathbf{x})} + V_{eff}(\mathbf{x}), \tag{4}$$

with

$$V_{eff}(\mathbf{x}) = V(\mathbf{x}) + \frac{\hbar^2}{2m_0}\left[\frac{1}{2}(1 - \delta - \lambda)f(\mathbf{x})\nabla^2 f(\mathbf{x}) + \left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right)(\nabla f(\mathbf{x}))^2\right]. \tag{5}$$

3. Bohr Hamiltonian with deformation-dependent effective mass

The original Bohr Hamiltonian [1] is

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)}\right] + V(\beta, \gamma), \tag{6}$$

where β and γ are the usual collective coordinates (β being a deformation coordinate measuring departure from spherical shape, and γ being an angle measuring departure from axial symmetry), while Q_k ($k = 1, 2, 3$) are the components of angular momentum in the intrinsic frame, and B is the mass parameter, which is usually considered constant.

We wish to construct a Bohr equation with a mass depending on the deformation coordinate β, in accordance with the formalism described above,

$$B(\beta) = \frac{B_0}{(f(\beta))^2}, \tag{7}$$

where B_0 is a constant. We then need the usual Pauli–Podolsky prescription [14]

$$(\nabla\Phi)^i = g^{ij}\frac{\partial\Phi}{\partial x^j}, \qquad \nabla^2\Phi = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\partial_j\Phi, \tag{8}$$

in order to construct a Schrödinger equation of the form of Eq. (4) in a 5-dimensional space equipped with the Bohr–Wheeler coordinates β, γ. Since the deformation function f depends only on the radial coordinate β, only the β part of the resulting equation will be affected, the final result reading

$$\left[-\frac{1}{2}\frac{\sqrt{f}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4 f\frac{\partial}{\partial\beta}\sqrt{f} - \frac{f^2}{2\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{f^2}{8\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} + v_{eff}\right]\Psi = \epsilon\Psi, \tag{9}$$

where reduced energies $\epsilon = B_0 E/\hbar^2$ and reduced potentials $v = B_0 V/\hbar^2$ have been used, with

$$v_{eff} = v(\beta, \gamma) + \frac{1}{4}(1 - \delta - \lambda)f\nabla^2 f + \frac{1}{2}\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right)(\nabla f)^2. \tag{10}$$

4. The γ-unstable Davidson potential

The solution of the above Bohr-like equation can be reached for certain classes of potentials using techniques developed in the context of SUSYQM [9–11]. In order to achieve separation of variables we assume that the potential $v(\beta, \gamma)$ depends only on the variable β, i.e. $v(\beta, \gamma) = u(\beta)$ [15]. Potentials of this kind are called γ-unstable potentials, since they are appropriate for the description of nuclei which can depart from axial symmetry without any energy cost. Furthermore, we are going to use as an example the Davidson potential [16]

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \tag{11}$$

where the parameter β_0 indicates the position of the minimum of the potential, the special case of $\beta_0 = 0$ corresponding to the simple harmonic oscillator. (Note that the term containing β_0 offers no additional complication to the solution.)

One then seeks wave functions of the form [15,17]

$$\Psi(\beta, \gamma, \theta_i) = F(\beta)\Phi(\gamma, \theta_i), \tag{12}$$

where θ_i ($i = 1, 2, 3$) are the Euler angles. Separation of variables gives

$$\left[-\frac{1}{2} \frac{\sqrt{f}}{\beta^4} \frac{\partial}{\partial\beta} \beta^4 f \frac{\partial}{\partial\beta} \sqrt{f} + \frac{f^2}{2\beta^2} \Lambda + \frac{1}{4}(1 - \delta - \lambda) f \nabla^2 f + \frac{1}{2}\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right)(\nabla f)^2 + u(\beta) \right] F(\beta) = \epsilon F(\beta), \tag{13}$$

$$\left[-\frac{1}{\sin 3\gamma} \frac{\partial}{\partial\gamma} \sin 3\gamma \frac{\partial}{\partial\gamma} + \frac{1}{4} \sum_k \frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)} \right] \Phi(\gamma, \theta_i) = \Lambda \Phi(\gamma, \theta_i). \tag{14}$$

Eq. (14) has been solved by Bès [18]. $\Lambda = \tau(\tau + 3)$ represents the eigenvalues of the second order Casimir operator of SO(5), while τ is the seniority quantum number, characterizing the irreducible representations of SO(5). The values of angular momentum L occurring for each τ are provided by a well-known algorithm and are listed in [6,15]. Within the ground state band (gsb) one has $L = 2\tau$. The $L = 2$ member of the quasi-γ_1 band is degenerate with the $L = 4$ member of the gsb, the $L = 3, 4$ members of the quasi-γ_1 band are degenerate to the $L = 6$ member of the gsb, the $L = 5, 6$ members of the quasi-γ_1 band are degenerate to the $L = 8$ member of the gsb, and so on.

