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Effectivity and Density in Domains: A Survey

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Abstract

This article surveys the main results on effectivity and totality in domain theory and its applications. A more abstract and informative proof of Normann's generalized density theorem for total functionals of finite type over the reals is presented.

1 Introduction

Domain Theory, introduced by D. Scott [22,23] as a semantics for data types and functional programming languages, has attracted many researchers in Mathematics and Computer Science because of its rich structural properties and its numerous connections with other research disciplines like e.g. recursion theory, topology, lattice theory, category theory, and type theory. This paper aims to survey classical and recent results in set-theoretic domain theory focussing on aspects of effectivity and totality. No attempt is made to bring this work into the context of synthetic or axiomatic domain theory.

We will work with a rather special class of domains, namely bounded complete countably based algebraic dcpos with least element, often called *Scott domains*. In presence of a countable basis of compact elements there is a natural notion of effectivity or computability induced by an enumeration of the compacts. It is interesting to compare this notion of computability imported by ordinary recursion theory with definability in an appropriate functional programming language. A famous theorem of Plotkin [20] says that when enriching the programming language PCF with some parallel features both notions of computability coincide.

A related theorem, recently proved by Normann [17], brings totality into play, a notion which is of inherent interest in Computer Science since it is the denotational counterpart to termination. Normann showed that pure PCF (without parallel features) suffices to define all total computable functionals

[1] Thanks to Pino Rosolini for his patience

of finite type over the integers. A crucial property of the total objects of a domain is their (topological) density. We will discuss this property and its counterpart, co-density, in some detail, and state the main results on density as well as its applications.

Another recent result of Normann's is concerned with a representation of the reals by the domain of nestings of closed intervals with rational endpoints. Normann [16] proves that the total continuous functionals of finite types over the reals are dense in the partial functionals. This solves a problem for which the abstract machinery developed so far did not work. We will give a more abstract and informative proof of Normann's result.

In order to fix notation we give below some definitions concerning the basics of domains. For a thorough introduction into domain theory we recommend [8].

We call a partially orderd set $(D, \sqsubseteq)$ a *Scott-domain*, *domain* for short, if it is

1. *directed complete*, i.e. every directed set $A \subseteq D$ has a least upper bound $\bigsqcup A \in D$ (A is *directed* if $A \neq \emptyset$ and $\forall x, y \in A \exists z \in A(x, y \sqsubseteq z)$),
2. *algebraic*, i.e. for every $x \in D$ the set $\{x_0 \in D \colon x_0 \text{ compact and } x_0 \sqsubseteq x\}$ is directed and has x as its least upper bound ($x_0 \in D$ is *compact* if for every directed set $A \subseteq D$ such that $x \sqsubseteq \bigsqcup A$ we have $x_0 \sqsubseteq y$ for some $y \in A$),
3. *countably based*, i.e. the set of compact elements is countable,
4. *bounded complete*, i.e. every nonempty bounded subset of D has a least upper bound in D,
5. equipped with a *least element*, usually denoted $\perp$.

$(D, \sqsubseteq)$ is called a *quasi-domain* if 1.-4. are satisfied, i.e. there need not be a least element. Hence every domain is a quasi-domain, but not vice versa. For example every countable set S, partially ordered by the equality relation, is a quasi-domain. By adding a new, least, element $\perp$ it becomes a so-called 'flat' domain $S^{\perp} := S \cup \{\perp\}$.

For convenience we will also assume that all quasi-domains are *coherent*, which means that a nonempty subset is bounded whenever all its two element subsets are bounded. Although all results presented in this paper also hold without this assumptions, many notions have an easier definition and some proofs become less clumsy when coherency is assumed.

The set of compact elements of a quasi-domain D is denoted by D_0. The *Scott-topology* on a quasi-domain D is generated by the basic open sets $\{x \in D \colon x \sqsupseteq x_0\}$, where $x_0 \in D_0$. For $x, y \in D$ we define *binary consistency* by

$$x \uparrow y :\Leftrightarrow \{x, y\} \text{ is bounded (in } D).$$

It is easy to see that $x \uparrow y$ holds iff x and y are topologically inseparable, i.e. x and y do not have disjoint neighbourhoods.

Products. If D and E are quasi-domains then $D \times E$ with the pointwise ordering is a quasi-domain, which is a domain if D and E happen to be domains. The Scott-topology on $D \times E$ is the product topology. Clearly $(D \times E)_0 = D_0 \times E_0$.

