Applied Mathematics Letters 24 (2011) 1938–1944

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Dynamical behaviors and synchronization in the fractional order hyperchaotic Chen system

A.S. Hegazi [a], A.E. Matouk [a,b,*]

[a] Mathematics Department, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt

[b] Mathematics Department, Faculty of Science, Hail University, Hail, 2440, Saudi Arabia

ARTICLE INFO

Article history:
Received 2 September 2009
Received in revised form 17 May 2011
Accepted 17 May 2011

Keywords:
Fractional order
Hyperchaotic Chen system
Routh–Hurwitz conditions
Hyperchaos
Synchronization

ABSTRACT

Some dynamical behaviors are studied in the fractional order hyperchaotic Chen system which shows hyperchaos with order less than 4. The analytical conditions for achieving synchronization in this system via linear control are investigated theoretically by using the Laplace transform theory. Routh–Hurwitz conditions and numerical simulations are used to show the agreement between the theoretical and numerical results. To the best of our knowledge this is the first example of a hyperchaotic system synchronizable just in the fractional order case, using a specific choice of controllers.

1. Introduction

The idea of fractional calculus has been known of since the work of Leibniz and L'Hospital in 1695 [1]. It has useful applications in physics, engineering [2] and mathematical biology [3,4].

The fractional order derivatives have many definitions. The Caputo definition of fractional derivative [5] is used throughout this work and is given as follows:

$$D^{\alpha}f(x) = I^{m-\alpha}f^{(m)}(x), \quad \alpha > 0, \tag{1}$$

where $f^{(m)}$ represents the m-order derivative of $f(x)$, $m = [\alpha]$ is the first integer which is not less than α, and the operator

$$I^{q}g(x) = \frac{1}{\Gamma(q)}\int_0^x (x-t)^{q-1}g(t)\mathrm{d}t, \quad q > 0, \tag{2}$$

is the q-order Riemann–Liouville integral operator, where $\Gamma(q)$ is the gamma function. The operator D^{α} is called the "α-order Caputo differential operator". The geometric and physical interpretation of the fractional derivatives was given in [6,7]. It has recently been found that chaos has important applications in fractional order systems, especially chaos synchronization [8]. A regular chaotic system has one positive Lyapunov exponent. However, a hyperchaotic system has more than one positive Lyapunov exponent, which shows more complex behaviors and abundant dynamics than the chaotic system. Therefore, hyperchaotic systems can be better for applications in secure communications than chaotic ones [9]. Recently, some fractional order hyperchaotic systems have been investigated [10–14].

In this study, some Routh–Hurwitz conditions are introduced in order to discuss local stability in some fractional order hyperchaotic systems. The proposed conditions are applied successfully to the fractional order hyperchaotic Chen system.

* Corresponding author at: Mathematics Department, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt.
E-mail address: aematouk@hotmail.com (A.E. Matouk).

0893-9659/$ – see front matter
doi:10.1016/j.aml.2011.05.025

The numerical results show that hyperchaos does exist in the proposed system with order less than 4, and Lyapunov exponents are also calculated for this system to verify the existence of hyperchaos. Moreover, the Laplace transform theory is used to achieve synchronization between two identical fractional order hyperchaotic Chen systems via the linear control technique. Furthermore, chaos synchronization of the hyperchaotic Chen system is found only in the fractional order case when using a specific choice of nonlinear control functions.

2. Some Routh–Hurwitz conditions for the fractional order hyperchaotic systems

Consider the four-dimensional fractional order hyperchaotic system

$$\begin{aligned} D^\alpha x_1(t) &= f_1(x_1, x_2, x_3, x_4), \qquad D^\alpha x_2(t) = f_2(x_1, x_2, x_3, x_4), \\ D^\alpha x_3(t) &= f_3(x_1, x_2, x_3, x_4), \qquad D^\alpha x_4(t) = f_4(x_1, x_2, x_3, x_4), \end{aligned} \tag{3}$$

where the fractional derivative in (3) is in the sense of Caputo and $0 \le \alpha < 1$. Let $\bar{E} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ be an equilibrium solution of (3); then $\bar{E}$ is locally asymptotically stable if all the eigenvalues of the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij},$$