Eq. (13) can be simplified by performing the derivations

$$\frac{1}{2} f^2 F'' + \left(ff' + \frac{2f^2}{\beta} \right) F' + \left(\frac{(f')^2}{8} + \frac{ff''}{4} + \frac{ff'}{\beta} \right) F - \frac{f^2}{2\beta^2} \Lambda F + \epsilon F - v_{eff} F = 0, \tag{15}$$

with

$$v_{eff} = u + \frac{1}{4}(1 - \delta - \lambda) f \left(\frac{4f'}{\beta} + f'' \right) + \frac{1}{2}\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right)(f')^2. \tag{16}$$

The difference in the numerical coefficient of f' observed in comparison to Eq. (2.27) of Ref. [8] is due to the different dimensionality of the space used in each case.

Setting

$$F(\beta) = \frac{R(\beta)}{\beta^2}, \tag{17}$$

Eq. (15) is put into the form

$$HR = -\frac{1}{2}\left(\sqrt{f} \frac{d}{d\beta} \sqrt{f} \right)^2 R + u_{eff} R = \epsilon R, \tag{18}$$

where

$$u_{eff} = v_{eff} + \frac{f^2 + \beta ff'}{\beta^2} + \frac{f^2}{2\beta^2} \Lambda. \tag{19}$$

Based on the results for the 3-dimensional harmonic oscillator reported in Ref. [11], in order to find analytical results for Eq. (18) we are going to consider for the deformation function the special form

$$f(\beta) = 1 + a\beta^2. \tag{20}$$

This choice is made in order to lead to an exact solution. Its physical implications will be discussed in Section 8.

Using these forms for the potential and the deformation function one obtains

$$u_{eff} = \beta^2 + a^2\beta^2\left[\frac{5}{2}(1 - \delta - \lambda) + 2\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} - \lambda\right) + 3 + \frac{\Lambda}{2} \right] + \frac{1}{\beta^2}\left(1 + \frac{1}{2}\Lambda + \beta_0^4 \right) + a\left[\frac{5}{2}(1 - \delta - \lambda) + 4 + \Lambda \right]. \tag{21}$$

5. Factorization

Following the general method used in supersymmetric quantum mechanics (SUSYQM) [9,10], one should take the following steps:

(i) Factorize the Hamiltonian.
(ii) Write a hierarchy of Hamiltonians starting from the first one.
(iii) Impose the shape invariance conditions, which are integrability conditions guaranteeing exact solvability.

Thus one first tries to put the Hamiltonian in Eq. (18) in the form

$$H_0 = B_0^+ B_0^- + \varepsilon_0. \tag{22}$$

Then this Hamiltonian can be considered as the first member of a hierarchy of Hamiltonians

$$H_i = B_i^+ B_i^- + \sum_{j=0}^{i} \varepsilon_j, \tag{23}$$

expressed in terms of the generalized ladder operators B_i^+, B_i^-, which are determined recursively, along with ε_i [which are in general different from the energy eigenvalues ϵ appearing in Eq. (18), the only exception being $\varepsilon_0 = \epsilon_0$], through the shape invariance (SI) condition

$$B_i^- B_i^+ = B_{i+1}^+ B_{i+1}^- + \varepsilon_{i+1}. \tag{24}$$

In the present case one can start with

$$B_0^\pm = \mp \frac{1}{\sqrt{2}} \left(\sqrt{f} \frac{d}{d\beta} \sqrt{f} \right) + \frac{1}{\sqrt{2}} \left(c_0 \beta + \bar{c}_0 \frac{1}{\beta} \right), \tag{25}$$

where the second term resembles the superpotential occurring in the case of the 3-dimensional harmonic oscillator, reported in Ref. [11]. Substituting into Eq. (22) one obtains

$$H_0 = -\frac{1}{2} \left(\sqrt{f} \frac{d}{d\beta} \sqrt{f} \right)^2 + \frac{\beta^2}{2} (c_0^2 - ac_0) + \frac{(\bar{c}_0^2 + \bar{c}_0)}{2\beta^2} + \left(-\frac{1}{2} c_0 + \frac{1}{2} a\bar{c}_0 + c_0 \bar{c}_0 + \varepsilon_0 \right). \tag{26}$$

