Monotonicity characteristic of Köthe–Bochner spaces

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A B S T R A C T

Estimates of the characteristic of monotonicity in Köthe–Bochner function spaces $E(X)$
with some consequences are given. Characterizations of strict and uniform monotonicity
of the sequence lattice $e((X_n)_{n=1}^{\infty})$ are obtained.

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1. Introduction

The problem of finding exact values or obtaining optimal estimates of various kind of characteristics in various Banach spaces or Banach lattices has been considered in literature by many authors. We will cite below the most important results connected with the characteristic of convexity, the characteristic of orthogonal convexity and the characteristic of monotonicity, respectively. We start with the characteristic of convexity (called also convexity coefficient). So far this characteristic has been studied for some Köthe spaces (see [11,17,19,21,24] or [25]) and its relation to the fixed point theory has been observed (see [13,28,30] or [42]). Besides L.L. Fang and H. Hudzik presented a new formula for the convexity coefficient of Orlicz spaces in [14] and S.H. Shu showed in [51] a relationship between the convexity characteristic and a smooth modulus. It is also worth to mention about Lindenstrauss formulae, which show us some relationship between Lindenstrauss modulus of smoothness of a Banach space $X$ (respectively $X^*$) and the modulus of convexity of $X^*$ (respectively $X$). Consequently, relationships between the characteristic of smoothness of a Banach space $X$ (respectively $X^*$) and the characteristic of convexity of its dual space $X^*$ (respectively $X$) are given (see [40] or [41]). For more information about the modulus and the characteristic of monotonicity we refer to [30] or [16]. The coefficient of orthogonal convexity in Köthe spaces was considered in [31] by P. Kolwicz and S. Rolewicz. Moreover, they showed in the paper the best estimates of the coefficient of orthogonal convexity in Köthe–Bochner spaces $E(X)$. Finally, let us mention a few words about the characteristic of monotonicity. Namely, Y. Lü, J. Wang and T. Wang in [44] calculated the values of the characteristic of monotonicity of Orlicz spaces equipped with the Luxemburg and with the Orlicz norm as well. Some of their results were improved in [20], where the characteristic was calculated for Orlicz spaces with the Luxemburg norm not excluding cases when an Orlicz function $\Phi$ vanishes outside zero or has infinite values. A. Betiuk-Pilarska and S. Prus have recently showed a crucial theorem, which says that if $X$ is a weakly orthogonal Banach lattice with $\varepsilon_{q,m}(X) < 1$, then $X$ has the weak normal structure. Consequently, $X$ has then the weak fixed point property (see [2]). Recall also that for any real Köthe space $E$ strict monotonicity and

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uniform monotonicity of $E$ coincide, respectively, with complex rotundity and complex uniform rotundity of the complexification $E^\ast$ of $E$. This result was obtained independently by H. Hudzik and A. Narloch (see [27]) and H. Ju Lee (see [37]).

Recall that complex rotundity was introduced by E. Thorp and R. Whitley (see [52]), while the notion of complex uniform rotundity was introduced by J. Globevnik (see [15]). Recall also that complex rotundity properties have applications in the theory of vector-valued analytic functions. Namely, it is known that if $f$ is a function from the unit disc $B(C)$ in the field of convex numbers $C$ into a complex Banach space $X$ and $f$ is analytic, i.e. $x^* \circ f$ is analytic in the classical sense for any $x^* \in X^\ast$ (= the dual space of $X$) and the supremum of the function $F(z) = \|f(z)\|_X$ is attained in an interior point $z_0 \in B(C)$, then $f$ is a constant function. But in the case when $X$ is $C$-rotund we can also deduce that $f$ is a constant function. For others geometric problems in Köthe–Bochner spaces we refer to [5–10,22,24,26,29,31–33,37–39,46–48] and [49].

Let $(\Omega, \Sigma, \mu)$ be a complete and $\sigma$-finite measure space and let $E$ denote a Köthe space on the measure space $(\Omega, \Sigma, \mu)$. For a given Banach space $X$, the Köthe–Bochner space $E(X)$ is the linear space of all (equivalence classes of) strongly $\Sigma$-measurable functions $x$ from $\Omega$ to $X$ such that the real function $|x| : \Omega \to \mathbb{R}$ defined by

$$
|x| (\omega) = \|x(\omega)\|_X
$$

belongs to $E$, endowed with the norm $\|x\|_{E(X)} = \||x||_E$. Obviously, $E(X)$ is a Banach space. If additionally, $X$ is a Banach lattice, then $E(X)$ is a Banach lattice (with $x \leq y$ iff $x(\omega) \leq y(\omega)$ a.e.) as well. For more facts about Köthe–Bochner spaces and their geometry, we refer to [8] and [39].

We introduce the following notations: $S_X$ and $B_X$ denote the unit sphere and the unit ball of $X$, respectively, and for $x \in X \setminus \{0\}$ we write $\hat{x} = x/\|x\|$ and $\bar{x} = \overline{x/\|x\|}$. By the notation $1_A$ we will mean the characteristic function of a given set $A$. Moreover, let us denote by $X_+$ a Banach lattice $(X, \|\cdot\|, \leq)$ and by $X_+$ the positive cone of $X$. Let $S_+ = S(X) \cap X_+$ and let us denote by $a \oplus b - c \ominus d$ the segment of $S_+ (\mathbb{R}^2)$ linking the points $a \oplus b := (a, b)$ and $c \ominus d := (c, d)$ in $\mathbb{R}^2$. A Banach lattice $X$ is said to be strictly monotone (in $(SM)$) if for all $x, y \in X_+$ such that $y \leq x$ and $x \neq y$, we have $\|y\| < \|x\|$. Equivalently, we say that $X$ is strictly monotone, if for all $y \in X_+$ and $x \in S_+(X)$ such that $y \leq x$ and $x \neq y$, we have $\|x - y\| < \|x\|$. A Banach lattice $X$ is said to be uniformly monotone (in $(UM)$) if for any $\epsilon \in (0, 1)$ there is $\delta(\epsilon) \in (0, 1)$ such that $\|x - y\| \leq 1 - \delta(\epsilon)$ whenever $0 \leq y \leq x$, $\|x\| = 1$ and $\|y\| \geq \epsilon$. It is worth noticing that in the case of finite dimensional Banach lattices, thanks to the compactness of the unit ball in such lattices and continuity of the function $1 - \|x - y\|$, uniform monotonicity is not distinguished from strict monotonicity. Besides monotonicity properties of Banach lattices $X$ are strictly related to their convexity properties on the positive cone of $X$ (see [18]). Recall also that monotonicity properties have various applications, among others in the ergodic theory (see [1]), in the dominated best approximation as well as in approximation problems (see [4,23,35] and [36]) or in the fixed point theory (see [2,12]).