evaluated at $\bar{E}$ satisfy Matignon's condition [15], i.e., the eigenvalues λ_{i_s} of J evaluated at the equilibrium point $\bar{E}$ are given as

$$|\arg(\lambda_i)| > \alpha\pi/2, \quad (i = 1, 2, 3, 4).$$

The eigenvalue equation of the equilibrium point $\bar{E}$ is given as

$$P(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{4}$$

whose discriminant $D(P)$ is given by

$$\begin{aligned} D(P) = &-4a_3^3a_1^3 + a_2^2a_1^2a_3^2 + 18a_3^3a_2a_1 - 6a_1^2a_3^2a_4 - 4a_2^3a_1^2a_4 - 80a_3a_2^2a_1a_4 + 144a_3^2a_2a_4 - 192a_3a_4^2a_1 \\ &+ 144a_2a_1^2a_4^2 + 18a_3a_2a_1^3a_4 - 27a_3^4 - 4a_2^3a_3^2 - 128a_4^2a_2^2 - 27a_1^4a_4^2 + 256a_4^3 + 16a_2^4a_4. \end{aligned} \tag{5}$$

Proposition 1. (i) *If c_1, c_2, c_3 are Routh–Hurwitz determinants which are defined as follows:*

$$c_1 = a_1, \qquad c_2 = a_1a_2 - a_3, \qquad c_3 = a_1a_2a_3 - a_1^2a_4 - a_3^2, \tag{6}$$

then for $\alpha = 1$, the equilibrium point $\bar{E}$ of system (3) is locally asymptotically stable if and only if

$$c_1 > 0, \qquad c_2 > 0, \qquad c_3 = 0, \qquad a_4 > 0. \tag{7}$$

Moreover, the conditions of (7) are sufficient conditions for the equilibrium point $\bar{E}$ to be locally asymptotically stable for all $\alpha \in [0, 1)$.

(ii) *If $D(P) > 0$, $a_1 > 0$, $a_2 < 0$ and $\alpha > 2/3$ then the equilibrium point $\bar{E}$ is unstable.*

(iii) *If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$, and $\alpha < 1/3$, then the equilibrium point $\bar{E}$ is locally asymptotically stable. Also, if $D(P) < 0$, $a_1 < 0$, $a_2 > 0$, $a_3 < 0$, $a_4 > 0$, then the equilibrium point $\bar{E}$ is unstable.*

(iv) *If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$ and $a_2 = \frac{a_1a_4}{a_3} + \frac{a_3}{a_1}$, then the equilibrium point $\bar{E}$ is locally asymptotically stable, for all $\alpha \in (0, 1)$.*

(v) *$a_4 > 0$ is the necessary condition for the equilibrium point $\bar{E}$ to be locally asymptotically stable.*

The proof of Proposition 1 has been given in [14].

3. The fractional order hyperchaotic Chen system

In the following, we investigate the stability conditions and hyperchaos in the fractional order hyperchaotic Chen system. This system will be integrated numerically to show hyperchaos using an efficient method for solving fractional order differential equations, that is the predictor–corrector scheme or, more precisely, the *PECE* (predict, evaluate, correct, evaluate) technique which has been investigated in [16–18], and represents a generalization of the Adams–Bashforth–Moulton algorithm. It is used throughout this work. To explain this method, we consider the following fractional order differential equation:

$$D^\alpha y(t) = g(t, y(t)), \quad 0 \le t \le T, \qquad y^{(k)}(0) = y_0^{(k)}, \quad k = 0, \dots, m-1,$$

which is equivalent to the Volterra integral equation of the second kind:

$$y(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau, y(\tau)) \mathrm{d}\tau. \tag{8}$$

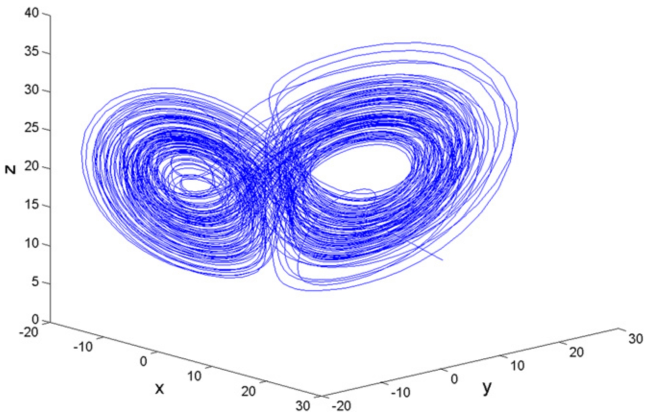

Fig. 1. 3D plot of the fractional order hyperchaotic Chen attractor in x–y–z space..