Comparing this result to Eq. (18) with the effective potential of Eq. (21) and equating powers of β one gets the following:

$$c_0 = \frac{1}{2} \left(a \pm \sqrt{a^2 + 8P_1} \right), \tag{27}$$

$$\bar{c}_0 = \frac{1}{2} \left(-1 \pm \sqrt{9 + 4\Lambda + 8\beta_0^4} \right), \quad \text{and} \tag{28}$$

$$\varepsilon_0 = \frac{1}{2} c_0 - \frac{1}{2} a\bar{c}_0 - c_0 \bar{c}_0 + \frac{5}{2} (1 - \delta - \lambda) a + 4a + a\Lambda, \tag{29}$$

where

$$P_1 = 1 + a^2 \left[\frac{5}{2} (1 - \delta - \lambda) + 2 \left(\frac{1}{2} - \delta \right) \left(\frac{1}{2} - \lambda \right) + 3 + \frac{\Lambda}{2} \right]. \tag{30}$$

Upon substitution of c_0 and $\bar{c}_0$ from Eqs. (27) and (28), Eq. (29) leads to

$$\varepsilon_0 = 7a \left(\frac{29}{4} - \frac{5}{2} (\delta + \lambda) + \Lambda \right) \pm \frac{1}{2} \sqrt{a^2 + 8P_1} \mp \frac{a}{2} \sqrt{9 + 4\Lambda + 8\beta_0^4} + \sigma \frac{1}{4} \sqrt{(a^2 + 8P_1)(9 + 4\Lambda + 8\beta_0^4)}, \tag{31}$$

where $\pm$ refers to the sign in Eq. (27), $\mp$ corresponds to the sign selection in Eq. (28), while σ is $+1$ for the selections $(+, -)$ and $(-, +)$ in Eqs. (27) and (28) respectively, while it is -1 for the selections $(+, +)$ and $(-, -)$.

The limiting case of $a = 0$, in which $P_1 = 1$, can be used for reducing the number of possibilities. Since ε_0 should be increasing with Λ (members of the ground state band should exhibit energies increasing with the angular momentum L), it turns out that one should have $\sigma = +1$, thus only the choices $(+, -)$ and $(-, +)$ are acceptable.

In Section 7 it will be shown that good behaviour of the wavefunctions at $\beta = 0$ leaves $(+, -)$ as the only choice.

6. Shape invariance

In the next step the hierarchy of Hamiltonians of Eq. (23) should be considered, with

$$B_i^\pm = \mp \frac{1}{\sqrt{2}} \left(\sqrt{f} \frac{d}{d\beta} \sqrt{f} \right) + \frac{1}{\sqrt{2}} \left(c_i \beta + \frac{\bar{c}_i}{\beta} \right). \tag{32}$$

Substituting these expressions in the shape invariance condition of Eq. (24) and equating powers of β leads to the following results:

$$c_i^2 + c_i a = c_{i+1}^2 - c_{i+1} a, \tag{33}$$
$$\bar{c}_i^2 - \bar{c}_i = \bar{c}_{i+1}^2 + \bar{c}_{i+1}, \tag{34}$$

and

$$2\varepsilon_{i+1} = c_i + c_{i+1} - a\bar{c}_i - a\bar{c}_{i+1} + 2c_i \bar{c}_i - 2c_{i+1} \bar{c}_{i+1}. \tag{35}$$

Keeping from the first two of these only the solutions $c_{i+1} = c_i + a$ (leading to $c_i = c_0 + ia$) and $\bar{c}_{i+1} = \bar{c}_i - 1$ (leading to $\bar{c}_i = \bar{c}_0 - i$), in accordance with the results obtained for the 3-dimensional harmonic oscillator [11], we get

$$\varepsilon_i = 2 \left[c_0 - a\bar{c}_0 + a(2i - 1) \right]. \tag{36}$$

One then easily finds for the energy

$$\epsilon_\nu = \sum_{i=0}^{\nu} \varepsilon_i = \epsilon_0 + 2\nu (c_0 - a\bar{c}_0) + 2a\nu^2. \tag{37}$$

The ground state band is obtained for $\nu = 0$, while the quasi-β_1 band corresponds to $\nu = 1$.

