The \(k\)-tuple twin domination in de Bruijn and Kautz digraphs

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Abstract

A vertex \(u\) in a digraph \(G\) out-dominates itself and all vertices \(v\) such that \((u, v)\) is an arc of \(G\), similarly, \(u\) in-dominates both itself and all vertices \(w\) such that \((w, u)\) is an arc of \(G\). A set \(D\) of vertices of \(G\) is a twin dominating set of \(G\) if every vertex of \(G\) is out-dominated by a vertex of \(D\) and in-dominated by a vertex in \(D\). In this paper, we introduce the \(k\)-tuple twin domination in directed graphs. A set \(D\) of vertices of \(G\) is a \(k\)-tuple twin dominating set if every vertex of \(G\) is out-dominated by at least \(k\) vertices in \(D\) and in-dominated by at least \(k\) vertices in \(D\). We consider the problem of the \(k\)-tuple twin domination in de Bruijn and Kautz digraphs, and give construction methods for constructing minimum \(k\)-tuple twin dominating sets in these digraphs.

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1. Introduction

Domination in graphs has been studied extensively in recent years since it has many applications. For comprehensive treatment of domination and its variation, see the book by Haynes, Hedetniemi, and Slater [7]. This paper deals with a problem of domination in digraphs (directed graphs). The concept of domination in undirected graphs is naturally transferred to the out-domination in digraphs. A vertex \(u\) in a digraph out-dominates itself and all vertices \(v\) such that \((u, v)\) is an arc of the digraph. We can similarly define in-domination in digraphs. Another natural concept of domination in digraphs, twin domination, was introduced by Chartrand, Dankelmann, Schultz, and Swart [4]. A twin dominating set is both out-dominating set and in-dominating set in a digraph, and the twin domination number is the cardinality of a minimum twin dominating set. In [4], sharp upper bounds of the twin domination numbers for digraphs were given, and also a Nordhaus–Gaddum type inequality for the twin domination number was presented.

Another variation of the domination in (undirected) graphs, the \(k\)-tuple domination, was introduced by Harary and Haynes [5]. For a positive integer \(k\), a \(k\)-tuple dominating set of a graph is a subset \(D\) of vertices such that every vertex is dominated by at least \(k\) vertices in \(D\). The \(k\)-tuple domination in graphs have been studied in [3,9–12].

From the concept of the \(k\)-tuple domination, we introduce a natural generalization of the twin domination in digraphs. A \(k\)-tuple twin dominating set of a digraph is a subset \(D\) of vertices such that every vertex is in-dominated by at least \(k\) vertices and out-dominated by at least \(k\) vertices in \(D\). We consider the \(k\)-tuple twin domination in
particular classes of digraphs, de Bruijn and Kautz digraphs. The de Bruijn and Kautz digraphs have been studied as interconnection networks because of various good properties [2]. Kikuchi and Shibata [8] considered the out-domination problem for generalized de Bruijn and Kautz digraphs. The author investigated the problem of the k-tuple out-domination in de Bruijn and Kautz digraphs [1].

The rest of this paper is organized as follows: In Section 2, some definitions and notations used in this paper are given. In Sections 3 and 4, we present the k-tuple twin domination numbers of de Bruijn and Kautz digraphs, respectively. Finally we conclude the paper in Section 5.

2. Preliminaries

In this paper, we deal with simple digraphs which admit self-loops but no multiple arcs. For a digraph G, the vertex and arc set of G are denoted by V(G) and A(G), respectively. The open out-neighborhood of a vertex u is denoted by N_G^+(u) = {v | (u, v) ∈ A(G)} \ {u}. Similarly, the open in-neighborhood is N_G^−(u) = {v | (v, u) ∈ A(G)} \ {u}. The closed out-neighborhood and closed in-neighborhood of u are N_G^+_C(u) = {u} ∪ N_G^+(u) and N_G^−_C(u) = {u} ∪ N_G^−(u), respectively. Note that if u has a self-loop, the open out- and in-neighborhoods of u do not contain u itself. For a subset S ⊆ V(G), the open out-neighborhood of S is N_G^+(S) = ∪_{u ∈ S} N_G^+(u) \ S and the open in-neighborhood is N_G^−(S) = ∪_{u ∈ S} N_G^−(u) \ S. The outdegree and indegree of u are deg_G^+(u) = |N_G^+(u)| and deg_G^−(u) = |N_G^−(u)|, respectively. We use δ^+(G) and δ^−(G) to denote the minimum outdegree and the minimum indegree of G, respectively. N_G^+(u) and N_G^−(u) are also denoted by N^+(u) and N^−(u) if G is clear from the context.