For a given Banach lattice $X$, the function $\delta_{m,X} : [0, 1] \to [0, 1]$ defined as

$$
\delta_{m,X}(\epsilon) = \inf \left\{ 1 - \|x - y\| : 0 \leq y \leq x, \quad \|x\| = 1, \quad \|y\| \geq \epsilon \right\}
$$

is said to be the modulus of monotonicity of $X$. Obviously, $X$ is uniformly monotone if and only if $\delta_{m,X}(\epsilon) > 0$ for every $\epsilon \in (0, 1)$. It is easy to see that a Banach lattice $X$ is strictly monotone if and only if $\delta_{m,X}(1) = 1$. The modulus of monotonicity $\delta_{m,X}$ is a convex function on the interval $[0, 1]$ which is continuous on the interval $[0, 1)$ and nondecreasing with respect to $\epsilon \in [0, 1]$ (see [34]). The number $\varepsilon_{0,m}(X) \in [0, 1]$ defined by

$$
\sup \left\{ \epsilon \in [0, 1] : \delta_{m,X}(\epsilon) = 0 \right\} = \inf \left\{ \epsilon \in [0, 1] : \delta_{m,X}(\epsilon) > 0 \right\}
$$

is said to be the characteristic of monotonicity of $X$. A Banach lattice $X$ is uniformly monotone if and only if $\varepsilon_{0,m}(X) = 0$. More information and facts about Banach lattices can be found in [3,4,41,43,45] or [50].

2. Auxiliary facts

In the following $E$ denotes a Köthe space and $X$ denotes a Banach lattice even if it is not indicated.

Remark 1. Let $\epsilon \in (0, 1)$. Then

$$
\delta_{m,X}(\epsilon) = \inf \left\{ 1 - \|x - y\| : x \in S_+(X), \quad 0 \leq y \leq x, \quad \|y\| \geq \epsilon \right\} = \inf \left\{ 1 - \|z\| : 0 \leq z \leq x \in S_+(X), \quad \|x - z\| \geq \epsilon \right\}.
$$

Fact 1. The following conditions are equivalent:

(a) For all $\epsilon \in (0, 1)$ there is $\bar{\delta}(\epsilon) \in (0, 1)$ such that $\|y\| \geq \epsilon$ then $\|x - y\| \leq 1 - \bar{\delta}(\epsilon)$ for all $0 \leq y \leq x$ and $x \in S_+(X)$.

(b) For all $(x_n)_{n=1}^{\infty} \subset S_+(X)$ and $(y_n)_{n=1}^{\infty} \subset X_+$ if $0 \leq y_n \leq x_n$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \|y_n\| = 1$ then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Since the proof of the above fact is easy, we omitted it here. Now we will prove some Lemmas that will be used in the proof of our main Theorem.

Lemma 1. Let $X$ be a Banach lattice, $x, y \in X_+$ and $0 \leq y \leq x$. Then

$$
\|x - y\| \leq \|x\| - \|y\| + \|\hat{x} - \bar{y}\|.
$$

(1)
Proof. Since \( \|y\| \leq \|x\| \) we have
\[
x - y \leq x - \frac{\|y\|}{\|x\|} y = x + \left( \frac{\|y\|}{\|x\|} x - \frac{\|y\|}{\|x\|} y \right) - \frac{\|y\|}{\|x\|} x = x \left( \frac{\|x\| - \|y\|}{\|x\|} \right) + \|y\| \left( \frac{x}{\|x\|} - \frac{y}{\|x\|} \right).
\]
Therefore
\[
\|x - y\| \leq \|x\| - \|y\| + \|y\| \left( \frac{\|x\| - \|y\|}{\|x\|} \right) = \|x\| - \|y\| + \|y\| \|\hat{x} - \hat{y}\|.
\]
which ends the proof. \(\square\)

Corollary 1. If additionally \( \|\hat{x} - \hat{y}\| \leq \gamma \), then
\[
\|x - y\| \leq \|x\| - \|y\| + \|y\| \gamma = \|x\| - (1 - \gamma) \|y\|.
\]

Proof. It is obvious. \(\square\)

Lemma 2. Let \( X \) be a Banach lattice, \( x, y \in X_+ \), \( 0 \leq y \leq x \) and \( \varepsilon \in (0, 1) \). If \( \|\tilde{y}\| \geq \varepsilon \), then
\[
\|x - y\| \leq \|x\| - \|y\| \delta_{m,X}(\varepsilon).
\]

Proof. Assume that \( x, y \in X_+ \), \( 0 \leq y \leq x \) and \( \|\tilde{y}\| \geq \varepsilon \), where \( \varepsilon \in (0, 1) \). Then
\[
\delta_{m,X}(\varepsilon) = 1 - \|\hat{x} - \hat{y}\|.
\]
Thus \( \|\hat{x} - \hat{y}\| \leq 1 - \delta_{m,X}(\varepsilon) \). Applying inequality (1) we get
\[
\|x - y\| \leq \|x\| - \|y\| + \|y\| (1 - \delta_{m,X}(\varepsilon)).
\]
Hence \( \|x - y\| \leq \|x\| - \|y\| \delta_{m,X}(\varepsilon) \). \(\square\)

Lemma 3. Let \( \varepsilon \in (0, 1) \). If \( \delta_{m,X}(\varepsilon) > 0 \), then for all sequences \( (x_n) \) in \( S_+(X) \) and \( (y_n) \) in \( B_+(X) \) such that \( 0 \leq y_n \leq x_n \) for every \( n \in \mathbb{N} \) and \( \|y_n\| \rightarrow 1 \), we have \( \limsup_{n \rightarrow \infty} \|x_n - y_n\| \leq \varepsilon \).

Proof. Fix arbitrary \( \varepsilon \in (0, 1) \). We only need to observe that in view of our assumption and Remark 1 we have
\[
\delta_{m,X}(\varepsilon) = \inf \left\{ 1 - \|y\| : 0 \leq y \leq x \in S_+(X), \|x - y\| \geq \varepsilon \right\} > 0,
\]
whence our thesis follows. Indeed, if there were sequences \( (x_n) \) in \( S_+(X) \) and \( (y_n) \) in \( B_+(X) \) such that \( 0 \leq y_n \leq x_n \) for every \( n \in \mathbb{N} \), \( \|y_n\| \rightarrow 1 \) and \( \limsup_{n \rightarrow \infty} \|x_n - y_n\| > \varepsilon \), then we would find subsequences \( (x_{n_k}) \) and \( (y_{n_k}) \) of given sequences such that \( \lim_{k \rightarrow \infty} \|x_{n_k} - y_{n_k}\| > \varepsilon \). Hence there is \( l \in \mathbb{N} \) such that \( \|x_{n_k} - y_{n_k}\| > \varepsilon \) for every \( k \geq l \). By virtue of (2) we get
\[
0 \leq \delta_{m,X}(\varepsilon) \leq \inf_{k \geq l} \left( 1 - \|y_{n_k}\| \right).
\]
Therefore \( \delta_{m,X}(\varepsilon) = 0 \). \(\square\)

Lemma 4. If for some \( \varepsilon \in (0, 1) \) the condition that for all sequences \( (x_n) \) in \( S_+(X) \) and \( (y_n) \) in \( B_+(X) \) such that \( 0 \leq y_n \leq x_n \) for every \( n \in \mathbb{N} \) and \( \|y_n\| \rightarrow 1 \) implies that \( \limsup_{n \rightarrow \infty} \|x_n - y_n\| \leq \varepsilon \), then \( \delta_{m,X}(\varepsilon) > 0 \) for all \( \varepsilon \in (\varepsilon, 1) \).