Eq. (37) only provides a formal solution to the bound-state energy spectrum. The range of ν values is actually determined by the existence of corresponding physically acceptable wavefunctions, to be discussed in the next section.

7. Wave functions

To be physically acceptable, the bound-state wavefunctions should satisfy two conditions [11]:

(i) As in conventional (constant-mass) quantum mechanics, they should be square integrable on the interval of definition of u_{eff}, i.e.,

$$\int_0^\infty d\beta \left| R_\nu(\beta) \right|^2 < \infty. \tag{38}$$

(ii) Furthermore, they should ensure the Hermiticity of H. For such a purpose, it is enough to impose that the operator $\sqrt{f}(d/d\beta)\sqrt{f}$ be Hermitian, which amounts to the restriction

$$\left| R_\nu(\beta) \right|^2 f(\beta) \to 0 \quad \text{for } \beta \to 0 \quad \text{and} \quad \beta \to \infty, \tag{39}$$

or, equivalently,

$$\left| R_\nu(\beta) \right|^2 \to 0 \quad \text{for } \beta \to 0 \quad \text{and} \quad \left| R_\nu(\beta) \right|^2 \beta^2 \to 0 \quad \text{for } \beta \to \infty. \tag{40}$$

As condition (40) is more stringent than condition (38), we should only be concerned with the former.

The ground state wave function can be determined from the differential equation [11]

$$B_0^-(\beta; c_0, \bar{c}_0) R_0(\beta; c_0, \bar{c}_0) = 0, \tag{41}$$

where B_0^- is given by Eq. (25). Trying the solution

$$R_0 = C_0 \beta^n f^{\bar{n}}, \tag{42}$$

where C_0 is a normalization constant, the powers of β lead to the conditions

$$n = -\bar{c}_0, \tag{43}$$

and

$$\bar{n} = \frac{a\bar{c}_0 - c_0 - a}{2a}. \tag{44}$$

For $\beta \to 0$, the function $|R_0(\beta)|^2$ behaves as β^{2n}. Condition (40) imposes that $n > 0$, i.e. $\bar{c}_0 < 0$. From Eq. (28) it is clear that this is guaranteed if the minus sign is retained in it. From the discussion given at the end of Section 5 it is clear that the only possibility remaining for the signs in Eqs. (27) and (28) is the $(+, -)$ one. One can easily see that this choice guarantees $\bar{n} < 0$.

For $\beta \to \infty$, $|R_0(\beta)|^2 \beta^2$ behaves as $\beta^{-2c_0/a}$, since in this case $f \approx a\beta^2$. Condition (40) therefore imposes that $c_0 > 0$. This restriction is already satisfied, since we have been led in the previous paragraph to keep the upper sign choice in (27).

Wave functions of excited states can then be obtained from the recursion relation [11]

$$R_{\nu+1}(\beta; c_0, \bar{c}_0) = \frac{1}{\sqrt{\epsilon_{\nu+1}(c_0, \bar{c}_0) - \epsilon_0(c_0, \bar{c}_0)}} B^+(\beta; c_0, \bar{c}_0) R_\nu(\beta; c_1, \bar{c}_1), \tag{45}$$

where the different coefficients appearing in the last term should be noticed. From the recursion relation one obtains

$$R_1(\beta; c_0, \bar{c}_0) = \frac{C_1}{2^{1/2}\sqrt{\epsilon_1 - \epsilon_0}} \left[(2c_0 + a)\beta + \frac{2\bar{c}_0 - 1}{\beta} \right] \beta^{n+1} f^{\bar{n}-1}, \tag{46}$$

$$R_2(\beta; c_0, \bar{c}_0) = \frac{C_2}{2^{3/2}\sqrt{(\epsilon_2 - \epsilon_0)(\epsilon_1 - \epsilon_0)}}$$

$$\times \left[(2c_0 + 3a)(2c_0 + a)\beta^2 + (2\bar{c}_0 - 3)(2\bar{c}_0 - 1)\frac{1}{\beta^2} + 2(2c_0 + 3a)(2\bar{c}_0 - 3) \right] \beta^{n+2} f^{\bar{n}-2}, \tag{47}$$

where C_1, C_2 are normalization constants.