Set $h = T/N$, $t_n = nh$, $n = 0, 1, \ldots, N \in Z^+$. Then (8) can be discretized as follows:

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} g(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum \rho_{j,n+1} g(t_j, y_h(t_j)),$$

where

$$\rho_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases}$$

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \sigma_{j,n+1} g(t_j, y_h(t_j)), \qquad \sigma_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha).$$

The error estimate is $\max_{j=0,1,\ldots,N} |y(t_j) - y_h(t_j)| = O(h^p)$, in which $p = \min(2, 1+\alpha)$.

Now, we consider the fractional order hyperchaotic Chen system as follows:

$$D^\alpha x = a(y-x) + u, \qquad D^\alpha y = \gamma x - xz + cy, \qquad D^\alpha z = xy - bz, \qquad D^\alpha u = yz + du. \tag{9}$$

The integer order form of system (9) was studied in [19]. System (9) has only the equilibrium point $E_0 = (0, 0, 0, 0)$. Using the parameter values $(a, b, c, d, \gamma) = (35, 3, 12, 0.3, 7)$ and $\alpha = 0.97$, system (9) has two positive Lyapunov exponents, $\lambda_1 \approx 1.214$ and $\lambda_2 \approx 0.138$, which are calculated using an algorithm developed in [20]. System (9) is numerically integrated using the above-mentioned discretization scheme with the given values of the parameters. Fig. 1 shows the hyperchaotic attractor of system (9) for the above-mentioned parameter values and fractional order $\alpha = 0.97$. However, the lowest fractional order at which system (9) exhibits a chaotic attractor is $\alpha = 0.94$, i.e., the lowest order found to yield chaos for system (9) is 3.76.

The characteristic polynomial for the equilibrium point E_0 is given by

$$\lambda^4 + (-c + a + b - d)\lambda^3 + (cd - \gamma a - ad - bd + ab - cb - ac)\lambda^2 + (-acb + acd + cbd + \gamma ad - abd - \gamma ab)\lambda + abd(c+\gamma) = 0. \tag{10}$$

Eq. (10) has the roots $\lambda_1 = d$, $\lambda_2 = -b$ and $\lambda_{3,4} = \frac{c-a\pm\sqrt{(a+c)^2+4a\gamma}}{2}$. If all these eigenvalues satisfy the conditions $|\arg(\lambda_i)| > \alpha\pi/2$ $(i = 1, 2, 3, 4)$, then system (9) is locally asymptotically stable at the equilibrium point E_0. Moreover, using the above-mentioned parameter values, it is easy to verify that $D(P) > 0$, $a_1 > 0$, $a_2 < 0$, $a_4 > 0$. Thus, Proposition 1 part (ii) implies that the equilibrium point E_0 is unstable for $\alpha > 2/3$.

4. Synchronization of the fractional order hyperchaotic Chen system

In the following, we are going to achieve chaos synchronization of the fractional order hyperchaotic Chen system using linear and nonlinear control techniques. It should be noted that the term "synchronization" here denotes "complete synchronization".

4.1. Synchronization of the fractional order hyperchaotic Chen system via linear control

Let the drive fractional order hyperchaotic Chen system be given as follows:

$$D^\alpha x_m = a(y_m - x_m) + u_m, \qquad D^\alpha y_m = \gamma x_m - x_m z_m + c y_m,$$
$$D^\alpha z_m = x_m y_m - b z_m, \qquad D^\alpha u_m = y_m z_m + \mathrm{d}u_m, \tag{11}$$

and the response system be given by

$$D^\alpha x_s = a(y_s - x_s) + u_s + v_1, \qquad D^\alpha y_s = \gamma x_s - x_s z_s + c y_s + v_2,$$
$$D^\alpha z_s = x_s y_s - b z_s + v_3, \qquad D^\alpha u_s = y_s z_s + \mathrm{d}u_s + v_4, \tag{12}$$

where v_1, v_2, v_3 and v_4 are the linear controllers. Define the error variables as

$$e_1 = x_s - x_m, \qquad e_2 = y_s - y_m, \qquad e_3 = z_s - z_m, \qquad e_4 = u_s - u_m. \tag{13}$$

