Persistence and extinction of a stochastic delay Logistic equation under regime switching

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**A R T I C L E    I N F O**

Article history:
Received 29 September 2011
Received in revised form 13 April 2012
Accepted 14 April 2012

Keywords:
Logistic equation
Markov switching
Delay
Persistence
Extinction

**A B S T R A C T**

An a stochastic delay Logistic equation under regime switching is proposed and studied. Sufficient conditions for extinction, non-persistence in the mean and weak persistence of the solutions are established. The critical value between weak persistence and extinction is obtained.

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1. Introduction

The Logistic model is one of the important models in mathematical ecology. The classical Logistic equation with time delay is

\[
dx(t)/dt = x(t)\left[r - ax(t) - nx(t - \tau)\right]
\] (1)

where \(x(t)\) is the population size at time \(t\); \(\tau\) is a positive constant which represents the time delay; \(r\), \(a\) and \(n\) are positive constant. There is an extensive literature concerned with the properties of system (1) and we here mention [1–4] among many others.

On the other hand, population models are inevitably affected by environmental noises. As we know, there are various types of environmental noise. First of all, let us consider the famous telegraph noise. It has been noticed that [see, e.g. [5]] the carrying capacities and the growth rates are often affected by telegraph noise. For example, the growth rates of some populations in the dry season will be different from those in the rainy season. Moreover, according to the changes in nutrition or food resources, the carrying capacities often vary. Several authors (see e.g. [6–9]) have pointed out that we can model telegraph noise by a continuous-time Markov chain \(\gamma (t)\), \(t \geq 0\) with finite-state space \(S = \{1, \ldots, m\}\). Let the Markov chain \(\gamma (t)\) be generated by \(Q = (q_{ij})\), that is, \(P[\gamma (t + \Delta t) = j|\gamma (t) = i] = \begin{cases} q_{ij}\Delta t + o(\Delta t), & \text{if } j \neq i; \\ 1 + q_{ii}\Delta t + o(\Delta t), & \text{if } j = i, \end{cases}\) where \(q_{ij} \geq 0\) for \(i, j = 1, 2, \ldots, m\) with \(j \neq i\) and \(\sum_{j=1}^{m} q_{ij} = 0\) for \(i = 1, 2, \ldots, m\). Then Eq. (1) becomes

\[
dx(t)/dt = x(t)\left[r(\gamma (t)) - a(\gamma (t))x(t) - n(\gamma (t))x(t - \tau)\right].
\] (2)

The mechanism of Eq. (2) can be explained as follows. Assume that initially, \(\gamma (0) = \kappa \in S\), then (2) obeys \(dx(t)/dt = x(t)\left[r(\kappa) - a(\kappa)x(t) - n(\kappa)x(t - \tau)\right]\) for a random amount of time until the Markov chain \(\gamma (t)\) jumps to another state, say, \(\varsigma \in S\). Then
the model satisfies \( \frac{dx(t)}{dt} = x(t)[r(\zeta) - a(\zeta)x(t) - n(\zeta)x(t - \tau)] \) for a random amount of time until \( \gamma(t) \) jumps to a new state again.

Let us now take a further step by considering the white noise. Recall that \( r(i) \) denotes the growth rate in regime \( i (i \in S) \). We usually estimate it by an error term plus an average value. Frequently, the error term follows a normal distribution. Therefore we can replace \( r(i) \) by \( r(i) + \sigma_1(i) B_1(t) \) (see e.g. [7–10]), where \( B_1(t) \) is a white noise, and \( \sigma_1^2(i) \) represents the intensity of the white noise. Similarly, \( -a(i) \) and \( n(i) \) will become \( -a(i) + \sigma_2(i) B_2(t) \) and \( -n(i) + \sigma_3(i) B_3(t) \) (see e.g. [11]), respectively. Then we obtain the following stochastic delay Logistic system under regime switching:

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t)[r(\gamma(t)) - a(\gamma(t))x(t) - n(\gamma(t))x(t - \tau)]dt \\
&\quad + \sigma_1(\gamma(t))x(t)dB_1(t) + \sigma_2(\gamma(t))x(t)dB_2(t) + \sigma_3(\gamma(t))x(t)dB_3(t)
\end{align*}
\]

where \( B(t) = (B_1(t), B_2(t), B_3(t))^T \) denotes a three-dimensional Brownian motion defined on a complete probability space \( (\Omega, \mathcal{F}, P) \). Assume that the Markov chain \( \gamma(\cdot) \) is independent of \( B(t) \). As a standing hypothesis, we assume moreover in this paper that \( \gamma(\cdot) \) has a unique stationary distribution \( \pi = (\pi_1, \pi_2, \ldots, \pi_m) \) which can be obtained by solving the following linear equation \( \pi Q = 0 \) subject to \( \sum_{i=1}^m \pi_i = 1 \) and \( \pi_i > 0 \), \( i \in S \). Throughout this paper, suppose that \( \min_{i \in S} \sigma(i) > 0 \), \( \min_{i \in S} \sigma(i) > 0 \), \( \theta(i) > 0 \), \( i \in S \). For the sake of convenience, we define the following notions: \( \hat{\nu} = \max_{i \in S} \nu(i), \hat{\nu} = \inf_{i \in S} \nu(i) \).

Since Eq. (3) describes a population system, it is critical to find out when the population goes to extinction and when does not. As far as we know, there are no persistent and extinct results for Eq. (3). The aim of this work is to investigate this problem. We shall show that:

**Theorem 1.** If \( \sum_{i=1}^m \pi_i b(i) < 0 \), then the population \( x(t) \) represented by model (3) goes to extinction a.s. (almost surely), i.e., \( \lim_{t \to +\infty} x(t) = 0 \), where \( b(\gamma) = r(\gamma) - 0.5\sigma_1^2(\gamma) \).

**Theorem 2.** If \( \sum_{i=1}^m \pi_i b(i) = 0 \), then the population is nonpersistent in the mean (see e.g. [12]) a.s., i.e., \( \lim_{t \to +\infty} t^{-1} \int_0^t x(s)ds = 0 \) a.s.

**Theorem 3.** If \( \sum_{i=1}^m \pi_i b(i) > 0 \), then the population is weakly persistent (see e.g. [12]) a.s., i.e., \( \lim_{t \to +\infty} x(t) > 0 \). a.s.

**2. Proofs**

**Theorem 4.** Eq. (3) has a unique and positive solution on \( t \geq -\tau \) a.s. (almost surely) for any initial data \( x(t) : -\tau \leq t \leq 0 \in C([-\tau, 0], R_+) \) and \( \gamma(0) \), where \( R_+ = \langle 0, +\infty \rangle \).

The proof is standard (see e.g. Mao et al. [11]) and hence is omitted.

**Proof of Theorem 1.** Applying generalized Itô’s formula to Eq. (3) gives

\[
\begin{align*}
\frac{d\ln x}{x} &= \frac{dx}{x} - \frac{(dx)^2}{2x^2} = \left[ b(\gamma) - a(\gamma)x - n(\gamma)x(t - \tau) - 0.5\sigma_1^2(\gamma)x^2 - 0.5\sigma_2^2(\gamma)x^2(t - \tau) \right]dt \\
&\quad + \sigma_1(\gamma)x dB_1(t) + \sigma_2(\gamma)x dB_2(t) + \sigma_3(\gamma)x(t - \tau) dB_3(t).
\end{align*}
\]

Integrating both sides from 0 to \( t \), we get

\[
\begin{align*}
\ln x(t) - \ln x(0) &= \int_0^t \left[ b(\gamma(s)) - a(\gamma(s))x(s) - n(\gamma(s))x(s - \tau) \\
&\quad - 0.5\sigma_1^2(\gamma(s))x^2(s) - 0.5\sigma_2^2(\gamma(s))x^2(s - \tau) \right]ds + M_1(t) + M_2(t) + M_3(t),
\end{align*}
\]

where

\[
\begin{align*}
M_1(t) &= \int_0^t \sigma_1(\gamma(s)) dB_1(s), \quad M_2(t) = \int_0^t \sigma_2(\gamma(s))x(s) dB_2(s), \quad M_3(t) = \int_0^t \sigma_3(\gamma(s))x(s - \tau) dB_3(s).
\end{align*}
\]