From the above it is clear that wave functions of the states belonging to the ground state band are obtained from Eq. (12), substituting in it Eqs. (17) and (42), while wave functions of the states belonging to the quasi-β_1 band are obtained from Eq. (12), substituting in it Eqs. (17) and (46).

It is easy to see that the conditions imposed above in order to guarantee the physically acceptable behaviour of the ground state wavefunctions, also guarantee the physically acceptable behaviour of the wavefunctions for excited states. In particular:

(i) In order to examine the behaviour at $\beta \to 0$, it suffices to examine in R_ν the behaviour of the polynomial term containing the lowest power of β (β^{-1} in R_1, β^{-2} in R_2). In both cases the function $|R_\nu(\beta)|^2$ behaves as β^{2n}, i.e. it exhibits the same behaviour as $|R_0(\beta)|^2$.
(ii) In order to examine the behaviour at $\beta \to \infty$, it suffices to examine in R_ν the behaviour of the polynomial term containing the highest power of β (β in R_1, β^2 in R_2). In both cases the function $|R_\nu(\beta)|^2 \beta^2$ behaves as $\beta^{-2c_0/a}$, i.e. it exhibits the same behaviour as $|R_0(\beta)|^2 \beta^2$.

Therefore the wavefunctions of the excited states given above are forced to exhibit physically acceptable behaviour by the same conditions which guarantee the physically acceptable behaviour of the ground state wavefunctions.
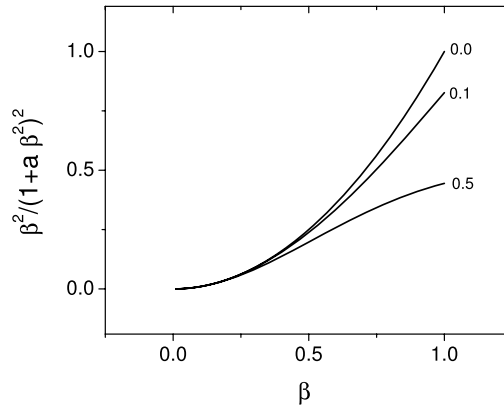
Fig. 1. The function $\beta^2/f^2(\beta) = \beta^2/(1 + a\beta^2)^2$, to which moments of inertia are proportional as seen from Eq. (9), plotted as a function of the nuclear deformation β for different values of the parameter a. See Section 8 for further discussion.

Table 1

Comparison of theoretical predictions of the γ-unstable Bohr Hamiltonian with β-dependent mass (with $\delta = \lambda = 0$) to experimental data [19] of Xe and Ba isotopes. The $R_{4/2} = E(4_1^+)/E(2_1^+)$ ratios, as well as the quasi-β_1 and quasi-γ_1 bandheads, normalized to the 2_1^+ state and labelled by $R_{0/2} = E(0_\beta^+)/E(2_1^+)$ and $R_{2/2} = E(2_\gamma^+)/E(2_1^+)$ respectively, are shown. The angular momenta of the highest levels of the ground state, quasi-β_1 and quasi-γ_1 bands included in the rms fit are labelled by L_g, L_β, and L_γ respectively, while n indicates the total number of levels involved in the fit and σ is the quality measure of Eq. (48). The theoretical predictions are obtained from the formulae mentioned below Eq. (48). See Section 8 for further discussion.