By subtracting (11) from (12) and using (13), we obtain

$$D^\alpha e_1 = a(e_2 - e_1) + e_4 + v_1, \qquad D^\alpha e_2 = \gamma e_1 - z_m e_1 - x_m e_3 - e_1 e_3 + c e_2 + v_2,$$
$$D^\alpha e_3 = -b e_3 + y_m e_1 + x_m e_2 + e_1 e_2 + v_3, \qquad D^\alpha e_4 = d e_4 + z_m e_2 + y_m e_3 + e_2 e_3 + v_4. \tag{14}$$

Now, we choose the controllers as

$$v_1 = -k_1 e_1 - e_4, \qquad v_2 = -k_2 e_2, \qquad v_3 = -k_3 e_3, \qquad v_4 = -k_4 e_4, \tag{15}$$

where $k_1, k_2, k_3, k_4 \geq 0$. Hence, the error system (14) is reduced to

$$D^\alpha e_1 = -(a + k_1) e_1 + a e_2, \qquad D^\alpha e_2 = \gamma e_1 + (c - k_2) e_2 - z_m e_1 - x_m e_3 - e_1 e_3,$$
$$D^\alpha e_3 = -(b + k_3) e_3 + y_m e_1 + x_m e_2 + e_1 e_2, \qquad D^\alpha e_4 = (d - k_4) e_4 + z_m e_2 + y_m e_3 + e_2 e_3. \tag{16}$$

Proposition 2. *The drive fractional order hyperchaotic Chen system (11) and the response system (12) with linear controllers $v_1 = -k_1 e_1 - e_4$, $v_2 = -k_2 e_2$, $v_3 = -k_3 e_3$, $v_4 = -k_4 e_4$ are synchronized under the conditions $E_1(s) \leq \eta < \infty$, $E_2(s) \leq \eta < \infty$, $k_2 \neq c$ and $k_4 \neq d$.*

Proof. By taking the Laplace transform on both sides of (16), letting $E_i(s) = L\{e_i(t)\}$ where $i = 1, 2, 3, 4$, and applying $L\{D^\alpha e_i(t)\} = s^\alpha E_i(s) - s^{\alpha-1} e_i(0)$, we obtain

$$\begin{aligned}
E_1(s) &= \frac{a E_2(s)}{s^\alpha + a + k_1} + \frac{s^{\alpha-1} e_1(0)}{s^\alpha + a + k_1},\\
E_2(s) &= \frac{\gamma E_1(s)}{s^\alpha - c + k_2} - \frac{L\{z_m e_1\}}{s^\alpha - c + k_2} - \frac{L\{x_m e_3\}}{s^\alpha - c + k_2} - \frac{E_1(s)E_3(s)}{s^\alpha - c + k_2} + \frac{s^{\alpha-1} e_2(0)}{s^\alpha - c + k_2},\\
E_3(s) &= \frac{L\{y_m e_1\}}{s^\alpha + b + k_3} + \frac{L\{x_m e_2\}}{s^\alpha + b + k_3} + \frac{E_1(s)E_2(s)}{s^\alpha + b + k_3} + \frac{s^{\alpha-1} e_3(0)}{s^\alpha + b + k_3},\\
E_4(s) &= \frac{L\{y_m e_3\}}{s^\alpha - d + k_4} + \frac{L\{z_m e_2\}}{s^\alpha - d + k_4} + \frac{E_2(s)E_3(s)}{s^\alpha - d + k_4} + \frac{s^{\alpha-1} e_4(0)}{s^\alpha - d + k_4}.
\end{aligned} \tag{17}$$