Function space. A function $f: D \to E$ between quasi-domains D, E is continuous w.r.t. the Scott-topology iff it is monotone and preserves suprema of directed sets. Hence f is uniquely determined by its values on compact arguments. If D is a quasi-domain and E is a domain then the set of continuous functions, denoted $D \to E$, is again a domain under the pointwise order. The Scott-topology on $D \to E$ coincides with the pointwise topology and also with the compact-open topology. When writing $D \to E$ it is always understood that D is a quasi-domain and E is a domain (exception: in '$f: D \to E$' E may be a quasi-domain as well).

In order to describe the compacts of $D \to E$ we consider a finite set $\mathcal{G} = \{(x_i, y_i) \mid i \in I\} \subseteq D_0 \times E_0$ satisfying the following consistency condition:

$$\forall i, j \in I \, (\, x_i \uparrow x_j \Rightarrow y_i \uparrow y_j \,).$$

Define $\chi_{\mathcal{G}}: D \to E$ by $\chi_{\mathcal{G}}(x) := \bigsqcup\{y_i \mid i \in I, \, x_i \sqsubseteq x\}$. It is easy to see that $(D \to E)_0$ consists precisely of such functions $\chi_{\mathcal{G}}$.

The interest of computer scientists in domain is mainly due to the following folklore theorems.

Theorem 1.1 *Every continuous function $f: D \to D$ has a least fixed point depending continuously on f.*

Theorem 1.2 *The category of domains and continuous functions is cartesian closed. $D \times E$ and $D \to E$ are the categorical product and exponential respectively.*

The cartesian closed subcategory generated by a domain D is usually called the hierarchy of *partial continuous functionals of finite types* over D. The objects of this category are a family of domains D_ρ, where

$$D_0 := D, \quad D_{\rho \times \sigma} := D_\rho \times D_\sigma, \quad D_{\rho \to \sigma} := D_\rho \to D_\sigma.$$

In this paper we will be mainly interested in the cases $D = \mathbf{N}^\perp$ and $D = R$, where the latter is intended to model the 'partial real numbers'. R is the *ideal completion* of the partial order

$$I_{\mathbf{Q}} := \{[a, b] \mid a \in \{-\infty\} \cup \mathbf{Q}, \, b \in \mathbf{Q} \cup \{+\infty\}, \, a \leq b\},$$

where $\mathbf{Q}$ is the set of rational numbers. The ordering on $I_{\mathbf{Q}}$ corresponds to reverse inclusion of closed intervals, i.e. $[a, b] \leq [a', b']$ iff $a \leq a'$ and $b' \leq b$. The elements of R are downward closed directed subsets $A \subseteq I_{\mathbf{Q}}$ (ideals). The ordering on R is set inclusion. In ideal $A \in R$ which is 'converging', i.e. $\delta(A) := \inf\{b - a \mid [a, b] \in A\} = 0$ in a natural way represents the real number $r := \sup\{a \mid [a, b] \in A\} = \inf\{b \mid [a, b] \in A\}$.

By the theorems above the partial continuous functionals over $\mathbf{N}^\perp$ form a model for the functional programming language PCF which however is not fully abstract as has been shown by Plotkin [20].

If we consider as morphisms between domains not continuous functions, but embedding projection pairs, we get another interesting category which can be used to solve 'domain equations' for constructing useful date types, even paradoxical ones like $D = \mathbf{N}^{\perp} + (D \to D)$ (Scott's D_∞) yielding a model of type free lambda-calculus. Here '+' denotes the separated sum operation forming the disjoint sum of two domains and adding a new least element.

Theorem 1.3 *The category of domains with embedding projection pairs has an initial object and direct colimits. The cartesian product, separated sum, and function space constructions define continuous functors in this category. Hence domain equations built up from these constructions always have a least solution.*

2 Effectivity

An *effective quasi-domain* is a quasi-domain $(D, \sqsubseteq)$ together with a numbering $\nu_0 : \mathbf{N} \to D$, called *effectivation*, such that

1. the sets $\{(n, m) \mid \nu_0 n \sqsubseteq \nu_0 m\}$, and $\{(n, m) \mid \nu_0 n \sqcup \nu_0 m \text{ exists}\}$ are decidable,
2. there is a recursive function $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that $\nu_0 f(n, m) = \nu_0 n \sqcup \nu_0 m$ whenever the supremum exists.

