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Constructing r-matrices on simple Lie superalgebras

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Abstract

We construct r-matrices for simple Lie superalgebras with non-degenerate Killing forms using Belavin–Drinfeld type triples. This construction gives us the standard r-matrices and some non-standard ones.

Keywords: Lie superalgebras; Yang–Baxter equation; Classical r-matrices

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate $\mathfrak{g}$-invariant bilinear form $(\,,\,)$. Then the *classical Yang–Baxter equation* (*CYBE*) for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

A solution r to the classical Yang–Baxter equation is called a *classical r-matrix* (or simply an *r-matrix*). r is called *non-degenerate* if it satisfies

$$r^{12} + r^{21} \neq 0.$$

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In [1] and [2] Belavin and Drinfeld classified such r-matrices. Their classification is given by a discrete parameter called an *admissible* (or a *Belavin–Drinfeld*) *triple*, and a continuous parameter r_0 which satisfies certain relations depending on the given admissible triple.

In this paper, we aim to develop a similar theory for simple Lie superalgebras. We start in Section 2 with an overview of the Belavin–Drinfeld result for simple Lie algebras. In Section 3, we recall some basic definitions and results about simple Lie superalgebras, and after developing the necessary ingredients we state our main theorem. The next three sections of the paper are devoted to the proof of this theorem. Then in Section 7 we construct various r-matrices for the Lie superalgebra $sl(2,1)$ using the main theorem.

This theorem is very much in the spirit of the Belavin–Drinfeld result. It tells us that, given a Belavin–Drinfeld type triple, one can construct a non-degenerate r-matrix in a way similar to the construction in the Lie algebra case. However, unlike in the Lie algebra case, this is not a complete classification result. In fact, in the last section, we construct an r-matrix that cannot be obtained by this theorem.

Recall that a non-degenerate r-matrix r on a simple Lie algebra defines a Lie bialgebra structure by $\delta(x) = [r, x \otimes 1 + 1 \otimes x]$. Therefore, the results of Belavin and Drinfeld give us the classification of Lie bialgebra structures and the corresponding Poisson–Lie structures associated to a simple Lie algebra [3,4]. Hence a study of non-degenerate r-matrices on Lie superalgebras may be a natural step towards a theory of super Poisson–Lie groups.

2. Classification theorem for Lie algebras

Here we recall briefly the main result of [1] and [2] for Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra. Denote by Ω the element of $(\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ that corresponds to the quadratic Casimir element in the universal enveloping algebra $\mathfrak{U}\mathfrak{g}$ of $\mathfrak{g}$. Fix a positive Borel subalgebra $\mathfrak{b}_+$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}_+$. Let $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be the set of simple roots of $\mathfrak{g}$. An *admissible triple* is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_i \subset \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$ is a bijection such that

(1) for any $\alpha, \beta \in \Gamma_1$, $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$;
(2) for any $\alpha \in \Gamma_1$ there exists a $k \in \mathbb{N}$ such that $\tau^k(\alpha) \notin \Gamma_1$.

Fix a system of Weyl–Chevalley generators $X_\alpha, Y_\alpha, H_\alpha$ for $\alpha \in \Gamma$. Recall that these elements generate the Lie algebra $\mathfrak{g}$ with the defining relations: $[X_{\alpha_i}, Y_{\alpha_j}] = \delta_{ij} H_{\alpha_j}$, $[H_{\alpha_i}, X_{\alpha_j}] = a_{ij} X_{\alpha_j}$ and $[H_{\alpha_i}, Y_{\alpha_j}] = -a_{ij} Y_{\alpha_j}$ for all $\alpha_i, \alpha_j \in \Gamma$ (where $a_{ij} = \alpha_j(H_{\alpha_i}) = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$), along with the well-known Serre relations.

Denote by $\mathfrak{g}_i$ the subalgebra of $\mathfrak{g}$ generated by the elements X_α, Y_α, H_α for all $\alpha \in \Gamma_i$. We define a map φ by

$$\varphi(X_\alpha) = X_{\tau(\alpha)}, \qquad \varphi(Y_\alpha) = Y_{\tau(\alpha)}, \qquad \varphi(H_\alpha) = H_{\tau(\alpha)}$$

for all $\alpha \in \Gamma_1$. Then this can be extended uniquely to an isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ because the relations between X_α, Y_α, H_α for $\alpha \in \Gamma_1$ will be the same as the relations between

$X_{\tau(\alpha)}, Y_{\tau(\alpha)}, H_{\tau(\alpha)}$ for $\alpha \in \Gamma_1$ (τ is an isometry). Next extend τ to a bijection $\bar{\tau} : \overline{\Gamma}_1 \to \overline{\Gamma}_2$, where $\overline{\Gamma}_i$ is the set of those roots which can be written as a non-negative integral linear combination of the elements of Γ_i. In each root space $\mathfrak{g}_\alpha$, choose an element e_α such that $(e_\alpha, e_{-\alpha}) = 1$ for any α and $\varphi(e_\alpha) = e_{\bar{\tau}(\alpha)}$ for all $\alpha \in \overline{\Gamma}_1$.

Finally, define a partial order on the set of all positive roots

$$\alpha \prec \beta \text{ if and only if there exists a } k \in \mathbb{N} \text{ such that } \beta = \bar{\tau}^k(\alpha).$$

Note that if $\alpha \prec \beta$, then necessarily $\alpha \in \overline{\Gamma}_1$, $\beta \in \overline{\Gamma}_2$.