A vertex u in a digraph G is said to out-dominate the vertices in N_G^+(u), while u in-dominates the vertices in N_G^−(u). A set D of vertices in a digraph G is a twin dominating set if |N_G^+(u) ∩ D| ≥ 1 and |N_G^−(u) ∩ D| ≥ 1 for every vertex u [4]. The twin domination number γ^t(G) of G is the cardinality of a minimum twin dominating set. In this paper, we introduce a generalization of the twin domination in digraphs as follows.

Definition 2.1. A set D of vertices in a digraph G is a k-tuple twin dominating set if |N_G^+(u) ∩ D| ≥ k and |N_G^−(u) ∩ D| ≥ k for every vertex u. The k-tuple twin domination number γ^t_k(G) of G is the cardinality of a minimum k-tuple twin dominating set.

The special case when k = 1 is a usual twin domination. Notice that a digraph G has a k-tuple twin dominating set if and only if k ≤ δ^−(G) + 1 and k ≤ δ^+(G) + 1.

For positive integers d ≥ 2 and n ≥ 1, the de Bruijn digraph B(d, n) has vertex set \{x_1x_2...x_n | 0 ≤ x_i ≤ d − 1, i = 1, 2, ..., n\} and a vertex x_1x_2...x_n is adjacent to x_2x_3...x_{n+1}α for 0 ≤ α ≤ d − 1. From the definition, B(d, n) has d^n vertices. The de Bruijn digraph B(d, 1) is the complete digraph of d vertices that have self-loops. A vertex x has a self-loop if and only if x = i^n = ii...i for i = 0, ..., d − 1. The set of vertices that have self-loops in B(d, n) is denoted by I_{d,n}. If x = i^n ∈ I_{d,n}, we have deg^+(x) = deg^−(x) = d − 1, otherwise deg^+(x) = deg^−(x) = d. Hence B(d, n) has a k-tuple twin dominating set if and only if k ≤ d.

For positive integers d ≥ 2 and n ≥ 1, the Kautz digraph K(d, n) has vertex set \{x_1x_2...x_n | 0 ≤ x_i ≤ d and x_i ≠ x_{i+1} for any i\}. A vertex x_1x_2...x_n is adjacent to d vertices x_2...x_nα for 0 ≤ α ≤ d and α ≠ x_n. The Kautz digraph K(d, n) has d^n + d^{n−1} vertices. The Kautz digraph K(d, 1) is the complete digraph of d + 1 vertices. Since deg^+(u) = deg^−(u) = d for every vertex u, K(d, n) has a k-tuple twin dominating set if and only if k ≤ d + 1. Fig. 1 shows B(2, n) and K(2, n) for n = 1, 2, 3.

A complete bipartite digraph K_{m,n} has a bipartition L ∪ R such that |L| = m and |R| = n, and arc set \{(u, v) | u ∈ L, v ∈ R\}. We define K_{m,n} as a digraph obtained by contracting an arc of K_{m,n}. Also, let K_{m,n}^∞ be a digraph obtained by adding an arc from a vertex in R to a vertex in L. For example, K_{3,3}^∞, K_{3,3} and K_{3,3}^∞ are shown in Fig. 2.

The following is an important property for considering twin domination in de Bruijn and Kautz digraphs. For a set S of vertices, the subgraph induced by S is denoted by (S).

Property 2.2. Suppose that d ≥ 2 and n ≥ 3. For a vertex x of B(d, n), let B_x = N^+(x) ∪ N^−(N^+(x)).

1. If x = ji^{n−1} for 0 ≤ i, j ≤ d − 1, then (B_x) is isomorphic to K_{d,d}.
Suppose that $d = x$. Otherwise, $L = x$. If $x = ki^n$, then $B_k = \{i^n\} \cup \{i^n-k \mid 0 \leq k \leq d-1, k \neq i\}$ \cup $\{\ell i^{n-1} \mid 0 \leq \ell \leq d-1, \ell \neq i\}$. Hence $\langle B_k \rangle$ is isomorphic to $\overrightarrow{K}_{d,d}$.