An element $x \in D$ is *computable* iff the set $\{n \mid \nu_0 n \sqsubseteq x\}$ is recursively enumerable. We let D_{comp} denote the set of computable elements of D.

In view of the representation of compacts in $D \to E$ by a finite set of pairs of compacts, as shown in the introduction, it is obvious how to construct effectivations of $D \times E$ and $D \to E$ from effectivations of D and E.

Still, when writing $D \to E$ it is understood that E is a domain whereas D only needs to be a quasi-domain. Nevertheless in the following we will say in such a situation 'let D and E be effective domains' for sake of readability.

Given an effective quasi-domain $(D, \sqsubseteq, \nu_0)$ one constructs by standard techniques of elementary recursion theory a numbering $\nu : \mathbf{N} \to D_{\text{comp}}$, called *principal constructivation* such that

1. the set $\{(n, m) \mid \nu_0 n \sqsubseteq \nu m\}$ is recursively enumerable,
2. there is a recursive function $g : \mathbf{N} \to \mathbf{N}$ such that $\nu_0 n = \nu g(n)$ for all n,
3. For any other numbering $\nu' : \mathbf{N} \to D_{\text{comp}}$ satisfying 1. and 2. there is a recursive function $h : \mathbf{N} \to \mathbf{N}$ such that $\nu' n = \nu h(n)$ for all n.

As an example of an effective domain consider the set of partial functions on the natural numbers ordered by inclusion of graphs. The compact elements are the finite functions which can be numbered in an obvious way. The computable elements are precisely the partial recursive functions. As principal constructivation we may take the usual Kleene brackets $\{\cdot\}$. Another interesting example is provided by the computable and converging elements in the domain R of partial reals, which correspond precisely to the recursive reals.

The partial continuous functionals of finite type over an effective domain D form a hierarchy of effective domains. In the case $D = \mathbf{N}^{\perp}$ this gives rise to several rival notions of recursiveness for these functionals. 1. Domain theoretic computability, 2. PCF-definability, 3. Computability in the sense of Kleene's Schemata (S1-S9) [10]. The following theorem is due to Platek [19] (first part) and Plotkin [20] (second part).

Theorem 2.1 *On the partial continuous functionals PCF-definability and (S1-S9) computability coincide, but are weaker than domain theoretic computability.*

For example the parallel or, POR, is compact and hence domain computable, but not PCF definable. To remedy this one can either try to restrict the continuous functionals to some 'sequential' fragment, or extend PCF. Plotkin [20] showed how to do the latter:

Theorem 2.2 *In PCF+POR every compact functional is definable and hence the partial continuous functionals form a fully abstract model for PCF+POR. Adding further the parallel existential quantifier E yields full domain theoretic computability.*

Problem. Does PCF (or PCF+POR, or PCF+POR+E) get weaker if the fixed point operators are replaced by minimization?

Remark: Kleene [10] has shown that on the full set-theoretic hierarchy of functionals of finite types (S1-S9) computability gets weaker if S9 is replaced by minimization. However it seems that his proof does not carry over to our situation, since it essentially uses discontinuous functionals.

Next we consider two theorems from elementary recursion theory, generalized to domain theory by Ershov [7]. They establish a surprising connection between recursion theory and topology.

Theorem 2.3 (Generalized Rice-Shapiro Theorem) *Let $U \subseteq D_{comp}$ be such that the set $\{n \mid \nu n \in U\}$ is recursively enumerable. Then U is an open subset of D_{comp} (w.r.t. the relativized Scott-topology).*

This theorem can be proved by either employing a recursively enumerable nonrecursive set, or by using the recursion theorem. If one is interested in constructive meta theory it is important to note that the proof requires Markov's principle.

In order to state an important corollary, we need the notion of an *effective operation* between effective domain D and E. By this we mean a function $f\colon D_{\mathrm{comp}} \to E_{\mathrm{comp}}$ which is 'traced' by some recursive function $\hat{f}\colon \mathbf{N} \to \mathbf{N}$, i.e. $f \circ \nu = \mu \circ \hat{f}$, where ν and μ are the principal constructivations of D and E respectively.