Note that \( M_1(t) \) is a local martingale, whose quadratic variation is \( \langle M_1(t), M_1(t) \rangle = \int_0^t \sigma_1^2(\gamma(s))ds \leq \sigma_1^2 t \). Making use of the strong law of large numbers for local martingales (see e.g. [6, p. 16]) yields

\[
\lim_{t \to +\infty} M_1(t)/t = 0 \quad a.s.
\]

On the other hand, \( \langle M_2(t), M_2(t) \rangle = \int_0^t \sigma_2^2(\gamma(s))x^2(s)ds, \quad \langle M_3(t), M_3(t) \rangle = \int_0^t \sigma_3^2(\gamma(s))x^2(s - \tau)ds \). In view of the exponential martingale inequality (see e.g. [6, p. 74]), for any positive constants \( T, \alpha \) and \( \beta \), we have

\[
\mathbb{P} \left\{ \sup_{0 \leq s \leq T} \left| \frac{M_i(t)}{2} - \frac{\alpha}{2} \langle M_i(t), M_i(t) \rangle \right| > \beta \right\} \leq e^{-\alpha \beta}, \quad i = 2, 3.
\]
Choose \( T = k, \alpha = 1, \beta = 2 \ln k \), then it follows that
\[
\mathcal{P}\left\{ \sup_{0 \leq t \leq k} \left[ M_i(t) - \frac{1}{2} \langle M_i(t), M_i(t) \rangle \right] > 2 \ln k \right\} \leq 1/k^2, \quad i = 2, 3.
\]

Using the Borel–Cantelli Lemma (see e.g. [6, p. 10]) leads to that for almost all \( \omega \in \Omega \), there is a random integer \( k_0 = k_0(\omega) \) such that for \( k \geq k_0, \sup_{0 \leq t \leq k} \left[ M_i(t) - \frac{1}{2} \langle M_i(t), M_i(t) \rangle \right] \leq 2 \ln k \). Then
\[
M_2(t) \leq 2 \ln k + \frac{1}{2} \langle M_2(t), M_2(t) \rangle = 2 \ln(k + 0.5 \int_0^t \sigma_2^2(\gamma(s))x^2(s)ds),
\]
\[
M_3(t) \leq 2 \ln k + \frac{1}{2} \langle M_3(t), M_3(t) \rangle = 2 \ln(k + 0.5 \int_0^t \sigma_3^2(\gamma(s))x^2(s - \tau)ds)
\]
for all \( 0 \leq t \leq k, \ k \geq k_0 \) a.s. Substituting these inequalities into (4) gives
\[
\ln x(t) - \ln x(0) \leq \int_0^t b(\gamma(s))ds -\int_0^t a(\gamma(s))x(s)ds - \int_0^t n(\gamma(s))x(s - \tau)ds + M_1(t) + 4 \ln k
\]
\[
\leq \int_0^t b(\gamma(s))ds + M_1(t) + 4 \ln k \tag{7}
\]
for all \( 0 \leq t \leq k, \ k \geq k_0 \) almost surely. In other words, we have shown that for \( 0 < k - 1 \leq t \leq k \),
\[
t^{-1}[\ln x(t) - \ln x(0)] \leq t^{-1} \int_0^t b(\gamma(s))ds + 4 \ln k/t + M_1(t)/t \leq t^{-1} \int_0^t b(\gamma(s))ds + 4 \ln k/t - k - 1 + M_1(t)/t.
\]

Making use of (5) and the ergodicity of \( \gamma(\cdot) \), we have
\[
\limsup_{t \to +\infty} t^{-1} \ln x(t) \leq \limsup_{t \to +\infty} t^{-1} \int_0^t b(\gamma(s))ds = \sum_{i=1}^m \pi_i b(i).
\]
That is to say, if \( \sum_{i=1}^m \pi_i b(i) < 0 \), then \( \lim_{t \to +\infty} x(t) = 0 \). \( \square \)

**Proof of Theorem 2.** For given \( \varepsilon > 0 \), there exists a constant \( T_1 = T_1(\varepsilon) \) such that
\[
t^{-1} \int_0^t b(\gamma(s))ds \leq \limsup_{t \to +\infty} t^{-1} \int_0^t b(\gamma(s))ds + \varepsilon/2 = \sum_{i=1}^m \pi_i b(i) + \varepsilon/2 = \varepsilon/2, \quad t \geq T_1.
\]