According to the final-value theorem of the Laplace transform, it follows that

$$\begin{aligned}
\lim_{t\to\infty} e_1(t) &= \lim_{s\to 0^+} sE_1(s) = \frac{a}{a + k_1} \lim_{s\to 0^+} sE_2(s) = \frac{a}{a + k_1} \lim_{t\to\infty} e_2(t),\\
\lim_{t\to\infty} e_2(t) &= \lim_{s\to 0^+} sE_2(s) = \frac{\gamma \lim_{t\to\infty} e_1(t)}{k_2 - c} - \frac{\lim_{s\to 0^+} sL\{x_m e_3\}}{k_2 - c} - \frac{\lim_{s\to 0^+} sL\{z_m e_1\}}{k_2 - c}\\
&\quad - \frac{\lim_{t\to\infty} e_1(t).\lim_{t\to\infty} e_3(t)}{k_2 - c},\\
\lim_{t\to\infty} e_3(t) &= \lim_{s\to 0^+} sE_3(s) = \frac{\lim_{s\to 0^+} sL\{y_m e_1\}}{b + k_3} + \frac{\lim_{s\to 0^+} sL\{x_m e_2\}}{b + k_3} + \frac{\lim_{t\to\infty} e_1(t).\lim_{t\to\infty} e_2(t)}{b + k_3}\\
\lim_{t\to\infty} e_4(t) &= \lim_{s\to 0^+} sE_4(s) = \frac{\lim_{s\to 0^+} sL\{y_m e_3\}}{k_4 - d} + \frac{\lim_{s\to 0^+} sL\{z_m e_2\}}{k_4 - d} + \frac{\lim_{t\to\infty} e_2(t).\lim_{t\to\infty} e_3(t)}{k_4 - d}.
\end{aligned} \tag{18}$$

Assume that $E_1(s), E_2(s)$ are bounded and $k_2 - c \neq 0$; then $\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = 0$. Now, owing to the attractiveness of the attractor, there exists $\varepsilon > 0$ such that $|x_i(t)| \leq \varepsilon < \infty$, $|y_i(t)| \leq \varepsilon < \infty$, $|z_i(t)| \leq \varepsilon < \infty$ and

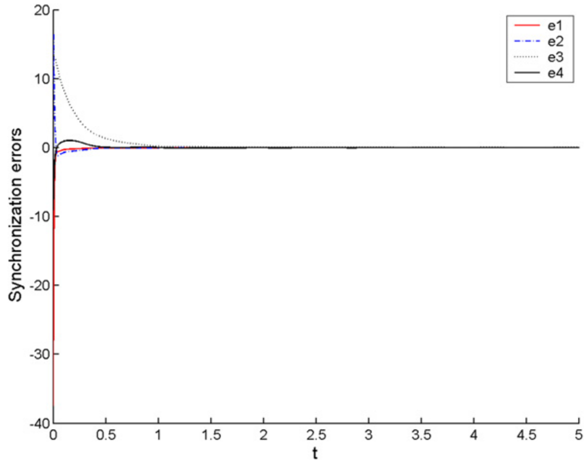

Fig. 2. Synchronization errors between the drive and response systems (11) and (12) tend to zero when using fractional order $\alpha = 0.97$ and feedback control gains $k_1 = 100$, $k_2 = 100$, $k_3 = 1$, $k_4 = 30$.

$|u_i(t)| \leq \varepsilon < \infty$ where i refers to the subscript of the drive or response variables. Consequently, $\lim_{t\to\infty} e_3(t) = 0$ and $\lim_{t\to\infty} e_4(t) = 0$ provided that $k_4 - d \neq 0$. Hence, we have proved that

$$\lim_{t\to\infty} e_i(t) = 0, \quad i = 1, 2, 3, 4. \tag{19}$$

Thus, the synchronization between the drive and response systems (11) and (12) is achieved. □

On the basis of the *PECE* scheme, the drive and response systems (11) and (12) are integrated numerically using the above-mentioned parameter values and fractional order $\alpha = 0.97$, with the initial values $x_m(0) = 30$, $y_m(0) = 7$, $z_m(0) = 10$, $u_m(0) = 40$ and $x_s(0) = -15$, $y_s(0) = 25$, $z_s(0) = 25$, $u_s(0) = 30$. From Fig. 2, it is clear that synchronization is achieved when $k_1 = 100$, $k_2 = 100$, $k_3 = 1$ and $k_4 = 30$. □