Theorem 2.4 (Generalized Myhill-Shepherdson Theorem) *Every effective operation between effective domains is continuous (w.r.t. the relativized*

Scott-topologies).

Starting with a principal constructivation of $\mathbf{N}^{\perp}$ one may construct in a purely recursion theoretic way a hierarchy of partial effective operations of finite type [2]. The function space simply consists of effective operations, which are partially numbered by Kleene-indices of partial recursive functions tracing them. The Generalized Myhill-Shepherdson Theorem entails:

Theorem 2.5 *The computable partial continuous functionals and the partial effective operations are effectively isomorphic.*

Since effective operations presumably embody the weakest notion of computation on an abstract structure, this theorem says that continuity does not restrict computability. This theorem together with theorem 2.2 entails that PCF+POR+E is complete in the sense that no further constructs can enlarge its computational power.

3 Totality

Most domains D of interest contain a distinguished subset $\overline{D} \subseteq D$ consisting of those elements naturally to be considered as 'total'. For example the total elements in a flat domain $S^{\perp}$ are the elements of S, i.e. $\overline{S^{\perp}} := S$, the total elements in the domain of partial number-theoretic functions are the total functions, and the total elements of the domain R are the converging ideals, i.e. $\overline{R} := \{A \in R \mid \delta(A) = 0\}$.

In all these examples the subset $\overline{D}$ is upwards closed, i.e. if $x \in \overline{D}$ and $x \sqsubseteq y$ then $y \in \overline{D}$. Furthermore binary consistency, $x \uparrow y$, is an equivalence relation on $\overline{D}$ (i.e. transitive). Following Normann [15] we call a domain D with a subset $\overline{D}$ having these properties a *domain with totality*, and call $\overline{D}$ a *totality on* D. We simply call x *total* if $x \in \overline{D}$, provided $\overline{D}$ is clear from the context. Of course this notions also apply to quasi-domains.

The second requirement on a domain with totality is motivated by the fact that $x \uparrow y$ holds iff x and y are topologically inseparable. Regarding open sets as the only observable properties this means that x and y are indistinguishable. Hence one would like to identify x and y, which of course requires $\uparrow$ to be an equivalence relation on $\overline{D}$.

For every total x we let $\mathbf{x} := \{y \in \overline{D} \mid x \uparrow y\}$, the equivalence class of x. Furthermore we let $\mathbf{D} := \overline{D}/\uparrow = \{\mathbf{x} \mid x \in \overline{D}\}$ denote the quotient structure endowed with the quotient topology.

For example the space $\mathbf{R}$ is homeomorphic to the reals. Rational numbers correspond to equivalence classes with 4 elements, since there are 4 ways of approximating a rational number by an ideal of closed rational intervals. Irrational numbers can be approximated in one way only, hence their equivalent classes are singletons.

In [5] and [6] there are numerous examples of interesting topological spaces represented in the form $\mathbf{D}$.

If the underlying domain D effective this also implements a natural notion of computability on $\mathbf{D}$: We call $\mathbf{x} \in \mathbf{D}$ *computable* (*PCF-definable*) if $\mathbf{x}$ contains a computable (PCF-definable) element.

If D and E are domains with totality (D need only be a quasi-domain) then it is natural to define the total elements of $D \to E$ as

$$\overline{D \to E} := \overline{D} \to \overline{E} := \{f \in D \to E : f[\overline{D}] \subseteq \overline{E}\}.$$

The elements of $\overline{D} \to \overline{E}$ are called *total functions*. However in general $\overline{D} \to \overline{E}$ will not be a totality on $D \to E$, since $\uparrow$ need not be transitive on $\overline{D} \to \overline{E}$. Moreover it is natural to consider $f, g \in \overline{D} \to \overline{E}$ as equivalent if $f(x) \uparrow g(x)$ for all $x \in \overline{D}$, but this notion of equivalence will in general not coincide with $\uparrow$ in $\overline{D} \to \overline{E}$. Just consider the example $D = E = \mathbf{N}^{\perp}$ with 0 as the only total element in D.