Nucleus	$R_{4/2}$ exp	$R_{4/2}$ th	$R_{0/2}$ exp	$R_{0/2}$ th	$R_{2/2}$ exp	$R_{2/2}$ th	β_0	a	L_g	L_β	L_γ	n	σ
^{118}Xe	2.40	2.32	2.5	2.6	2.8	2.3	1.27	0.103	16	4	10	19	0.319
^{120}Xe	2.47	2.36	2.8	3.4	2.7	2.4	1.51	0.063	26	4	9	23	0.524
^{122}Xe	2.50	2.40	3.5	3.3	2.5	2.4	1.57	0.096	16	0	9	16	0.638
^{124}Xe	2.48	2.36	3.6	3.5	2.4	2.4	1.55	0.051	20	2	11	21	0.554
^{126}Xe	2.42	2.33	3.4	3.1	2.3	2.3	1.42	0.064	12	4	9	16	0.584
^{128}Xe	2.33	2.27	3.6	3.5	2.2	2.3	1.42	0.000	10	2	7	12	0.431
^{130}Xe	2.25	2.21	3.3	3.1	2.1	2.2	1.27	0.000	14	0	5	11	0.347
^{132}Xe	2.16	2.00	2.8	2.0	1.9	2.0	0.00	0.000	6	0	5	7	0.467
^{134}Xe	2.04	2.00	1.9	2.0	1.9	2.0	0.00	0.000	6	0	5	7	0.685
^{130}Ba	2.52	2.42	3.3	3.2	2.5	2.4	1.60	0.118	12	0	6	11	0.352
^{132}Ba	2.43	2.29	3.2	2.8	2.2	2.3	1.29	0.059	14	0	8	14	0.619
^{134}Ba	2.32	2.16	2.9	2.7	1.9	2.2	1.12	0.000	8	0	4	7	0.332
^{136}Ba	2.28	2.00	1.9	2.0	1.9	2.0	0.00	0.000	6	0	2	4	0.250

8. Numerical results

From Eq. (9) it is clear that in the present case the moments of inertia are not proportional to $\beta^2 \sin^2(\gamma - 2\pi k/3)$ but to $(\beta^2/f^2(\beta)) \sin^2(\gamma - 2\pi k/3)$. The function $\beta^2/f^2(\beta)$ is shown in Fig. 1 for different values of the parameter a. It is clear that the increase of the moment of inertia is slowed down by the function $f(\beta)$, as it is expected as nuclear deformation sets in [4].

As a first testground of the present method we have used the Xe isotopes shown in Table 1. They have been chosen because:

(i) They are known to lie in a γ-unstable region [12].
(ii) At least the bandheads of the quasi-β_1 and quasi-γ_1 bands are known experimentally.
(iii) They extend from the borders of the neutron shell (^{134}Xe$_{80}$ is just below the $N = 82$ shell closure) to the midshell (^{120}Xe$_{66}$) and even beyond, exhibiting increasing collectivity (increasing $R_{4/2} = E(4_1^+)/E(2_1^+)$ ratios) from the border to the mishell.

For evaluating the rms fits performed, the quality measure

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(E_i(\exp) - E_i(th))^2}{(n-1)E(2_1^+)^2}} \tag{48}$$

has been used. The theoretical predictions for the levels of the ground state band are obtained from Eq. (31) (in which all terms with double signs are taken with positive signs, as explained at the end of Section 5), while the levels of the quasi-β_1 band are obtained from Eq. (37) for $\nu = 1$. The levels of the quasi-γ_1 band are obtained through their degeneracies to members of the ground state band, mentioned below Eq. (14).

Moving from the border of the neutron shell to the midshell, the following remarks apply

(i) ^{134}Xe and ^{132}Xe are almost pure vibrators. Therefore no need for deformation dependence of the mass exists, the least square fitting leading to $a = 0$. Furthermore, no β_0 term is needed in the potential, the fitting therefore leading to $\beta_0 = 0$, i.e., to pure harmonic behaviour.
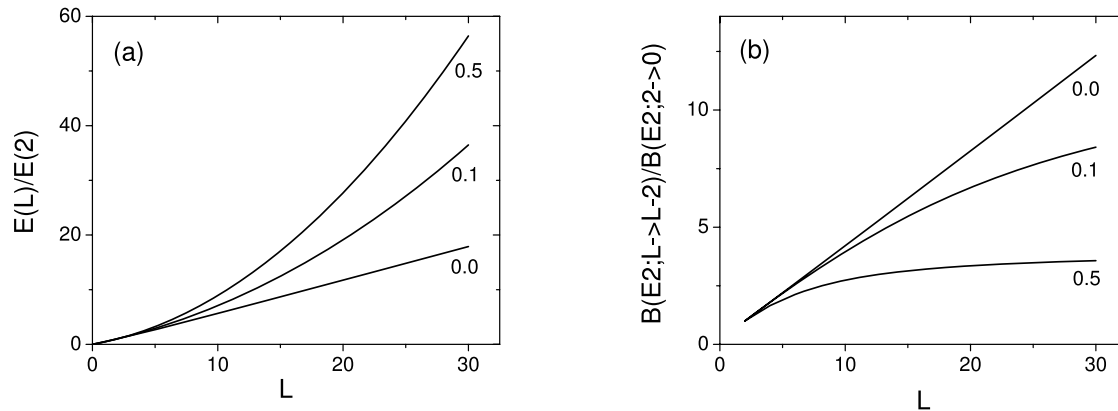
Fig. 2. Energy levels $E(L)$ [normalized to the energy of the first excited state, $E(2)$] and $B(E2; L \rightarrow L - 2)$ transition rates [normalized to the transition from the first excited state to the ground state, $B(E2; 2 \rightarrow 0)$] are shown for the ground state band as functions of the angular momentum L for $\beta_0 = 1$ and varying values of a (0.0, 0.1, 0.5). See Section 8 for further discussion.