Assume that $n$ is odd. If $x = kij \ldots i$ for $i \neq j$, then $L = \{kij \ldots ij \mid 0 \leq k \leq d-1\}$ and $R = \{ij \ldots ij\ell \mid 0 \leq \ell \leq d-1\}$. Every vertex in $L$ is adjacent to every vertex in $R$, and a vertex in $R$ is adjacent to a vertex in $L$ if and only if $k = j$ and $\ell = i$. Hence $\langle B_k \rangle$ is isomorphic to $\overrightarrow{K}_{d,d}$.

For other cases, if $x = x_1x_2 \ldots x_n$, then $R = \{ix_2 \ldots x_n \mid 0 \leq i \leq d-1\}$ and $L = \{x_2 \ldots x_n \mid 0 \leq j \leq d-1\}$. If a vertex $x_2 \ldots x_n i \in R$ is adjacent to $jx_2 \ldots x_n \in L$, we obtain $j = x_3 = \cdots = x_{n-2} = x_n$ and $i = x_2 = \cdots = x_{n-3} = x_{n-1}$. This implies that $x = ki^{n-1}$ or $x = kij \ldots i$. Hence there is no arc from $R$ to $L$, and thus $\langle B_k \rangle$ is isomorphic to $\overrightarrow{K}_{d,d}$.

We show examples of $\langle B_n \rangle$ of $B(3, 4)$ in Fig. 3. For Kautz digraphs, a similar property holds as follows.

**Property 2.3.** Suppose that $d \geq 2$ and $n \geq 3$. For a vertex $x$ of $K(d,n)$, let $K_x = N^+(x) \cup N^-(N^+(x))$. If $x = kiij \ldots$, then $\langle K_x \rangle$ is isomorphic to $\overrightarrow{K}_{d,d}$. Otherwise, $\langle B_k \rangle$ is isomorphic to $\overrightarrow{K}_{d,d}$.

For digraphs $G$ and $H$, a mapping $\varphi : V(G) \to V(H)$ is called a homomorphism of $G$ onto $H$ if it is surjective and arc-preserving (if $(u, v) \in A(G)$ then $(\varphi(u), \varphi(v)) \in A(H)$). For a homomorphism $\varphi$, we define

- $\varphi^{-1}(y) = \{x \in V(G) \mid \varphi(x) = y\}$ for $y \in V(H)$,
Assume that a homomorphism $\varphi$ of $G$ onto $H$ satisfies the conditions (1) $\varphi(N_G^+(x)) = N_H^+(\varphi(x))$ and (2) $\varphi(N_G^-(x)) = N_H^-(\varphi(x))$ for every $x \in V(G)$. If $D_H$ is a $k$-tuple twin dominating set of $H$, then $D_G = \varphi^{-1}(D_H)$ is a $k$-tuple twin dominating set of $G$.

**Proof.** Let $y = \varphi(x)$ for a vertex $x \in V(G)$. Since $D_H$ is a $k$-tuple twin dominating set of $H$, we have $|D_H \cap N_H^+(y)| \geq k$ and $|D_H \cap N_H^-(y)| \geq k$. For any $y_1 \in N_H^+(y)$, there exists a vertex $x_1 \in N_G^+(x)$ such that $\varphi(x_1) = y_1$ since $N_H^+(y) = \varphi(N_G^+(x))$. By the definition of $D_G$, a vertex $x \in V(G)$ is a member of $D_G$ if and only if $\varphi(x) \in D_H$. Thus we obtain $|D_G \cap N_G^+(x)| \geq |D_H \cap N_H^+(y)| \geq k$. Similarly, we can show that $|D_G \cap N_G^-(x)| \geq k$. Hence $D_G$ is a $k$-tuple twin dominating set of $G$. □

3. Twin domination in de Bruijn digraphs

The purpose of this section is to prove the following theorem.

**Theorem 3.1.** For $d \geq 2$, $n \geq 1$, and $1 \leq k \leq d - 1$,

$$\gamma_{x_k}^*(B(d, n)) = kd^{n-1}.$$  

The lower bound of $\gamma_{x_k}^*(B(d, n))$ is shown by the next lemma.

**Lemma 3.2.** $\gamma_{x_k}^*(B(d, n)) \geq kd^{n-1}$ for $1 \leq k \leq d - 1$.

**Proof.** Suppose that $D$ is a $k$-tuple twin dominating set of $B(d, n)$. For a vertex $x$ in $B(d, n)$, let $B_x = N^+(x) \cup N^-(N^+(x))$ (see Property 2.2), and let $c(x) = |D \cap B_x|$. Consider the value of $c(V) = \sum_{x \in V} c(x)$. Let $y \in D$.