Obviously a further property of total elements is required. This property is *density*. Note that $\overline{D} \subseteq D$ is dense iff

$$\forall x_0 \in D_0 \, \exists x \in \overline{D} \; x_0 \sqsubseteq x.$$

One immediately checks that if D, E are domains with totality such that $\overline{D}$ is dense (in D), then $\overline{D} \to \overline{E}$ is a totality on $D \to E$. Moreover for $f, g \in \overline{D} \to \overline{E}$ we have $f \uparrow g$ iff $f(x) \uparrow g(x)$ for all $x \in \overline{D}$. The latter amounts to a principle of *extensionality*: Two total functions are identified if they are extensionally equal on total arguments, that is, for $f, g \in \overline{D} \to \overline{E}$ we have $\mathbf{f} = \mathbf{g}$ iff $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ for all $x \in \overline{D}$, where, of course, $\mathbf{f}(\mathbf{x})$ is the equivalence class of $f(x)$.

We are still not satisfied, since, even if $\overline{D}$ and $\overline{E}$ are both dense, $\overline{D} \to \overline{E}$ need not be dense. Consider for example $D := \mathbf{N}^{\perp}$ with $\overline{D} := D$, and $E := \mathbf{N}^{\perp}$ with $\overline{E} := \mathbf{N}$. Both are dense totalities, but $\overline{D} \to \overline{E}$ contains only constant functions and hence is not dense. What is wrong here is the fact that we declared $\perp \in D$ to be total. To exclude this we consider a property of $\overline{D}$ forcing the elements of $\overline{D}$ to be in some sense 'large'.

A set $\overline{D} \subseteq D$ is *co-dense* if for every pair of inconsistent compacts x_0, y_0, i.e. $x_0 \not\uparrow y_0$, there is a total continuous function $t \colon D \to \mathbf{B}^{\perp}$, where $\mathbf{B} := \{\mathtt{tt}, \mathtt{ff}\}$, i.e. $t \in \overline{D} \to \mathbf{B}$, such that $t(x_0) = \mathtt{tt}$ and $t(y_0) = \mathtt{ff}$.

The following abstract density theorem is proved in [2,3].

Theorem 3.1 *Let D and E be domains with totality*

1. *If $\overline{D}$ is co-dense and $\overline{E}$ is dense, then $\overline{D} \to \overline{E}$ is dense.*
2. *If $\overline{D}$ is dense and $\overline{E}$ is co-dense, then $\overline{D} \to \overline{E}$ is co-dense.*

This theorem is a domain-theoretic abstraction of the density theorem of Ershov [7] which in turn has forerunners due to Kleene [9] and Kreisel [11].

Given a domain D with totality $\overline{D}$ we define for every finite type ρ the set $\overline{D}_\rho \subseteq D_\rho$ of total continuous functionals over $\overline{D}$ of type ρ

$$\overline{D}_0 := \overline{D}, \quad \overline{D}_{\rho\to\sigma} := \overline{D}_\rho \to \overline{D}_\sigma, \quad \overline{D}_{\rho\times\sigma} := \overline{D}_\rho \times \overline{D}_\sigma.$$

As an immediate corollary to theorem 3.1 we get:

Theorem 3.2 *Let D be a domain with dense and co-dense totality. Then $\overline{D}_\rho$ is a well-defined dense and co-dense totality on D_ρ for every finite type ρ.*

In particular this holds for $D := \boldsymbol{N}^\perp$ with $\overline{\boldsymbol{N}^\perp} := \boldsymbol{N}$, and in that case the quotients $\boldsymbol{D}_\rho$ are isomorphic with the so-called Kleene-Kreisel functionals [9,11].

The second part of the theorem (which is not so immediate) has been shown by Ershov [7].

The following characterization of co-density shows that it is the weakest possible property ensuring density for the function space [2]: A totality $\overline{D}$ is co-dense iff $\overline{D} \to \overline{E}$ is dense for all dense totalities $\overline{E}$. We also have: $\overline{D}$ is dense iff $\overline{D} \to \overline{E}$ is dense for all co-dense $\overline{E}$.

By the characterization theorem above co-density seems to be an indispensable property of a totality. But unfortunately the totality $\overline{R} \subseteq R$ is *not* co-dense. In fact, for example, the set $\overline{R} \to \mathbf{B} \subseteq R \to \mathbf{B}^\perp$ contains only constant functions and hence is not dense (any non-constant $t \in \overline{R} \to \mathbf{B}$ would divide the reals into two nonempty disjoint open sets). Nevertheless $\overline{R} \to \overline{R}$ is dense in $R \to R$, as can be seen fairly easily. The question, whether density holds for *all* finite types, was was recently answered by Normann [16]:

Theorem 3.3 *The total functionals over the reals, $\overline{R}_\rho$, are dense in R_ρ for all finite types ρ.*

In the following we prove a slightly more abstract form of this theorem which gives explicit information about the way a dense and total sequence in the domain under consideration can be constructed.