Proof. Assume that the assumptions of the lemma are satisfied and there exists \( \tilde{\varepsilon} \in (\varepsilon, 1) \) such that \( \delta_{m,X}(\tilde{\varepsilon}) = 0 \). Then
\[
0 = \delta_{m,X}(\tilde{\varepsilon}) = \inf \left\{ 1 - \|y\| : x \in S_+(X), 0 \leq y \leq x, \|x - y\| \geq \tilde{\varepsilon} \right\}.
\]
Therefore we can find sequences \( (x_n), (y_n) \) in \( X_+ \) such that \( \|x_n\| = 1, 0 \leq y_n \leq x_n, \|x_n - y_n\| \geq \tilde{\varepsilon} \) for all \( n \in \mathbb{N} \) and \( 1 - \|y_n\| \rightarrow 0 \), i.e., \( \|y_n\| \rightarrow 1 \). Hence
\[
\limsup_{n \rightarrow \infty} \|x_n - y_n\| \geq \tilde{\varepsilon} > \varepsilon,
\]
a contradiction, which finishes the proof. \(\square\)

Lemma 5. Let \( X \) be a Banach lattice and \( \varepsilon_{0,m}(X) = \alpha \), where \( \alpha \neq 1 \). There are sequences \( (x_n) \) in \( S_+(X) \) and \( (y_n) \) in \( X_+ \) such that \( 0 \leq y_n \leq x_n \) for all \( n \in \mathbb{N} \), \( \lim_{n \rightarrow \infty} \|y_n\| = 1 \) and \( \lim_{n \rightarrow \infty} \|x_n - y_n\| = \alpha \).
Proof. If $\varepsilon_{0,m}(X) = 0$, then Fact 1 ends the proof. Assume that $0 < \alpha < 1$ and take a sequence $(\varepsilon_n)$ in the interval $(0, 1)$ with $\varepsilon_n \not\to \alpha$. Since for arbitrary $n \in \mathbb{N}$,

$$0 = \delta_{m,X}(\varepsilon_n) = \inf \left\{ \|x - y\| : 0 \leq y \leq x \in S_+(X), \|x - y\| = \varepsilon_n \right\},$$

we can find sequences $(x_n)$ in $S_+(X)$ and $(y_n)$ in $X_+$ such that $\lim_{n \to \infty} \|x_n - y_n\| = \lim_{n \to \infty} \varepsilon_n = \alpha$ and $\lim_{n \to \infty} \|y_n\| = 1$. □

Recall also two results from [20], which will be useful in proofs of some corollaries that will follow from the main theorem. More precisely, we will need Corollary 2 in order to prove nontrivial corollaries of the main theorem.

**Theorem 1.** For any Banach lattice $X$ the following formula for its characteristic of monotonicity holds true:

$$\varepsilon_{0,m}(X) = \sup \left\{ \limsup_{n \to \infty} \|x_n - y_n\| : x_n \in S_+(X), (x_n) \subseteq B_+(X), 0 \leq x_n \leq x \forall n \in \mathbb{N} \right\}. \quad (3)$$

**Corollary 2.** In a finite dimensional Banach lattice $X$ the characteristic of monotonicity is just the length of the longest interval lying in the intersection of the unit sphere of $X$ and $X_+$, i.e.

$$\varepsilon_{0,m}(X) = \sup \left\{ \|x - y\| : 0 \leq y \leq x, \|x\| = \|y\| = 1 \right\} = \max \left\{ \|x - y\| : 0 \leq y \leq x, \|x\| = \|y\| = 1 \right\}.$$

The proof of this corollary follows from Theorem 1 and from compactness of $B(X)$ and $S(X)$ if $X$ is finite dimensional.

3. Results

Now we are ready to present the main result of this paper.

**Theorem 2.** For arbitrary Köthe space $E$ and Banach lattice $X$ the following estimates hold true:

$$\varepsilon_{0,m}(E) \lor \varepsilon_{0,m}(X) \leq \varepsilon_{0,m}(E(X)) \leq \varepsilon_{0,m}(E) - \varepsilon_{0,m}(E)\varepsilon_{0,m}(X) + \varepsilon_{0,m}(X).$$

**Proof.** The lower bound is clear since both spaces $E$ and $X$ are order-isometrically embedded into $E(X)$. We will prove the upper bound. Without loss of generality, we can assume that $\eta := \varepsilon_{0,m}(E) < 1$ and $\alpha := \varepsilon_{0,m}(X) < 1$. Choose $(x_n)$ in $S_+(E(X))$ and $(y_n)$ in $E_+(X)$ such that $0 \leq y_n \leq x_n$ for all $n \in \mathbb{N}$ with $\lim_{n \to \infty} \|y_n\|_{\varepsilon(E(X))} = 1$ and $\lim_{n \to \infty} \|x_n - y_n\|_{\varepsilon(E(X))} = \varepsilon_{0,m}(E(X)) = \varepsilon_{0,m}(E(X))$. Denote: $u_n = [x_n], v_n = [y_n], d_n = [x_n - y_n], D_n = [x_n] - [y_n] = u_n - v_n$, where $[x_n]_0 = \varepsilon_{0,m}(X)$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \|D_n\|_{\varepsilon(X)} = 0$, where $\gamma_n = \eta \delta_{m,X}(\varepsilon_n)$. The last observation need to be explained. From the assumption that $\|v_n\| \to 1$ as $n \to \infty$ it follows that $\lim_{n \to \infty} (\|u_n\| - \|v_n\|) = 0$. Denote $\alpha = \varepsilon_{0,m}(X)$ and let $\alpha < \varepsilon_n < 1$ for arbitrary $n \in \mathbb{N}$. Without loss of generality we can assume that $\delta_{m,X}(\varepsilon_n) > 0$. Let us denote $\delta = \delta_{m,X}(\varepsilon_n)$. Since $\lim_{n \to \infty} (\|u_n\| - \|v_n\|) = 0$, then there is $m \in \mathbb{N}$ such that for all $n \geq m$ we have $\|u_n\| - \|v_n\| < \delta$. Darboux property of the function $\delta_{m,X}(\cdot)$ on any compact interval contained in the interval $(0, 1)$ and the fact that $\delta \in (0, 1]$ yield that

$$\forall n \in \mathbb{N} \exists \alpha < \varepsilon_n < 1: \delta_{m,X}(\varepsilon_n) = \frac{\|u_n\| - \|v_n\|}{\varepsilon_n} = \frac{1}{\|u_n\| - \|v_n\|} \leq \delta.$$

Therefore, from the fact that $\delta_{m,X}(\varepsilon_n) \leq \varepsilon_n$ for every $n \in \mathbb{N}$, we get that $\frac{\|u_n\| - \|v_n\|}{\varepsilon_n} \leq \varepsilon_n$ for arbitrary $n \in \mathbb{N}$.