1. If $y \in I_{d,n}$, then $y$ is contained in $B_y$, and is also contained in $B_{y^-}$, where $y^- \in N^-(y)$. Thus $y$ is counted in the summation $c(V) = \sum_{x \in V} c(x)$ when $x = y$ or $x \in N^-(y)$. Hence $y$ provides the value $d$ for $c(V)$.

2. If $y = j^{n-1} \in N^-(I_{d,n})$, $j \neq i$, then $y$ is counted in $c(V) = \sum_{x \in V} c(x)$ when $x \in N^-(y)$ or $x = j^{n-1}$ for $0 \leq j' \leq d - 1$. Hence $y$ provides the value $2d$ for $c(V)$.

3. Similarly, if $y \notin I_{d,n} \cup N^-(I_{d,n})$, then $y$ is counted in $c(V) = \sum_{x \in V} c(x)$ when $x \in N^-(y)$ or $x \in N^-(N^+(y))$. Hence $y$ provides the value $2d$ for $c(V)$.

From the above discussion, we have

$$c(V) = d \times |D \cap I_{d,n}| + 2d \times |D \setminus I_{d,n}|$$

$$= 2d|D| - dp, \quad \text{where} \quad p = |D \cap I_{d,n}|. \quad (1)$$

On the other hand, the value $c(x)$ is bounded as follows.

**Claim 3.3.** Let $x \in V(G)$. If $i^n \in D$ and $x = j^{n-1}$ for some $0 \leq j \leq d - 1$, then $c(x) \geq 2k - 1$. Otherwise, $c(x) \geq 2k$. 

Proof of the claim. Suppose that \( x = ji^{n-1} \) for some \( 0 \leq j \leq d - 1 \). Then \( B_x \) is isomorphic to \( \tilde{K}_{d,d} \). Let \( B_x = \{i^n\} \cup L \cup R \), where \( L = N^-(i^n) \) and \( R = N^+(i^n) \). Since \( D \) is a \( k \)-tuple twin dominating set, if \( i^n \in D \), either (i) \( |D \cap L| \geq k - 1 \) and \( |D \cap R| \geq k - 1 \), or (ii) \( |D \cap L| = d - 1 \geq k \) and \( |D \cap R| \geq k - 2 \), or (iii) \( |D \cap L| \geq k - 2 \) and \( |D \cap R| = d - 1 \geq k \). In every case, \( c(x) \geq 2k - 1 \) holds. If \( i^n \notin D \), then \( |D \cap L| \geq k \) and \( |D \cap R| \geq k \). Hence we obtain \( c(x) = |D \cap B_x| \geq 2k \).

If \( x \neq ji^{n-1} \), then \( B_x \) is isomorphic to \( \tilde{R}_{d,d} \) or \( \tilde{K}_{d,d} \). If \( L \cup R \) is the bipartition of \( B_x \), then either (i) \( |D \cap L| \geq k \) and \( |D \cap R| \geq k \), or (ii) \( |D \cap L| = d \geq k + 1 \) and \( |D \cap R| \geq k - 1 \), or (iii) \( |D \cap L| \geq k - 1 \) and \( |D \cap R| = d \geq k + 1 \). Hence we obtain \( c(x) \geq 2k \). \( \square \)

Now we go back to the proof of Lemma 3.2. By Claim 3.3,

\[
c(V) = \sum_{x \in V} c(x) = \sum_{x = ji^{n-1} \text{ and } i^n \in D} c(x) + \sum_{\text{otherwise}} c(x) \\
\geq (2k - 1)dp + 2k(d^n - dp), \quad \text{where } p = |D \cap I_{d,n}|. \\
= 2kd^n - dp.
\]

From Eq. (1) and the above inequality, we obtain \( 2d|D| - dp \geq 2kd^n - dp \), hence \( |D| \geq kd^{n-1} \). \( \square \)

Next we show the upper bound of \( \gamma_{x_k}^*(B(d, n)) \).