Theorem 3.4 *Let D be a domain with nonempty totality such that $(\boldsymbol{N} \to \overline{D}) \to \overline{D}$ is dense in $(\boldsymbol{N} \to D) \to D$. Then the total functionals of finite type over D are dense, i.e. $\overline{D}_\rho$ is dense in D_ρ for all finite types ρ.*

Moreover for every type ρ a dense sequence of elements in $\overline{D}_\rho$ is explicitely definable (i.e. using simply typed lambda terms only) from an element in $\overline{D}$, a dense sequence in $(\boldsymbol{N} \to \overline{D}) \to \overline{D}$, any surjective function $\beta\colon \boldsymbol{N} \to \boldsymbol{N}^2$, division by two with remainder, and zero test.

Theorem 3.3 follows from theorem 3.4, since, as shown in [16], $\overline{R}^n \to \overline{R}$ is dense for all n, and this in turn implies that $\overline{D}$ is nonempty and $(\mathbf{N} \to \overline{D}) \to \overline{D}$ is dense.

Furthermore theorem 3.4 reduces the proof of theorem 3.2 to the task of proving density of $(\mathbf{N} \to \overline{D}) \to \overline{D}$ from density and co-density of $\overline{D}$, which is a special case of theorem 3.1.

We prove theorem 3.4 in several steps.

Let D, E be quasi-domains and f a continuous function from D to E. f is called *dense* if its range, $f[D]$, is dense in E. f is called *separating* if it preserves inconsistencies, i.e. $\forall x, y \in D\, (x \not\uparrow y \Rightarrow f(x) \not\uparrow f(y))$.

Note that for a domain D with totality 1. $\overline{D}$ is dense iff there is a dense $f \in \mathbf{N} \to \overline{D}$. 2. $\overline{D}$ is co-dense iff there is a separating $f \in \overline{D} \to (\mathbf{N} \to \mathbf{B})$.

We define the *Yoneda function* $\mathsf{Y}_D\colon (E \to F) \to ((F \to D) \to (E \to D))$ by $\mathsf{Y}_D(f)(g) := g \circ f$.

Lemma 1 Let $f \in E \to F$.

(a) If f is separating, then $\mathsf{Y}_D(f)$ is dense.

(b) If f is dense, then $\mathsf{Y}_D(f)$ is separating.

Proof. (a) Let f be separating. In order to show that $\mathsf{Y}_D(f)$ is dense we take a compact $h_0 \in E \to D$ and try to find $g \in F \to D$ such that $\mathsf{Y}_D(f)(g) \sqsupseteq h_0$. Since h_0 is compact it is of the form $\chi_{\mathcal{H}}$ for some finite set $\mathcal{H} = \{(y_i, x_i) \mid i \in I\} \subseteq E_0 \times D_0$, i.e. $h_0(y) = \bigsqcup\{x_i \mid i \in I,\ y_i \sqsubseteq y\}$ for all $y \in E$. By algebraicity of F it is easy to find compacts $z_i \sqsubseteq f(y_i)$ for $(i \in I)$, such that $z_i \not\uparrow z_j$ whenever $f(y_i) \not\uparrow f(y_j)$. Now set $\mathcal{G} := \chi_{\{(z_i, x_i) \mid i \in I\}} \subseteq F_0 \times D_0$ and $g := \chi_{\mathcal{G}}$. g is well-defined, since if $z_i \uparrow z_j$ then $f(y_i) \uparrow f(y_j)$ and hence $y_i \uparrow y_j$, since f is separating. Hence, by the consistence property of $\mathcal{H}$, it follows that $x_i \uparrow x_j$. For verifying $\mathsf{Y}_D(f)(g) \sqsupseteq h_0$ we have to show $\mathsf{Y}_D(f)(g)(x_i) \sqsupseteq y_i$ for all $i \in I$. But $\mathsf{Y}_D(f)(g)(x_i) = g(f(x_i)) \sqsupseteq g(z_i) \sqsupseteq y_i$.