Table 2
Experimental $B(E2; L \rightarrow L - 2)$ transition rates [normalized to the transition from the first excited state to the ground state, $B(E2; 2 \rightarrow 0)$] within the ground state band of ^{128}Xe [19], compared to theoretical predictions using the parameters of Table 1. See Section 8 for further discussion.

L	exp	th
4	1.468 ± 0.201	1.632
6	1.940 ± 0.275	2.196
8	2.388 ± 0.398	2.751
10	2.736 ± 1.138	3.310

(ii) In the next two isotopes (^{130}Xe and ^{128}Xe) the need to depart from the pure harmonic oscillator becomes clear, the fitting leading therefore to nonzero β_0 values. However, there is still no need of dependence of the mass on the deformation, the fitting still leading to $a = 0$.

(iii) Beyond ^{126}Xe both the β_0 term in the potential and the deformation dependence of the mass become necessary, leading to nonzero values of both β_0 and a.

Exactly the same behaviour is seen in the Ba isotopes, also known to lie in a γ-unstable region [12] and shown in Table 1.

The results shown in Table 1 have been obtained for $\delta = \lambda = 0$. One can easily verify that different choices for δ and λ lead to a renormalization of the parameter values a and β_0, the predicted energy levels remaining practically the same.

In addition to energy spectra, $B(E2)$ transition rates should be calculated and compared to experiment. The details of this task are deferred to a longer publication. However, basic qualitative features can be seen in Fig. 2, where the systematic behaviour of energy ratios and $B(E2)$ ratios within the ground state band are shown. We remark that an increase of the a parameter leads to a more rapid increase of energies (normalized to the energy of the first excited state) within the ground state band as a function of the angular momentum L, while in parallel it slows down the increase of $B(E2)$s (normalized to the transition from the first excited state to the ground state) as a function of L.

Among the nuclei of Table 1, the only one exhibiting experimentally known increasing $B(E2)$s within the ground state band is ^{128}Xe. The theoretical predictions (using the parameters of Table 1, obtained by fitting the energy levels alone) fall withing the experimental error bars, as seen in Table 2.

The present results suggest that dependence of the mass on deformation becomes necessary as deformation increases. It is therefore desirable to provide a similar solution of the Bohr Hamiltonian applicable to axially symmetric well deformed nuclei. Work in this direction is in progress.

9. Conclusion

Motivated by the existence in the geometrical limit of the O(6) limiting symmetry and of the U(5)–O(6) transition region of the Interacting Boson Model of extra terms of the form $\beta^2 \pi^2$ (where β is the nuclear deformation) in addition to the kinetic energy term π^2, as well as by the existence of more complicated additional terms in the SU(3) limiting symmetry and in the U(5)–SU(3) and SU(3)–O(6) transition regions, we have modified the Bohr Hamiltonian describing the collective motion of atomic nuclei by allowing the mass to depend on the nuclear deformation. Using techniques of supersymmetric quantum mechanics we have obtained exact analytical expressions for spectra and wave functions for the case of the γ-unstable Davidson potential. A first numerical application in the Xe–Ba region gives encouraging results. Detailed comparisons to experiment, including $B(E2)$ transition rates, as well as extension of the method to deformed axial nuclei (with $\gamma \approx 0$) are deferred to a longer publication.

Acknowledgements

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