Substituting this inequality into (7), one can see that
\[
\ln x(t) - \ln x(0) \leq \int_0^t b(\gamma(s))ds - \int_0^t a(\gamma(s))x(s)ds + 4 \ln k + M_1(t) \leq \varepsilon t/2 - \tilde{a} \int_0^t x(s)ds + 4 \ln k + M_1(t)
\]
for all \( T_1 \leq t \leq k, \ k \geq k_0 \) almost surely. Note that there exists a \( T > T_1 \) such that for all \( T \leq k - 1 \leq t \leq k \) and \( k \geq k_0 \) we have \( (4 \ln k)/t \leq \varepsilon/4 \) and \( M_1(t)/t \leq \varepsilon/4 \). In other words, we have proved that \( \ln x(t) - \ln x(0) \leq \varepsilon t - \tilde{a} \int_0^t x(s)ds \) for sufficiently large \( T \). Let \( g(t) = \int_0^t x(s)ds \), then we obtain
\[
\ln(dg/dt) < \varepsilon t - \tilde{a} g(t) + \ln x(0), \quad t \geq T.
\]
That is to say \( \tilde{a}^{-1}[e^{\tilde{a}g(t)} - e^{\tilde{a}g(T)}] < x(0) e^{-1}[e^{\varepsilon t} - e^{\varepsilon T}] \). Rewriting this inequality, we get
\[
e^{\tilde{a}g(T)} < e^{\tilde{a}g(T)} + x(0) \tilde{a}^{-1} e^{\varepsilon t} - x(0) \tilde{a}^{-1} e^{\varepsilon T}.
\]
Taking logarithm of both sides leads to
\[
g(t) < \tilde{a}^{-1} \ln \left\{ x(0) \tilde{a}^{-1} e^{\varepsilon t} + e^{\tilde{a}g(T)} - x(0) \tilde{a}^{-1} e^{\varepsilon T} \right\}.
\]
In other words,
\[
\limsup_{t \to +\infty} t^{-1} \int_0^t x(s)ds \leq \limsup_{t \to +\infty} t^{-1} \ln \left[ x(0) \tilde{a}^{-1} e^{\varepsilon t} + e^{\tilde{a}g(T)} - x(0) \tilde{a}^{-1} e^{\varepsilon T} \right].
\]
An application of L’Hospital’s rule, one can derive
\[
\limsup_{t \to +\infty} t^{-1} \int_0^t x(s)ds \leq \limsup_{t \to +\infty} t^{-1} \ln \left[ x(0) \tilde{a}^{-1} e^{\varepsilon t} \right] = \varepsilon / \tilde{a}.
\]
Since \( \varepsilon \) is arbitrary, we get \( \limsup_{t \to +\infty} t^{-1} \int_0^t x(s)ds \leq 0 \). \( \square \)

**Proof of Theorem 3.** To begin with, let us show that

\[
\limsup_{t \to +\infty} t^{-1} \ln x(t) \leq 0 \quad \text{a.s.} \tag{8}
\]

In fact, applying generalized Itô's formula to Eq. (3) results in

\[
d(e^t \ln x) = e^t \ln x dt + e^t d\ln x
\]

\[
= e^t \left[ \ln x + b(\gamma) - a(\gamma)x - n(\gamma)x(t - \tau) - 0.5\sigma^2(\gamma) x^2(t - \tau) \right]dt
\]

\[
+ e^t \left[ \sigma_1(\gamma) dB_1(t) + \sigma_2(\gamma) x dB_2(t) + \sigma_3(\gamma)x(t - \tau) dB_3(t) \right].
\]