(b) Let f be dense. In order to show that $\mathsf{Y}_D(f)$ is separating we take inconsistent $g, g' \in F \to D$ and show that $\mathsf{Y}_D(f)(g)$, $\mathsf{Y}_D(f)(g')$ are inconsistent. For this it suffices to find $y \in E$ such that $\mathsf{Y}_D(f)(g)(y)$, $\mathsf{Y}_D(f)(g')(y)$ are inconsistent. Since $g \not\uparrow g'$, by algebraicity of F and continuity of g, g', there is some compact $z_0 \in Z_0$ such that $g(z_0) \not\uparrow g'(z_0)$. Since $f[E]$ is dense in F there is $y \in E$ such that $z_0 \sqsubseteq f(y)$. One readily verifies that y has the required property. $\square$

Note that any isomorphism between domains is dense and separating. Furthermore the property of being dense and the property of being separating are both preserved by composition $g \circ f$ and pairing $f \times g$, where $(f \times g)(x, y) := (f(x), g(y))$.

Lemma 2 Let D, E be domains and F a quasi-domain, let $\alpha \in F \to (E \to D)$ and $\beta\colon F \to F^2$ both be dense continuous functions, let $\gamma \in (F \to D) \to E$ and $\delta \in E^2 \to E$ both be separating, and let finally $y^* \in E$ be a fixed element. Then for every finite type ρ there are a dense $f_\rho \in F \to D_\rho$ and a separating $g_\rho \in D_\rho \to E$, both explicitely definable from $\alpha, \beta, \gamma, \delta$, and y^*.

Proof. Any D_σ is explicitely isomorphic to some D_ρ, where ρ is generated by the restricted rules: 0, $\rho \to 0$, $\rho_1 \times \rho_2$. Hence it suffices to define f_ρ and g_ρ for types of this special form. We set

$$f_0 := \lambda z.f(z)(y^*), \quad g_0 := \lambda x.g(\lambda z.x)$$

which are clearly dense respectively separating. The remaining cases are taken care of by lemma 1 and the remarks above:

$$f_{\rho\to 0} := \mathsf{Y}_D(g_\rho) \circ \alpha, \quad g_{\rho\to 0} := \gamma \circ \mathsf{Y}_D(f_\rho),$$

$$f_{\rho\ \times\rho_2} := (f_{\rho}\ \times f_{\rho_2}) \circ \beta, \quad g_{\rho\ \times\rho_2} := \delta \circ (g_{\rho}\ \times g_{\rho_2}).$$

□

Proof of theorem 3.4 By the assumptions of theorem 3.4 we are given an element $x^* \in \overline{D}$ and a dense $\alpha \in \mathbf{N} \to ((\mathbf{N} \to \overline{D}) \to \overline{D})$. Furthermore we are allowed to use some surjective function $\beta: \mathbf{N} \to \mathbf{N}^2$ as well as division by two with remainder and zero test. In order to apply lemma 2 we set $F := \mathbf{N}$ and $E := \mathbf{N} \to D$. Since α and β are given it remains to define separating $\gamma \in (\mathbf{N} \to D) \to (\mathbf{N} \to D)$ and $\delta \in (\mathbf{N} \to D)^2 \to (\mathbf{N} \to D)$, and finally some $y^* \in \mathbf{N} \to D$. For γ we take the identity. We define δ by $\delta(h_0, h_1)(2k + i) := h_i(k)$ $(i = 0, 1)$ which is clearly separating. Finally $y^* := \lambda k.x^*$. Now, by lemma 2 we get for every type ρ a dense sequence $f_\rho: \mathbf{N} \to D_\rho$ which is explicitely defined from $\alpha, \beta, \gamma, \delta$, and y^*. Since $\alpha, \beta, \gamma, \delta$, and y^* are total, and totality is preserved by explicit definitions, it follows that f_ρ is total. □

In his paper [16] Normann proves a more general theorem than theorem 3.3 connecting the discrete ($\mathbf{N}$) and the continuous ($\mathbf{R}$) case by admitting certain partial equivalence relations different from the consistency relation $\uparrow$.

In the same spirit a group around D. Scott [1] has recently developed a rather general theory of topological spaces endowed with a partial equivalence relation (equilogical spaces) which might yield a good framework for putting the work presented here into a more general (categorical) context.