Let \( \phi \) be a mapping from the vertex set of \( B(d, n) \) to that of \( B(d, n - 1) \) defined by

\[
\phi(x_1x_2\ldots x_n) = (x_1 \ominus_d x_2)(x_2 \ominus_d x_3)\ldots(x_{n-1} \ominus_d x_n), \tag{2}
\]

where \( \ominus_d \) is subtraction modulo \( d \). We can easily verify that the mapping \( \phi \) is a homomorphism of \( B(d, n) \) onto \( B(d, n - 1) \) [13].

Lemma 3.4. Let \( \phi \) be the homomorphism defined by Eq. (2).

1. \( |\phi^{-1}(y)| = d \) for any \( y \) in \( B(d, n - 1) \).
2. \( \phi(N^+[x]) = N^+[\phi(x)] \) and \( \phi(N^-[x]) = N^-[\phi(x)] \) for any \( x \) in \( B(d, n) \). (Note that the neighborhoods of the left-hand side of the equations are in \( B(d, n) \) and those of the right-hand side are in \( B(d, n - 1) \).)

Proof. The first claim follows from the set of \( n - 1 \) equations \( x_i \ominus_d x_{i+1} = y_i, 1 \leq i \leq n - 1 \), has exactly \( d \) distinct solutions for given \( y_1, y_2, \ldots, y_{n-1} \).

In order to show the second claim, let \( x = x_1x_2\ldots x_n \) be a vertex in \( B(d, n) \) and \( y = \phi(x) = y_1y_2\ldots y_{n-1} \). Suppose that \( x \notin I_{d,n} \). Then \( N^-(x) = \{x_1 \ominus_d x_{n-1} \mid 0 \leq \alpha \leq d - 1\} \). Hence we have \( \phi(N^-(x)) = \{(\alpha \ominus_d x_1)y_1 \ldots y_{n-2} \mid 0 \leq \alpha \leq d - 1\} = \{y_1 \ldots y_{n-2} \mid 0 \leq \alpha \leq d - 1\} = N^-(y) \). Similarly, we have \( N^+(x) = \{x_2 \ldots x_n \mid 0 \leq \beta \leq d - 1\} \). Hence \( \phi(N^+(x)) = \{y_2 \ldots y_{n-1}(x_n \ominus_d \beta) | 0 \leq \beta \leq d - 1\} = \{y_2 \ldots y_{n-1} \ominus_d \beta | 0 \leq \beta \leq d - 1\} = N^+(y) \).

If \( x = i^n \in I_{d,n} \), then \( y = 0^{n-1} \). Hence \( \phi(N^-(x)) = \{0^{n-2} | 1 \leq \alpha \leq d - 1\} = N^-(y) \), and \( \phi(N^+(x)) = \{0^{n-2} | 1 \leq \beta \leq d - 1\} = N^+(y) \). \( \square \)

Lemma 3.5. \( \gamma_{x_k}^*(B(d, n)) \leq kd^{n-1} \) for \( 1 \leq k \leq d \).

Proof. We proceed the proof by induction on \( n \). It is obvious that \( \gamma_{x_k}^*(B(d, 1)) = k \). For \( n \geq 2 \), we assume that \( B(d, n - 1) \) has a \( k \)-tuple twin dominating set \( D_{n-1} \) such that \( |D_{n-1}| \leq kd^{n-2} \). Let \( D_n = \phi^-(D_{n-1}) \). By Lemmas 2.4 and 3.4, the set \( D_n \) is a \( k \)-tuple twin dominating set of \( B(d, n) \), and we obtain \( |D_n| \leq kd^{n-1} \) by Lemma 3.4. \( \square \)

Theorem 3.1 follows from Lemmas 3.2 and 3.5.

Example. Fig. 4 shows the constructed twin dominating sets in \( B(2,3) \) and \( B(2,4) \). In this figure, \( D_3 = \{001, 100, 101, 111\} \). Since

\[
\phi(0001) = \phi((1110) = 001, \quad \phi(0111) = \phi(1000) = 100, \quad \phi(0110) = \phi(1001) = 101, \quad \phi(0101) = \phi(1010) = 111,
\]

we obtain \( D_4 = \{0001, 1110, 0111, 1000, 0110, 1001, 0101, 1010\} \) which is a twin dominating set of \( B(2, 4) \).
It should be noted that Theorem 3.1 and Lemma 3.2 do not hold when \( k = d \). For example, in \( B(2, 3) \), 
\( D = \{000, 001, 011, 100, 101, 110, 111\} \) is a minimum 2-tuple twin dominating set. Hence \( \gamma^*_2(B(2, 3)) = 7 \). The exact value for the \( d \)-tuple domination number for \( B(d, n) \) remains open.