Thus, we have shown that

\[
e^t \ln x(t) - \ln x(0) = \int_0^t e^s \left[ \ln x(s) + b(\gamma(s)) - a(\gamma(s))x(s) - n(\gamma(s))x(s - \tau) - 0.5\sigma^2(\gamma(s)) x^2(s - \tau) \right]ds + N_1(t) + N_2(t) + N_3(t), \tag{9}
\]

where \( N_1(t) = \int_0^t e^s \sigma_1(\gamma(s)) dB_1(s) \), \( N_2(t) = \int_0^t e^s \sigma_2(\gamma(s)) x dB_2(s) \), \( N_3(t) = \int_0^t e^s \sigma_3(\gamma(s)) x(s - \tau) dB_3(s) \). The quadratic variations of \( N_1(t), N_2(t) \) and \( N_3(t) \) are

\[
\langle N_1(t), N_1(t) \rangle = \int_0^t e^{2s} \sigma_1^2(\gamma(s)) ds,
\]

\[
\langle N_2(t), N_2(t) \rangle = \int_0^t e^{2s} \sigma_2^2(\gamma(s)) x^2(s) ds,
\]

\[
\langle N_3(t), N_3(t) \rangle = \int_0^t e^{2s} \sigma_3^2(\gamma(s)) x^2(s - \tau) ds.
\]

It then follows from (6) that (choose \( T = \gamma k, \alpha = e^{-\lambda k}, \beta = \theta e^{\lambda k} \log k \))

\[
\mathcal{P} \left\{ \sup_{0 \leq t \leq \lambda k} \left[ N_i(t) - 0.5 e^{-\lambda k} \langle N_i(t), N_i(t) \rangle \right] > \theta e^{\lambda k} \log k \right\} \leq k^{-\theta}, \quad i = 1, 2, 3
\]

where \( \theta > 1 \) and \( \lambda > 0 \) are arbitrary. By virtue of the Borel–Cantelli lemma, for almost all \( \omega \in \Omega \), there exists \( k_0(\omega) \) such that for every \( k \geq k_0(\omega) \),

\[
N_i(t) \leq 0.5 \exp(-\lambda k) \langle N_i(t), N_i(t) \rangle + \theta \exp(\lambda k) \log k, \quad 0 \leq t \leq \lambda k, \quad i = 1, 2, 3.
\]

In other words

\[
N_1(t) \leq 0.5 e^{-\lambda k} \int_0^t e^{2s} \sigma_1^2(\gamma(s)) ds + \theta e^{\lambda k} \log k,
\]

\[
N_2(t) \leq 0.5 e^{-\lambda k} \int_0^t e^{2s} \sigma_2^2(\gamma(s)) x^2(s) ds + \theta e^{\lambda k} \log k,
\]

\[
N_3(t) \leq 0.5 e^{-\lambda k} \int_0^t e^{2s} \sigma_3^2(\gamma(s)) x^2(s - \tau) ds + \theta e^{\lambda k} \log k
\]

for \( 0 \leq t \leq \gamma k \). Substituting these inequalities into (9) yields that

\[
e^t \ln x(t) - \ln x(0) \leq \int_0^t e^s \left[ \ln x(s) + b(\gamma(s)) - a(\gamma(s))x(s) - n(\gamma(s))x(s - \tau) - 0.5\sigma^2(\gamma(s)) x^2(s - \tau) \right]ds
\]

\[
+ 0.5 e^{-\lambda k} \int_0^t e^{2s} \sigma_1^2(\gamma(s)) ds + 0.5 e^{-\lambda k} \int_0^t e^{2s} \sigma_2^2(\gamma(s)) x^2(s - \tau) ds + 3 \theta e^{\lambda k} \log k
\]

\[
= \int_0^t e^s \left[ \ln x(s) + b(\gamma(s)) - a(\gamma(s))x(s) - n(\gamma(s))x(s - \tau) + 0.5\sigma^2(\gamma(s)) e^{\gamma k} x^2(s - \tau) \right]ds
\]

\[
- \int_0^t e^{0.5 \sigma_2^2(\gamma(s)) x^2(s) [1 - e^{\gamma k}]} ds - \int_0^t e^{0.5 \sigma_3^2(\gamma(s)) x^2(s - \tau) [1 - e^{\gamma k}]} ds + 3 \theta e^{\lambda k} \log k
\]

\[
\leq \int_0^t e^s [\ln x(s) + \hat{b} - \hat{a} x(s) + 0.5 \hat{\sigma}^2] ds + 3 \theta e^{\lambda k} \log k,
\]
where in the last inequality, we have used the facts that \( s \leq \lambda k \) and \( n(y) \geq 0 \). Since \( \tilde{a} > 0 \), then there exists a constant \( C \) independent of \( k \) such that \( \ln x + \tilde{a}t - \tilde{a}x + 0.5\sigma_1^2 \leq C \). In other words,

\[
e^{\tilde{a}t} \ln x(t) - \ln x(0) \leq C[e^{\tilde{a}t} - 1] + 3\theta e^{\lambda k} \ln k, \quad 0 \leq t \leq \lambda k.
\]