Define $\eta_n = (\|u_n\| - \|v_n\|)_{\varepsilon_n}$. Then $\eta_n = (\|u_n\| - \|v_n\|)_{\varepsilon_n} \leq \varepsilon_n$ and $\eta_n \delta_{m,X}(\varepsilon_n) = (\|u_n\| - \|v_n\|)_{\varepsilon_n}$ for all $n \in \mathbb{N}$, and

$$\frac{\|u_n\| - \|v_n\|}{\eta_n \delta_{m,X}(\varepsilon_n)} = \left( \frac{\|u_n\| - \|v_n\|}{\varepsilon_n} \right) \to 0,$$

as $n \to \infty$. Moreover $\varepsilon_n \to \alpha$ as $n \to \infty$. Indeed, in the interval $(\alpha, 1)$ the modulus of monotonicity $\delta_{m,X}(\cdot)$ is a strictly increasing function because it is convex. Therefore, $\varepsilon_n \in (\alpha, 1)$ and $\lim_{n \to \infty} \delta_{m,X}(\varepsilon_n) = 0$ imply that $\lim_{n \to \infty} \varepsilon_n = \alpha$. Now we can come back to the proof of our theorem.

Define the sets

$$A_n^\alpha = \left\{ \omega \in \Omega : \frac{(x_n - y_n)(\omega)}{\|x_n(\omega)\|} \geq \varepsilon_n \right\}.$$

Then

$$A_n^\leq = \Omega \setminus A_n^\alpha = \left\{ \omega \in \Omega : \frac{(x_n - y_n)(\omega)}{\|x_n(\omega)\|} \leq \varepsilon_n \right\}.$$

Next define

$$f_n = v_n 1_{A_n^\leq} - v_n 1_{A_n^\alpha} = v_n 1_{A_n^\leq}.$$
Then $0 \leq f_n \leq u_n 1_{\Omega}$ for all $n \in \mathbb{N}$. Applying Lemma 2 to $x_0(\omega), x_n(\omega) - y_n(\omega)$ for $\omega \in A_n^\circ$ and denoting $\gamma_n = \eta_n \delta_{m,X}(\varepsilon_n)$, we get for all $\omega \in A_n^\circ$:

$$
\|x_n(\omega) - (x_0(\omega) - y_n(\omega))\|_X = \|y_n(\omega)\|_X
\leq \|x_n\|_X - \|x_0(\omega) - y_n(\omega)\|_{X_{m,X}(\varepsilon_n)}
\leq \|x_0(\omega)\|_X (1 - \varepsilon_n \delta_{m,X}(\varepsilon_n))
\leq \|x_0(\omega)\|_X (1 - \eta_n \delta_{m,X}(\varepsilon_n))
= \|x_0(\omega)\|_X (1 - \gamma_n),
$$

i.e. between the functions $v_n$ and $u_n$ on the set $A_n^\circ$ we have the following inequality

$$v_n < u_n (1 - \gamma_n).$$

Hence

$$
\|v_n\|_E = \|v_n 1_{\Omega \setminus A_n^\circ} + v_n 1_{A_n^\circ}\|_E
\leq \|v_n 1_{\Omega \setminus A_n^\circ} + (1 - \gamma_n) u_n 1_{A_n^\circ}\|_E
= \|v_n 1_{A_n^\circ} + (1 - \gamma_n) u_n - (1 - \gamma_n) u_n 1_{A_n^\circ}\|_E
\leq \|v_n 1_{A_n^\circ} + (1 - \gamma_n) u_n - (1 - \gamma_n) v_n 1_{A_n^\circ}\|_E
\leq \|v_n 1_{A_n^\circ} + (1 - \gamma_n) u_n - v_n 1_{A_n^\circ} + \gamma_n v_n 1_{A_n^\circ}\|_E
\leq (1 - \gamma_n) \|u_n\|_E + \gamma_n \|v_n 1_{A_n^\circ}\|_E.
$$

whence

$$
\gamma_n \|v_n 1_{A_n^\circ}\|_E > \|v_n\|_E - (1 - \gamma_n) \|u_n\|_E
$$

i.e.

$$
\|v_n 1_{A_n^\circ}\|_E > \|u_n\|_E - \frac{1}{\gamma_n} (\|u_n\|_E - \|v_n\|_E) \to 1. \tag{4}
$$

Inequality (4) and $\|f_n\| \leq 1$ yield

$$
\|f_n\|_E = \|v_n 1_{A_n^\circ}\|_E \to 1.
$$

Moreover

$$
\|u_n - f_n\|_E = \|u_n - v_n 1_{A_n^\circ}\|_E = \|u_n 1_{A_n^\circ} + u_n 1_{A_n^\circ} - v_n 1_{A_n^\circ}\|_E \geq \|d_n 1_{A_n^\circ} + (u_n - v_n) 1_{A_n^\circ}\|_E. \tag{5}
$$

Applying Corollary 1 to $x_0(\omega)$ and $y_n(\omega)$ with $\omega \in A_n^\circ$, we conclude that on the set $A_n^\circ$ we have

$$d_n \leq u_n - (1 - \varepsilon_n) v_n.$$

Thus

$$
\|d_n\|_E \leq \|d_n 1_{\Omega \setminus A_n^\circ} + [u_n - (1 - \varepsilon_n) v_n] 1_{A_n^\circ}\|_E
\leq \|\varepsilon_n [d_n 1_{A_n^\circ} + u_n 1_{A_n^\circ}] + (1 - \varepsilon_n) [d_n 1_{A_n^\circ} + u_n 1_{A_n^\circ} - v_n 1_{A_n^\circ}]\|_E
\leq \varepsilon_n + (1 - \varepsilon_n) \|d_n 1_{A_n^\circ} + u_n 1_{A_n^\circ} - v_n 1_{A_n^\circ}\|_E
= \varepsilon_n + (1 - \varepsilon_n) \|d_n 1_{A_n^\circ} + D_n 1_{A_n^\circ}\|_E.
$$

whence

$$
\|d_n 1_{A_n^\circ} + D_n 1_{A_n^\circ}\|_E \geq \frac{\|d_n\|_E - \varepsilon_n}{1 - \varepsilon_n} \to \frac{E - \alpha}{1 - \alpha}. \tag{6}
$$

By virtue of inequalities (5) and (6), we get

$$
\liminf_{n \to \infty} \|u_n - f_n\|_E \geq \lim_{n \to \infty} \frac{\|d_n\|_E - \varepsilon_n}{1 - \varepsilon_n} = \frac{E - \alpha}{1 - \alpha}.
$$

Moreover, Lemma 3 gives that $\limsup_{n \to \infty} \|u_n - f_n\|_E \leq \eta$. Therefore

$$
\frac{E - \alpha}{1 - \alpha} \leq \eta.
$$
i.e. \( \varepsilon \leq \alpha + \eta - \alpha \eta \), whence
\[
\varepsilon \leq \varepsilon_0,m(E(X)) \leq \varepsilon \leq \varepsilon_0,m(E) - \varepsilon_0,m(E)\varepsilon_0,m(X) + \varepsilon_0,m(X),
\]
and the proof is finished. \( \square \)

Now we will present some corollaries of Theorem 2.