4. Twin domination in Kautz digraphs

In this section, we prove the following theorem.

**Theorem 4.1.** For \( d \geq 2 \) and \( 1 \leq k \leq d - 1 \),
\[
\gamma^*_k(K(d, n)) = \begin{cases} 
k & \text{if } n = 1, 
k(d^{n-1} + d^{n-2}) & \text{if } n \geq 2.
\end{cases}
\]

**Proof.** It is easy to see that \( \gamma^*_k(K(d, 1)) = k \). For \( n \geq 2 \), the claim follows from Lemmas 4.2 and 4.4. \( \square \)

In the remainder of this section, we assume that \( n \geq 2 \).

**Lemma 4.2.** \( \gamma^*_k(K(d, n)) \geq k(d^{n-1} + d^{n-2}) \) for \( 1 \leq k \leq d - 1 \).

**Proof.** The proof is similar to the proof of Lemma 3.2. For a vertex \( x \) in \( K(d, n) \), let \( K_x = N^+(x) \cup N^-(N^+(x)) \) (see Property 2.3).

Suppose that \( D \) is a \( k \)-tuple twin dominating set of \( K(d, n) \). For a vertex \( x \), let \( c(x) = |D \cap K_x| \). By Property 2.3, the subgraph induced by \( K_x \) is isomorphic to \( \overrightarrow{K}_{d,d} \) or \( \overleftarrow{K}_{d,d} \).

Let \( c(x) = |D \cap K_x| \) and \( c(V) = \sum_{x \in V} c(x) \). For a vertex \( y \in D \), it is counted in \( c(V) \) when \( x \in K_y \). Since \( |K_y| = 2d \), we obtain \( c(V) = 2d|D| \).

On the other hand, since \( c(x) \geq 2k \) for any \( x \in D \), we have \( 2d|D| \geq 2k(d^n + d^{n-1}) \). Hence \( |D| \geq k(d^{n-1} + d^{n-2}) \). \( \square \)

For considering the upper bound for Kautz digraphs, we consider a homomorphism of a Kautz digraph onto a de Bruijn digraph.

Define a mapping \( \tau \) from \( V(K(d, n)) \) to \( V(B(d, n - 1)) \) by
\[
\tau(x_1x_2\ldots x_n) = (x_1 \boxplus_d x_2)(x_2 \boxplus_d x_3)\cdots(x_{n-1} \boxplus_d x_n),
\tag{3}
\]
where \( x_i \boxplus_d x_{i+1} = (x_i - x_{i+1} - 1) \mod (d + 1) \). It is easy to see that \( \tau \) is a surjection and preserves adjacency. Hence \( \tau \) is a homomorphism of \( K(d, n) \) onto \( B(d, n - 1) \).

**Lemma 4.3.** Let \( \tau \) be the mapping defined by Eq. (3).
1. \(|\tau^{-}(y)| = d + 1\) for any \(y\) in \(B(d, n - 1)\).
2. For any \(x\) in \(K(d, n)\), \(\tau(N^{-}[x]) = N^{-}[\tau(x)]\) and \(\tau(N^{+}[x]) = N^{+}[\tau(x)]\).

**Proof.** The first claim follows from the set of \(n - 1\) equations \(x_{i} \sqcup \tau x_{i+1} = y_{i}\), \(1 \leq i \leq n\), has exactly \(d + 1\) distinct solutions for given \(y_{1}, y_{2}, \ldots, y_{n}\).

Let \(x = x_{1}x_{2}\ldots x_{n}\) be a vertex in \(K(d, n)\) and \(y = \tau(x) = y_{1}y_{2}\ldots y_{n-1}\). Then \(N^{-}[x] = \{x\} \cup \{\alpha x_{1}\ldots x_{n-1} \mid 0 \leq \alpha \leq d, \alpha \neq x_{1}\}\). Hence we have \(\tau(N^{-}[x]) = \{y\} \cup \{ \alpha \tau(x_{1})\ldots y_{n-2} \mid 0 \leq \alpha \leq d - 1, \alpha \neq x_{1}\}\) = \(\{y\} \cup \{\alpha y_{1}\ldots y_{n-2} \mid 0 \leq \alpha \leq d - 1\}\) = \(N^{-}[y]\). Similarly, we have \(N^{+}[x] = \{x\} \cup \{x_{2}\ldots x_{n}\beta \mid 0 \leq \beta \leq d - 1, \beta \neq x_{n}\}\). Hence \(\tau(N^{+}[x]) = \{y\} \cup \{y_{2}\ldots y_{n-2}\beta \mid 0 \leq \beta \leq d - 1\}\) = \(N^{+}[y]\).