If \( \lambda (k - 1) \leq t \leq \lambda k \) and \( k \geq k_0(\omega) \), we have \( \ln x(t)/t \leq e^{-\tilde{a}t} \ln x(0)/t + C[1 - e^{-\tilde{a}t}]/t + 3\theta e^{-\lambda (k - 1)} e^{\lambda k} \ln k/t \), which becomes the desired assertion (8) by letting \( t \rightarrow +\infty \).

Now suppose that \( \sum_{i=1}^m \pi_i b(i) > 0 \), we are going to prove that \( \lim \sup_{t \to +\infty} x(t) = 0 \). If this assertion is not true, then \( \mathcal{P}(F) > 0 \), where \( F = \{ \lim \sup_{t \to +\infty} x(t) = 0 \} \). It follows from (4) that

\[
t^{-1}[\ln x(t) - \ln x(0)] = t^{-1} \int_0^t b(\gamma(s))ds - t^{-1} \int_0^t a(\gamma(s))x(s)ds \\
- t^{-1} \int_0^t n(\gamma(s))x(s - \tau)ds - 0.5t^{-1} \int_0^t \sigma_2^2(\gamma(s))x^2(s)ds \\
- 0.5\tau^{-1} \int_0^t \sigma_3^2(\gamma(s))x^2(s - \tau)ds + M_1(t)/t + M_2(t)/t + M_3(t)/t.
\]

On the other hand, for \( \forall \omega \in F \), we have \( \lim_{t \to +\infty} x(t, \omega) = 0 \). Thus it follows from the law of large numbers for local martingales that \( \lim_{t \to +\infty} M_i(t)/t = 0, i = 1, 2, 3 \). Substituting these inequalities into (10) gives \( \lim \sup_{t \to +\infty} [\ln x(t, \omega)/t] = \sum_{i=1}^m \pi_i b(i) > 0 \). Then \( \mathcal{P}(\lim \sup_{t \to +\infty} [\ln x(t)/t] > 0) > 0 \), this contradicts (8). \( \square \)

### 3. Concluding remarks

This paper is concerned with a stochastic delay Logistic equation under regime switching. Sufficient conditions for extinction, non-persistence in the mean and weak persistence are established. The critical value between weak persistence and extinction is obtained.

Our results have some obvious and interesting biological interpretations. Clearly, the extinction or persistence of \( x(t) \) modeled by system (3) depends only on \( \sum_{i=1}^m \pi_i b_i \). If \( \sum_{i=1}^m \pi_i b(i) > 0 \), then the species \( x(t) \) is weakly persistent; if \( \sum_{i=1}^m \pi_i b(i) < 0 \), then the population \( x(t) \) goes to extinction. At the same time, it is easy to obtain that the white noise \( \sigma_1 \) is unfavorable for the persistence of the species. However, the white noises \( \sigma_2 \) and \( \sigma_3 \) as well as the time delay \( \tau \) have no impact on the extinction and persistence of \( x(t) \). Now let us see the impact of Markov chain \( \gamma(t) \). The distribution \( (\pi_1, \ldots, \pi_m) \) of \( \gamma(t) \) plays a very important role in determining extinction or persistence of the species. If \( \gamma(t) \) spends enough time in the "good" states (the state where \( b(\cdot) \) is positive) then the population is to survive. If \( \gamma(t) \) spends many time in the "bad" states (the state where \( b(\cdot) \) is negative) then the population goes to extinction.

Some interesting topics deserve further investigations, such as the stochastic persistence, the multi-dimensional stochastic systems and the systems with distributed delays. We leave these for further investigations.

### Acknowledgments

We thank the editor and the reviewer for these important and valuable comments. We also thank the NSFC of PR China (Nos. 11126219, 11171081, 11171056, 11001032 and 11101183).

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