Closing this section we state a simple but important application of density [11,21,3]:

Theorem 3.5 (Effective choice principle) *Let $(D_\rho)_\rho$ be the hierarchy of partial continuous functional over the integers. For all types ρ and σ there is a PCF+POR definable total functional of type $(\rho \times \sigma \to 0) \to (\rho \to \sigma)$ computing for every total functional f of type $\rho \times \sigma \to 0$ such that*

$$\forall x \in \overline{D}_\rho \exists y \in \overline{D}_\sigma\ f(x, y) = 0$$

a total functional g of type $\rho \to \sigma$ such that

$$\forall x \in \overline{D}_\rho\ f(x, g(x)) = 0.$$

4 Effectivity and totality

The results discussed in this section may in some sense be viewed as the total versions of the theorems 2.2, 2.4, and 2.5.

In [3] it had been shown that the fan-functional computing a modulus of uniform continuity of a type-2 functional restricted to a compact fan is PCF-definable (in contrast to previous results proving the contrary on the Kleene-Kreisel functionals). It was then conjectured that *every* computable total continuous functional over the integers is PCF-definable, in the sense that its equivalence class of total continuous functionals contains a PCF-definable element. Again it was Normann [17] who proved in 1998 this conjecture:

Theorem 4.1 *Every computable total continuous functional over the integers is PCF-definable.*

Moreover for every type ρ there is a PCF-computable functional of type $(0 \to 0) \to \rho$ computing from every enumeration of the compact approximations of total functional f of type ρ (where this enumeration is coded as sequence of integers) a total functional $\hat{f} \sqsubseteq f$

The proof uses the density theorem in an essential way.

In the proof of theorem 4.1 the fixed point operator of PCF is used in such a way that it seems not to be substitutable by minimization:

Problem. Does theorem 4.1 still hold if in PCF the fixed point operators are replaced by minimization?

The generalized Myhill-Shepherdson Theorem stating that every effective operation between effective domains is continuous has various total analogues [3] which may be viewed as a generalization of the *Kreisel-Lacombe-Shoenfield Theorem* [12]. Here, we only present one of them.

A subset $\overline{D}$ of a domain D is *effectively dense* iff there is a computable dense sequence $g: \mathbf{N} \to \overline{D}$.

An element y of a domain E is called *almost maximal* if it cannot be extended in two inconsistent ways, i.e. $\forall y', y''(y \sqsubseteq y', y'' \Rightarrow y' \uparrow y'')$. Using the axiom of choice this can be shown to be equivalent with the property that y has precisely one maximal extension (but we will not use this fact). For instance, the elements of a co-dense set are almost maximal. Also all elements of $\overline{R}$ are maximal (although $\overline{R}$ is not a co-dense set).

Theorem 4.2 *Let D, E be effective domains with totality. Assume that $\overline{D}$ is effectively dense and all elements of $\overline{E}$ are almost maximal. Then every effective operation $f: \overline{D} \to \overline{E}$ can be extended to an effective (and by the generalized Myhill-Shepherdson Theorem continuous) operation $f': D \to E$, in the sense that $f(x) \sqsubseteq f'(x)$ for all $x \in \overline{D}$.*

This theorem immediately entails the well-known theorem of Ceitin and Moschovakis saying that every effective operator on the reals is continuous.

Another generalization of the Kreisel-Lacombe-Shoenfield Theorem proved in [3] can be used to prove a total analogue of theorem 2.5:

Theorem 4.3 *The hereditarily effective operations of finite type [24] are effectively isomorphic with the hereditarily computable total functionals over the integers.*

5 Dependent domains and universes

So far we considered only the type constructors $\to$ (function space) and $\times$ (cartesian product). However most of the work described in the previous sections has been extended to dependent products and dependent sums, and also to universe operators in the sense of Martin-Löf type theory [18,14,13,25,4].

Palmgren and Stoltenberg-Hansen [18] developed the notion of a dependent domain and gave a domain interpretation a partial type theory. Kristiansen and Normann [14,13] used a universe of dependent domains with dense totality to represent computations relative to certain noncontinuous functionals like ${}^3E$. Waagbø modified Palmgren's and Stoltenberg-Hansen's work for interpreting (the usual) total type theory using dependent domains with totality. In [4] abstract density theorems for dependent types and universe operators are proved. Time and space does not permit us to go into any further details.

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