**Example.** In \(K(2, 3)\), we have

\[
\begin{align*}
\tau(021) &= \tau(102) = \tau(210) = 00, \\
\tau(020) &= \tau(101) = \tau(212) = 01, \\
\tau(010) &= \tau(121) = \tau(202) = 10, \\
\tau(012) &= \tau(120) = \tau(201) = 11.
\end{align*}
\]

So, \(\tau\) is a surjection. An arc \((021, 210) \in A(K(2, 3))\) is mapped to \((00, 00) \in A(B(2, 2))\), and \((020, 201) \in A(K(2, 3))\) is mapped to \((01, 11) \in A(B(2, 2))\) by \(\tau\). From the above equations,

1. \(|\tau^{-}(00)| = |\tau^{-}(01)| = |\tau^{-}(10)| = |\tau^{-}(11)| = 3.
2. (a) \(\tau(N^{-}[020]) = \tau([020, 102, 202]) = \{01, 00, 10\}\) and \(N^{-}[\tau(020)] = N^{-}[01] = \{01, 00, 10\}.

(b) \(\tau(N^{-}[021]) = \tau([021, 102, 202]) = \{00, 10\}\) and \(N^{-}[\tau(021)] = N^{-}[00] = \{00, 10\}.

**Lemma 4.4.** \(y_{x_{k}}^{\ast}(K(d, n)) \leq k(d^n - 1 + d^{n-2})\) for \(1 \leq k \leq d + 1\).

**Proof.** Since \(K(d, n)\) has \(d^n + d^{n-1}\) vertices, the inequality clearly holds for \(k = d\) and \(d + 1\). So we assume that \(1 \leq k \leq d - 1\).

Suppose that \(D_{B}\) is a minimum \(k\)-tuple twin dominating set of \(B(d, n - 1)\). By **Theorem 3.1**, \(|D_{B}| = kd^{n-2}\).

Let \(D_{K} = \tau^{-}(D_{B})\). By **Lemma 2.4**, the set \(D_{K}\) is a \(k\)-tuple twin dominating set of \(K(d, n)\), and we obtain \(|D_{K}| = kd^{n-2}(d + 1)\) by **Lemma 4.3**.

**Example.** Fig. 5 shows an example of the constructed twin dominating sets in \(K(2, 4)\) and \(B(2, 3)\). In this figure, \(D_{B} = \{001, 100, 101, 111\}\). Since

\[
\tau(0212) = \tau(1020) = \tau(2101) = 001,
\]
\[
\tau(0102) = \tau(1210) = \tau(2021) = 100,
\tau(0101) = \tau(1212) = \tau(2020) = 101,
\tau(0120) = \tau(1201) = \tau(2012) = 111,
\]
we obtain a twin dominating set \(D_K\) of \(K(2, 4)\) that consists of the 12 vertices in \(\tau^{-1}(D_B)\).

Note that Theorem 4.1 does not hold when \(k = d\) and \(d+1\). For example, \(D = \{10, 02, 20, 12, 21\}\) is a minimum 2-tuple twin dominating set in \(K(2, 2)\). Hence \(\gamma^*_2(K(2, 2)) = 5\). Determining the value of the \(d\)-tuple twin domination number for \(K(d, n)\) remains open. For \(k = d + 1\), clearly we have \(\gamma^*_{d+1}(K(d, n)) = d^n + d^{n-1}\).

5. Conclusion

In this paper, we have introduced a generalized concept of twin domination, the \(k\)-tuple twin domination in digraphs. Then we have presented the \(k\)-tuple twin domination numbers of de Bruijn and Kautz digraphs. We have described methods for constructing \(k\)-tuple twin dominating sets by using homomorphisms of de Bruijn/Kautz digraphs onto de Bruijn digraphs. If the reader is interested in studies for topological properties and algorithms of de Bruijn and Kautz digraphs by using homomorphism, see the papers [6,13]. The twin domination of classes of regular digraphs, for example, Cayley digraphs, is an interesting problem.

References