4.2. *Synchronization of the fractional order hyperchaotic Chen system via nonlinear control*

The master system (11) is used to drive the following slave system:

$$\begin{aligned} &D^\alpha x_s = a(y_s - x_s) + u_s + w_1, \qquad D^\alpha y_s = \gamma x_s - x_m z_s + c y_s + w_2, \\ &D^\alpha z_s = x_m y_s - b z_s + w_3, \qquad D^\alpha u_s = y_m z_s + \mathrm{d}u_s + w_4, \end{aligned} \tag{20}$$

where w_1, w_2, w_3 and w_4 are the nonlinear controllers. Let the error variables be defined as

$$e_1 = x_s - x_m, \qquad e_2 = y_s - y_m, \qquad e_3 = z_s - z_m, \qquad e_4 = u_s - u_m. \tag{21}$$

By subtracting (11) from (20) and using (21), we obtain

$$\begin{aligned} &D^\alpha e_1 = a(e_2 - e_1) + e_4 + w_1, \qquad D^\alpha e_2 = \gamma e_1 - x_m e_3 + c e_2 + w_2, \\ &D^\alpha e_3 = -b e_3 + x_m e_2 + w_3, \qquad D^\alpha e_4 = d e_4 + y_m e_3 + w_4. \end{aligned} \tag{22}$$

Proposition 3. *The drive fractional order hyperchaotic Chen system (11) and response system (20), with nonlinear controllers*

$$\begin{aligned} &w_1 = -k_1 e_1 - e_4, \qquad w_2 = 18 e_1 - (k_2 + 11) e_2 + x_m e_3 + e_4, \\ &w_3 = -x_m e_2 - (1 + k_3) e_3, \qquad w_4 = -100 e_2 - y_m e_3 - (d + k_4) e_4, \end{aligned} \tag{23}$$

are synchronized only if $0 < \alpha < 1$, *where the feedback control gains satisfy the conditions* $k_1 = 693.0293$, $k_2 = 1.2$, $k_3 = 1$ *and* $k_4 = 1$ *or the conditions* $k_1 = 838.8855$, $k_2 = 1$, $k_3 = 1$ *and* $k_4 = 1$.

Proof. The error dynamical system (22) with the controllers (23) has the equilibrium point $(0, 0, 0, 0)$ and its Jacobian matrix is given by

$$J = \begin{bmatrix} -(a + k_1) & a & 0 & 0 \\ \gamma + 18 & c - k_2 - 11 & 0 & 1 \\ 0 & 0 & -(b + k_3 + 1) & 0 \\ 0 & -100 & 0 & -k_4 \end{bmatrix}. \tag{24}$$

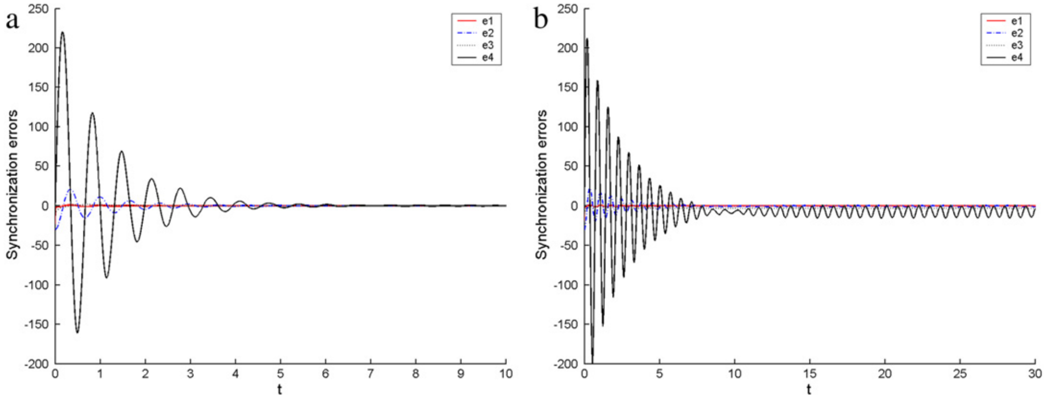

Fig. 3. Synchronization errors between the master and slave systems (11) and (20) with the controllers (23) and $k_1 = 693.0293$, $k_2 = 1.2$, $k_3 = k_4 = 1$; (a) the synchronization errors tend to zero when using fractional order $\alpha = 0.97$; (b) the synchronization errors do not approach zero when using $\alpha = 1.0$.