**Corollary 1.** If \( X \) is an uniformly monotone Banach lattice, then
\[
\varepsilon_0,m(E(X)) = \varepsilon_0,m(E)
\]
for any Köthe space.

**Corollary 2.** Let \( E \) be a Köthe space. If \( E \) is uniformly monotone, then
\[
\varepsilon_0,m(E(X)) = \varepsilon_0,m(X)
\]
for any Banach lattice \( X \).

**Corollary 3.** The Köthe–Bochner space \( E(X) \) is uniformly monotone if and only if both lattices \( E \) and \( X \) are uniformly monotone. In particular, every finite dimensional Banach lattice \( E(X) \) is strictly monotone if and only if both finite dimensional Banach lattices \( E \) and \( X \) are strictly monotone.

Proofs of Corollaries 1, 2 and 3 are clear.

**Corollary 4.** The characteristic of monotonicity of the Köthe–Bochner lattice \( E(X) \) is equal to one if and only if the characteristic of monotonicity of \( E \) is equal to one or the characteristic of monotonicity of \( X \) is equal to one.

**Proof.** If \( \varepsilon_0,m(E) = 1 \) or \( \varepsilon_0,m(X) = 1 \), then obviously \( \varepsilon_0,m(E(X)) = 1 \). Therefore, we only need to prove the opposite implication. In order to do it, assume that \( \varepsilon_0,m(E) < 1 \) and \( \varepsilon_0,m(X) < 1 \) and let us denote
\[
\varepsilon_0,m^\max = \max\{\varepsilon_0,m(X), \varepsilon_0,m(E)\}, \quad \varepsilon_0,m^\min = \min\{\varepsilon_0,m(X), \varepsilon_0,m(E)\}.
\]
Then \( \varepsilon_0,m^\max < 1 \), whence by virtue of Theorem 2, we obtain
\[
\varepsilon_0,m(E(X)) \leq \varepsilon_0,m^\max(1 - \varepsilon_0,m^\min) + \varepsilon_0,m^\min < 1 - \varepsilon_0,m^\min + \varepsilon_0,m^\min = 1. \quad \square
\]

**Corollary 5.** For every \( \alpha, \eta \in (0, 1) \) and \( \varepsilon \in (\alpha \vee \eta, \alpha - \alpha \eta + \eta] \) there exists a Köthe space \( E \) with \( \varepsilon_0,m(E) = \eta \) such that \( \varepsilon_0,m(E(X)) = \varepsilon \) for any Banach lattice \( X \) with \( \varepsilon_0,m(X) = \alpha \).

**Proof.** Let \( \alpha, \eta \in (0, 1) \) and take as \( E \) the normed space \( \mathbb{R}^2 \), which can be considered as a 2-dimensional subspace of the sequence space \( \ell^2 \), such that the positive part of its unit sphere is the set
\[
S_+(E) = 0 \oplus 1 - 1 \oplus 1 - \frac{1}{\varepsilon} \oplus \frac{\alpha}{\varepsilon} - \frac{1}{\eta} \oplus 0,
\]
where \( \varepsilon \in (\alpha \vee \eta, \alpha - \alpha \eta + \eta] \). Observe that the norm of \( S(E) \) is for all \((x_1, x_2) \in \mathbb{R}^2 \) defined by the formula
\[
\|(x_1, x_2)\| = \max\left\{|x_2|, \frac{\varepsilon - \alpha}{1 - \alpha} |x_1| + \frac{1 - \varepsilon}{1 - \alpha} |x_2|, \eta |x_1| + \frac{\varepsilon - \eta}{\alpha} |x_2|\right\},
\]
where \( 1 < \frac{1}{\varepsilon} < \frac{1}{\alpha}, 0 < \frac{\alpha}{\varepsilon}, \frac{\alpha}{\eta} < 1 \) and the slope of the straight lines \( y = -\frac{\alpha \eta}{\varepsilon - \eta} x + \frac{\alpha}{\varepsilon - \eta} \) and \( y = -\frac{\varepsilon - \alpha}{1 - \alpha} x + \frac{1 - \alpha}{1 - \varepsilon} \) satisfy the inequality \( \frac{\varepsilon - \alpha}{1 - \alpha} \leq \frac{\alpha \eta}{\varepsilon - \eta} \). Since
\[
\|1 \oplus 0 - 0 \oplus 1\|_E = \|1 \oplus 0\|_E = \eta \cdot \left|\frac{1}{\eta} \oplus \frac{0}{\eta}\right|_E = \eta
\]
and this is, in fact, the largest order interval on \( S_+(E) \), we have \( \varepsilon_0,m(E) = \eta \). Observe that since \( \varepsilon_0,m(E) = \alpha > 0 \), we find \( u_n \in S(X) \) and \( v_n \in X \) such that \( 0 \leq \varepsilon \leq u_n \) for all \( n \in \mathbb{N} \), \( \|v_n\|_X \to 1 \) and \( \|u_n - v_n\|_X \to \alpha \) as \( n \to \infty \). Let us define in \( X \times X \):
\[
x_n = u_n \oplus u_n, \quad y_n = \Theta \oplus v_n,
\]
where \( \Theta \) is just zero element in \( X \). Since the points \( 1 \oplus 1, 0 \oplus 1 \) and \( \frac{1}{\varepsilon} \oplus \frac{\alpha}{\varepsilon} \) belongs to \( S_+(E) \), we get \( \|x_n\|_E(X) = 1 \), \( \|y_n\|_E(X) = 1 \oplus 0 \oplus 1 \|v_n\|_X \to 1 \) and
\[
\|x_n - y_n\|_E(X) = \|u_n\|_X \oplus \|u_n - v_n\|_X \to 1 \oplus 0 \oplus 1 \|v_n\|_X \to \|1 \oplus \alpha\|_E = \varepsilon.
\]
Hence, by virtue of formula (3), \( \varepsilon_0,m(E(X)) \geq \varepsilon. \)
Conversely. Assume that \( (x_n) \) and \( (y_n) \) are sequences in \( E(X) \) satisfying the following conditions: \( \|x_n\|_{E(X)} = 1 \), \( 0 \leq y_n \leq x_n \) for each \( n \in \mathbb{N} \), \( \|y_n\|_{E(X)} \to 1 \) and \( \|x_n - y_n\|_{E(X)} \to \delta > 0 \) as \( n \to \infty \). The elements \( x_n \) and \( y_n \) are of the form

\[
x_n = u_n \oplus v_n, \quad y_n = w_n \oplus z_n,
\]

where \( u_n, v_n, w_n \) and \( z_n \) are from \( X \) for each \( n \in \mathbb{N} \). Then, passing to a subsequence of \( \mathbb{N} \) if necessary, we may assume without loss of generality that all conditions below are satisfied:

\[
\|u_n\|_{X} \to u, \quad \|v_n\|_{X} \to v.
\]

\[
\|w_n\|_{X} \to w, \quad \|z_n\|_{X} \to z.
\]

\[
\|u_n - w_n\|_{X} \to p, \quad \|v_n - z_n\|_{X} \to q.
\]

with \( 0 \oplus 0 \leq w \oplus z \leq u \oplus v \) in \( E \), \( \|u \oplus v\|_{E} = 1 = \|w \oplus z\|_{E} \) and \( \|p \oplus q\|_{E} = \delta \). By the construction of the positive part of the unit sphere \( S_+(E) \) and the fact that \( \|u \oplus v\|_{E} = 1 = \|w \oplus z\|_{E} \) and \( 0 \oplus 0 \leq w \oplus z \leq u \oplus v \) in \( E \), we get \( u, w \leq 1 = v = z \). From Lemma 3, the fact that \( \|v_n\|_{X} \to v = 1 \) and \( \varepsilon_{0,m}(X) = \alpha \), we get

\[
0 \leq q = \lim_{n \to \infty} \|v_n - z_n\|_{X} = \limsup_{n \to \infty} \left \| \frac{v_n}{\|v_n\|_{X}} - \frac{z_n}{\|v_n\|_{X}} \right \|_{X} \leq \alpha,
\]

i.e. \( 0 \leq q \leq \alpha \). Moreover, \( 0 \leq p = \lim_{m \to \infty} \|u_m - w_n\|_{X} \leq \lim_{m \to \infty} \|u_m\|_{X} = u \leq 1 \), i.e. \( 0 \leq p \leq u \leq 1 \). Hence and from monotonicity of the norm in \( E \) with respect to the partial order, we obtain \( \delta = \|p \oplus q\|_{E} \leq \|1 \oplus \alpha\|_{E} = \varepsilon \). Therefore \( \varepsilon_{0,m}(E(X)) \leq \varepsilon \). \( \square \)

**Corollary 6.** For every \( \eta \in (0, 1) \) there exists a Köthe space \( E \) with \( \varepsilon_{0,m}(E) = \eta \) such that \( \varepsilon_{0,m}(E(X)) = \eta \) for any Banach lattice with \( \varepsilon_{0,m}(X) \leq \eta^2 \).

**Proof.** Let \( \eta \in (0, 1) \) and \( E = \mathbb{R}^2 \). We define the norm \( \|\cdot\|_{E} \) in \( E \) such that the positive part of its unit sphere is the set

\[
S_+(E) = 0 \oplus 1 - 1 \oplus 1 - \frac{1}{\eta} \oplus 1 - \frac{\alpha}{\eta} \oplus 1 - \frac{1}{\eta} \oplus 0,
\]

where \( \alpha = \varepsilon_{0,m}(X) \in (0, 1) \) and \( \alpha \leq \eta^2 \). Observe that the norm with such \( S(E) \) is defined for all \( (x_1, x_2) \in \mathbb{R}^2 \) by the formula

\[
\|(x_1, x_2)\| = \max \left \{ \|x_1\|, |x_1|, \frac{\eta - \alpha}{1 - \alpha} |x_1| + 1 - \frac{\eta}{1 - \alpha} |x_2| \right \}.
\]

Since \( \varepsilon_{0,m}(X) \leq \eta^2 \), \( \|1 \oplus 1 - 1 \oplus 0\|_{E} = \|1 \oplus 0\|_{E} = \eta \) and

\[
\left \| \frac{1}{\eta} \oplus \frac{\alpha}{\eta} \oplus 1 - \frac{1}{\eta} \oplus 0 \right \|_{E} = \left \| 0 \oplus \frac{\alpha}{\eta} \right \|_{E} = \frac{\alpha}{\eta} \leq \eta,
\]

we have \( \varepsilon_{0,m}(E) = \eta \). By virtue of \( \varepsilon_{0,m}(X) = \alpha > 0 \), we can find sequences \( (u_n) \) in \( S(X) \) and \( (v_n) \) in \( X \) such that \( 0 \leq v_n \leq u_n \) for all \( n \in \mathbb{N} \), \( \|v_n\|_{X} \to 1 \) and \( \|u_n - v_n\|_{X} \to \alpha \) as \( n \to \infty \). Let us define in \( X \times X \):

\[
x_n = u_n \oplus v_n, \quad y_n = \Theta \oplus v_n,
\]

where \( \Theta \oplus \Theta \leq w_n \oplus z_n \leq u_n \oplus v_n \) in \( X \times X \) for each \( n \in \mathbb{N} \) (\( \Theta \) is zero element of \( X \)) and

\[
\|u_n\|_{X} \to u, \quad \|v_n\|_{X} \to v.
\]

\[
\|w_n\|_{X} \to w, \quad \|z_n\|_{X} \to z.
\]

\[
\|u_n - w_n\|_{X} \to p, \quad \|v_n - z_n\|_{X} \to q.
\]

are such that \( 0 \oplus 0 \leq w \oplus z \leq u \oplus v \) in \( E \), \( \|u \oplus v\|_{E} = 1 = \|w \oplus z\|_{E} \) and \( \|p \oplus q\|_{E} = \delta \). Notice that we have only two order intervals on \( S_+(E) \), namely the order interval linking points \( 0 \oplus 1 \) and \( 1 \oplus 1 \) and the order interval, endpoints of which are \( \frac{1}{\eta} \oplus 0 \) and \( \frac{1}{\eta} \oplus \frac{\alpha}{\eta} \). Taking into account the shape of the positive part of the unit sphere \( S_+(E) \) as well as the fact that \( \|u \oplus v\|_{E} = 1 = \|w \oplus z\|_{E} \) and \( 0 \oplus 0 \leq w \oplus z \leq u \oplus v \), we conclude that the points \( w \oplus z \) and \( u \oplus v \) lie either on the first order interval or on the second one, so we need to consider two cases separately.
Case (a) Let the points \( w \oplus z \) and \( u \oplus v \) lie on the first order interval with the endpoints 0 \( \oplus \) 1 and 1 \( \oplus \) 1 in \( E \). Then we have \( u, w \leq 1 = v = z \). By Lemma 3, the facts that \( \|v_n\|_X \to v = 1 \) and \( \varepsilon_{0,m}(X) = \alpha \), we get