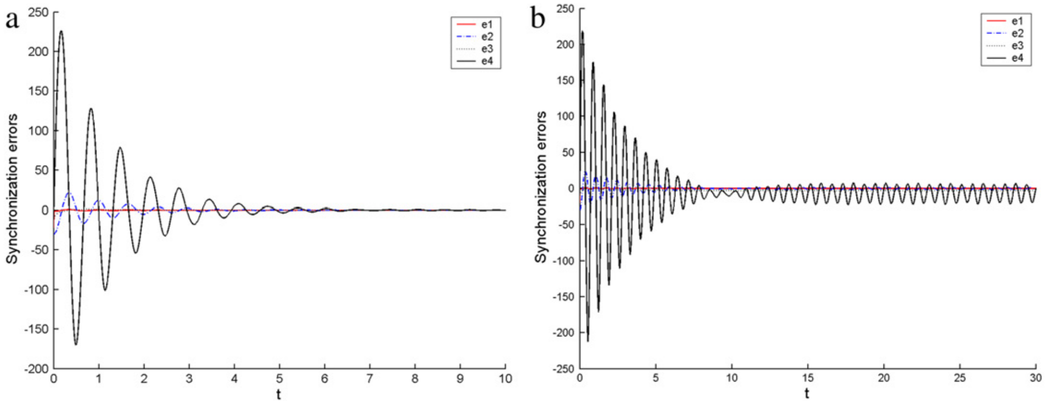

Fig. 4. Synchronization errors between the master and slave systems (11) and (20) with the controllers (23) and $k_1 = 838.8855$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$; (a) the synchronization errors converge to zero when using fractional order $\alpha = 0.97$; (b) the synchronization errors do not tend to zero when using $\alpha = 1.0$.

As using the above-mentioned parameter values, the characteristic equation for the Jacobian matrix (24) is given as

$$P(\lambda) = \lambda^4 + (41 + k_1)\lambda^3 + (-560 + 6k_1)\lambda^2 + (-1075 + 105k_1)\lambda + 13125 + 500k_1 = 0. \tag{25}$$

Using the feedback control gains $k_1 = 693.0293$, $k_2 = 1.2$, $k_3 = 1$, $k_4 = 1$, one can easily see that Eq. (25) satisfies $D(P) < 0, a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$ and $a_2 = \frac{a_1 a_4}{a_3} + \frac{a_3}{a_1}$; consequently, all the eigenvalues of the characteristic Eq. (25) lie in the stable region according to the part (iv) of Proposition 1. Thus, the zero solution of Eq. (22) is locally asymptotically stable, and the synchronization errors approach zero for the fractional orders $0 < \alpha < 1$. When $\alpha = 1$, it is easy to verify that (25) has one pair of complex eigenvalues with pure imaginary parts, so the zero equilibrium point of (22) is not asymptotically stable. Consequently, using the above-mentioned parameter values and feedback control gains, the fractional order master and slave systems (11) and (20) are synchronized, but their integer order counterparts are not synchronized (see Fig. 3(a)–(b)). Furthermore, we obtain the same results when using the feedback control gains $k_1 = 838.8855$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$ (see Fig. 4(a)–(b)). □

5. Conclusion

Some stability conditions in fractional order hyperchaotic systems have been introduced and applied to the fractional order hyperchaotic Chen system. Numerical simulations and Lyapunov exponents have been used to show that hyperchaos exists in this system with order less than 4. It has been shown that according to the Laplace transformation theory, one achieves synchronization of the fractional order hyperchaotic Chen system when choosing suitable linear controllers. Moreover, the fractional order hyperchaotic Chen system has been synchronized using a nonlinear feedback control method but its integer order counterpart has not been synchronized using the same nonlinear controllers. To the best of our knowledge this is the first example of a hyperchaotic system that can be synchronized in the fractional case while it is not synchronized in the integer order case using the same controllers. Numerical simulations have been used to verify the theoretical analysis.

Acknowledgments

The authors thank the anonymous reviewers for providing some helpful comments which helped to improve the style of this work.

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