\[
0 \leq q = \lim_{n \to \infty} \|v_n - z_n\|_X = \lim_{n \to \infty} \left\| \frac{v_n - z_n}{\|v_n\|_X} \right\|_X \leq \alpha,
\]

i.e. \( 0 \leq q \leq \alpha \). Moreover, \( 0 \leq p = \lim_{n \to \infty} \|u_n - w_n\|_X \leq \lim_{n \to \infty} \|u_n\|_X = u \leq 1 \), so \( 0 \leq p \leq u \leq 1 \). By virtue of monotonicity of the norm with respect to the partial order, we have \( \delta = \|p \oplus q\|_E \leq \|1 \oplus \alpha\|_E = \eta \). Hence \( \varepsilon_{0,m}(E(X)) \leq \eta \).

Case (b) Let the points \( w \oplus z \) and \( u \oplus v \) lie on the second order interval with the endpoints \( \frac{1}{\eta} \oplus 0 \) and \( \frac{1}{\eta} \oplus \alpha \). Then

\[
0 \leq v, z \leq \frac{\alpha}{\eta} \leq \eta < 1 \leq \frac{1}{\eta} = u = w.
\]

Therefore, \( \eta u = \eta w = 1 \) and \( \lim_{n \to \infty} \|\eta u_n\|_X = \eta u = 1 \). Hence, by virtue of Lemma 3, we obtain

\[
0 \leq \eta p = \lim_{n \to \infty} \|\eta u_n - w_n\|_X = \lim_{n \to \infty} \left\| \frac{\eta u_n - \eta w_n}{\|\eta u_n\|_X} \right\|_X = \lim_{n \to \infty} \left\| \frac{u_n - w_n}{u_n \|u_n\|_X} \right\|_X \leq \alpha,
\]

i.e. \( 0 \leq p \leq \frac{\alpha}{\eta} \). Moreover

\[
0 \leq q = \lim_{n \to \infty} \|v_n - z_n\|_X \leq \lim_{n \to \infty} \|v_n\|_X = v \leq \frac{\alpha}{\eta},
\]

so \( 0 \leq q \leq v \leq \frac{\alpha}{\eta} \). We also have \( \delta = \|p \oplus q\|_E \leq \frac{\alpha}{\eta} \|1 \oplus \alpha\|_E = \frac{\alpha}{\eta} \leq \eta \). Therefore \( \varepsilon_{0,m}(E(X)) \leq \eta \). \( \square \)

4. Some additional conclusions

The results of Theorem 2 suggest us to introduce two new parameters.

Definition 1. For each \( \alpha \in [0, 1] \) and each Köthe space \( E \) let us define the numbers

\[
e^m_{\alpha}(E) := \inf \{ \varepsilon_{0,m}(E(X)) : X \text{ satisfies } \varepsilon_{0,m}(X) = \alpha \},
\]

\[
e^m_{\alpha}(E) := \sup \{ \varepsilon_{0,m}(E(X)) : X \text{ satisfies } \varepsilon_{0,m}(X) = \alpha \},
\]

called the lower and upper \( \alpha \)-characteristic of the monotonicity of \( E \).

Fact 2. The following statements are true:

1. \( \alpha \vee \varepsilon_{0,m}(E) \leq \varepsilon^m_{\alpha}(E) \leq \varepsilon^m_{\alpha}(E) - \alpha \varepsilon_{0,m}(E) + \alpha \).
2. \( \varepsilon^m_{\alpha}(E) = \varepsilon^m_{\alpha}(E) \) for \( \alpha = 0 \).
3. \( \varepsilon^m_{\alpha}(E) = \varepsilon^m_{\alpha}(E) = 1 \).
4. \( \varepsilon^m_{\alpha}(E) = \varepsilon^m_{\alpha}(E) \) when \( \varepsilon_{0,m}(E) = 0 \).
5. For each \( \eta \in (0, 1) \), there exists a Köthe space \( E \) such that

\[
\varepsilon^m_{\alpha}(E) = \varepsilon^m_{\alpha}(E) = \varepsilon_{0,m}(E) = \eta
\]

for all \( \alpha \leq \eta^2 \).
6. For each \( \eta \in (0, 1) \), there is a Köthe space \( E \) with \( \varepsilon_{0,m}(E) = \eta \) such that \( \varepsilon^m_{\alpha}(E) = \varepsilon^m_{\alpha}(E) = \alpha - \alpha \eta + \eta \) for all \( \alpha \in (0, 1) \).

Proof. By virtue of Definition 1 and Theorem 2 the proof of thesis (1) is obvious. Statements (2)–(4) follows from Fact 2, statement (1). Statement (5) is clear by virtue of Definition 1 and Corollary 6, while statement (6) follows from Corollary 5 applied with \( \varepsilon = \alpha + \eta - \alpha \eta \). \( \square \)

5. Strict and uniform monotonicity of the lattice \( e((X_n)_{n=1}^\infty) \)

Let \( e \) be a Köthe sequence space and \( (X_n)_{n=1}^\infty \), be a sequence of real Banach lattices \( (X_n, \|\cdot\|_n) \). Sometimes we write shortly \( (X_n) \) instead of \( (X_n)_{n=1}^\infty \). By the notation \( e((X_n)_{n=1}^\infty) \) (or \( e((X_n)) \) for short) we will mean the space of all sequences \( (x(n))_{n=1}^\infty \) in the Cartesian product \( \prod_{n=1}^\infty X_n \) of all lattices \( X_n \) such that the sequence \( (\|x_n\|_n)_{n=1}^\infty \) belongs to \( e \). We consider the space \( e((X_n)) \) equipped with the norm

\[
\|x\|_e((X_n)) = \left\| (\|x_n\|_n)_{n=1}^\infty \right\|_e.
\]

Notice that the space \( e((X_n)_{n=1}^\infty) \) with the coordinatewise partial order is a Banach lattice.
Theorem 3. The space \( e((X_0)_{n=1}^\infty) \) is strictly monotone if and only if the Köthe sequence space \( e \) is strictly monotone and all real Banach lattices \( (X_n, \| \cdot \|_n) \) are strictly monotone.

Proof. Since \( e \) is order-isomertically embedded into \( e((X_0)_{n=1}^\infty) \) and for each \( j \in \mathbb{N} \), \( X_j \) is order-isometrically embedded into \( e((X_0)_{n=1}^\infty) \), it is enough to show the sufficiency. In order to do it, assume that all Banach lattices \( (X_n, \| \cdot \|_n) \) are strictly monotone, the Köthe sequence space \( e \) is strictly monotone, \( 0 \leq y \leq x, x \in e((X_0)_{n=1}^\infty), \| x \|_{e((X_0))} = 1 \) and \( y \neq 0 \). Then \( A = \{ n \in \mathbb{N} : y(n) \neq 0 \} \neq \emptyset \). Since all lattices \( X_n \) are strictly monotone, we get that \( \| x(n) - y(n) \|_n < \| x(n) \|_n \) for each \( n \in A \). Hence denoting \( |x| = ((x(n)))_{n=1}^\infty \), we have \( |x - y| \leq |x| \) and \( |x - y| \neq |x| \). By strict monotonicity of \( e \), we get \( \| x - y \|_{e((X_0))} = \| x - y \|_e < \| x \|_e \), which means that \( e((X_0)) \) is strictly monotone. \( \square \)

Theorem 4. The space \( e((X_0)_{n=1}^\infty) \) is uniformly monotone if and only if the Köthe sequence space \( e \) is uniformly monotone and all Banach lattices \( X_n \) are equi-uniformly monotone, i.e. \( \inf_{n \in \mathbb{N}} \delta_{m,X_n}(\varepsilon) > 0 \) for every \( \varepsilon \in (0,1) \).

Proof. Since \( e \) is order-isomertically embedded into \( e((X_0)_{n=1}^\infty) \) and any \( X_k \) embeds continuously and order-isometrically into \( e((X_0)_{n=1}^\infty) \), the following inequalities are true for arbitrary \( \varepsilon \in (0,1) \):

\[
\delta_{m,e((X_0))}(\varepsilon) \leq \delta_{m,e}(\varepsilon) \tag{7}
\]

and \( \delta_{m,e((X_0))}(\varepsilon) \leq \delta_{m,X_k}(\varepsilon) \). Thus

\[
\delta_{m,e((X_0))}(\varepsilon) \leq \inf_{k \in \mathbb{N}} \delta_{m,X_k}(\varepsilon). \tag{8}
\]

By the assumption and inequalities (7) and (8), we get for arbitrary \( \varepsilon \in (0,1) \) that

\[
0 < \delta_{m,e((X_0))}(\varepsilon) \leq \delta_{m,e}(\varepsilon) \wedge \inf_{k \in \mathbb{N}} \delta_{m,X_k}(\varepsilon),
\]

whence it follows that \( e \) is uniformly monotone and all \( X_k \) are equi-uniformly monotone.

Conversely, let \( e \) be uniformly monotone and all \( X_n \) be equi-uniformly monotone. Fix arbitrary \( \varepsilon \in (0,1) \) and let \( x \in S(e((X_0))) \) and \( y \in e((X_0)) \) be such that \( 0 \leq y \leq x \) and \( \| y \|_{e((X_0))} \geq \varepsilon \). Denote \( x = (x(n))_{n=1}^\infty \), \( y = (y(n))_{n=1}^\infty \), where \( x(n), y(n) \in X_n \) for every \( n \in \mathbb{N} \). Choose \( g = (g(n))_{n=1}^\infty \) in \( S(e) \) such that \( g(n) > 0 \) for every \( n \in \mathbb{N} \) and define the sets

\[
B = \left\{ n \in \mathbb{N} : \| x(n) \|_n < \frac{\varepsilon}{4} \| x(n) \|_n \right\},
\]

\[
C = \left\{ n \in \mathbb{N} : \| x(n) \|_n < \frac{\varepsilon}{4} g(n) \right\}
\]

and

\[
A = (B \cup C)^e = \left\{ n \in \mathbb{N} : \| y(n) \|_n \geq \frac{\varepsilon}{4} \| x(n) \|_n \wedge \| x(n) \|_n \geq \frac{\varepsilon}{4} g(n) \right\}.
\]

Then \( \| (y(n))_{B \cup C}^e \|_{n=1}^\infty \|_e \leq \frac{\varepsilon}{4} \) and \( \| (y(n))_{B \cup C}^e \|_{n=1}^\infty \|_e \leq \frac{\varepsilon}{4} \). Therefore

\[
\| (y(n))_{B \cup C}^e \|_{n=1}^\infty \|_e \leq \| (y(n))_{B}^e \|_{n=1}^\infty + \| y(n) \|_{C} \|_{n=1}^\infty \|_e \leq \frac{\varepsilon}{2}.
\]

Notice that the inequalities

\[
\varepsilon \leq \| y \|_{e(X_0)} \leq \| (y(n))_{X_A} \|_{n=1}^\infty \|_e \leq \| (y(n))_{X_{A'}} \|_{n=1}^\infty \|_e \leq \| (y(n))_{X_{A}} \|_{n=1}^\infty \|_e + \frac{\varepsilon}{2}
\]

yield that

\[
\| (x(n))_{X_A} \|_{n=1}^\infty \|_e \geq \| (y(n))_{X_A} \|_{n=1}^\infty \|_e \geq \varepsilon - \frac{\varepsilon}{2} > \frac{\varepsilon}{4}.
\]

Besides for every \( n \in A \), we have \( 0 \leq \frac{x(n)}{\| x(n) \|_n} \leq \frac{x(n)}{\| x(n) \|_n} \), \( \| x(n) \|_{X_{A}} \|_{n=1}^\infty = 1 \) and \( \| \frac{x(n)}{\| x(n) \|_n} \|_{n} \geq \frac{\varepsilon}{4} \). Denoting \( \tilde{\delta}_m(\varepsilon) = \inf \{ \delta_{m,X_n}(\varepsilon) : n \in \mathbb{N} \} \), the assumption that \( X_n \) are equi-uniformly monotone we get that \( \tilde{\delta}_m(\varepsilon) > 0 \) for arbitrary \( \varepsilon \in (0,1) \). Hence for every \( n \in A \) we have

\[
\| x(n) - y(n) \|_n \leq \left( 1 - \frac{\tilde{\delta}_m(\varepsilon)}{\frac{\varepsilon}{4}} \right) \| x(n) \|_n.
\]

Moreover, applying inequality (9), we obtain

\[
\left( \frac{\tilde{\delta}_m(\varepsilon)}{\frac{\varepsilon}{4}} \right) \| x(n) \|_{X_{A}} \|_{n=1}^\infty \|_e > \frac{\varepsilon}{4} \delta_m \left( \frac{\varepsilon}{4} \right).
\]
By virtue of the assumption of uniform monotonicity of $e$, the inequalities $\|x(n) - y(n)\|_n \leq \|x(n)\|_n$ for all $n \in \mathbb{A}^*$, and inequalities (10) and (11) yield

$$\|x - y\|_{((X_n))} = \|x - y\|_e \leq \left( \left( 1 - \frac{\epsilon}{4} \right) \left( 1 - \frac{\epsilon}{4} \right) \right),$$

which is the desired result. \qed

References