Now we can state the Belavin–Drinfeld theorem ([2]; also see [4]).

Theorem 1.

(1) *If $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfies*

$$r_0^{12} + r_0^{21} = \Omega_0, \tag{1}$$

$$\big(\tau(\alpha) \otimes 1\big)(r_0) + (1 \otimes \alpha)(r_0) = 0 \quad \textit{for all } \alpha \in \Gamma_1 \tag{2}$$

where $\Omega_0 \in \mathfrak{h} \otimes \mathfrak{h}$ is the $\mathfrak{h}$-component of Ω, then the element r of $\mathfrak{g} \otimes \mathfrak{g}$ defined by

$$r = r_0 + \sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha,\beta>0,\ \alpha\prec\beta} (e_{-\alpha} \otimes e_\beta - e_\beta \otimes e_{-\alpha})$$

is a solution to the system

$$r^{12} + r^{21} = \Omega, \tag{3}$$

$$\big[r^{12}, r^{13}\big] + \big[r^{12}, r^{23}\big] + \big[r^{13}, r^{23}\big] = 0. \tag{4}$$

(2) *Any solution to this system can be obtained as above from some admissible triple $(\Gamma_1, \Gamma_2, \tau)$ and some $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ that satisfies Eqs.* (1) *and* (2)*, by choosing a suitable triangular decomposition of $\mathfrak{g}$ and a set of Weyl–Chevalley generators.*

3. The construction theorem for Lie superalgebras

Now our aim is to develop a similar theory for super structures. Let $\mathfrak{g}$ be a simple Lie superalgebra with non-degenerate Killing form. (In fact, most of our results can be extended to the whole class of classical Lie superalgebras because most of the statements involving the Killing form may be asserted more generally for a non-degenerate invariant form.)

3.1. The quadratic Casimir element

Let $\{I_\alpha\}$ be a homogeneous basis for $\mathfrak{g}$ and denote by $\{I_\alpha^*\}$ the dual basis of $\mathfrak{g}$ with respect to the non-degenerate (Killing) form. Thus we have

$$\left(I_\alpha, I_\beta{}^*\right) = \delta_{\alpha\beta}.$$

Denote the parity of a homogeneous element $x \in \mathfrak{g}$ by $|x|$; then $|I_\alpha| = |I_\alpha^*|$, since the Killing form is consistent, and so the quadratic Casimir element of $\mathfrak{g}$ is

$$\Omega = \sum_\alpha (-1)^{|I_\alpha||I_\alpha^*|} I_\alpha \otimes I_\alpha^* = \sum_\alpha (-1)^{|I_\alpha|} I_\alpha \otimes I_\alpha^*.$$

For a definition of the Casimir element (and many other facts about Lie superalgebras used here), one can look at [6,8].

Example. Let $\mathfrak{g} = gl(m,n)$. Fix the basis $\{e_{ij} \mid 1 \leqslant i, j \leqslant m+n\}$, where $|e_{ij}| = 0$ if and only if $1 \leqslant i, j \leqslant m$ or $m+1 \leqslant i, j \leqslant m+n$. The dual basis is

$$e_{ij}^* = (-1)^{[i]} e_{ji}$$

where

$$[j] = \begin{cases} 0 & \text{if } j \leqslant m, \\ 1 & \text{if } j > m \end{cases}$$

and (,) is the supertrace form. Then this gives us

$$\Omega = \sum_\alpha (-1)^{|I_\alpha|} I_\alpha \otimes I_\alpha^* = \sum_{i,j} (-1)^{|e_{ij}|} e_{ij} \otimes (-1)^{[i]} e_{ji} = \sum_{i,j} (-1)^{[j]} e_{ij} \otimes e_{ji}.$$

3.2. Borel subsuperalgebras and Dynkin diagrams

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. By definition, $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}}$ is a Cartan subalgebra of the even part of $\mathfrak{g}$. Let $\Delta = \Delta_{\bar{0}} + \Delta_{\bar{1}}$ be the set of all roots of $\mathfrak{g}$ associated with the Cartan subalgebra $\mathfrak{h}$, where $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ are the even and odd roots respectively. Recall that a Lie subsuperalgebra $\mathfrak{b}$ of a Lie superalgebra $\mathfrak{g}$ is a *Borel subsuperalgebra* if there is some Cartan subsuperalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and some base Γ for Δ, such that

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

where Δ^+ is the set of all positive roots.

In the Lie algebra case, subalgebras given by this definition are all maximally solvable, and all maximally solvable subalgebras of a simple Lie algebra are of this type. Therefore, this definition agrees with the usual definition of a Borel subalgebra as a maximally solvable subalgebra. However Borel subsuperalgebras as defined above are not necessarily maximally solvable. For instance if α is a positive isotropic root of the simple Lie superalgebra $\mathfrak{g}$, and if $\mathfrak{b}$ is the sum of all the positive root spaces, then $\mathfrak{b}$ is a Borel subsuperalgebra but is not maximally solvable. The (parabolic) subsuperalgebra $\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha}$ is also solvable. In fact, maximally solvable subsuperalgebras may be more complicated than merely parabolic. (See [9] for maximally solvable subsuperalgebras of $gl(m,n)$ and $sl(m,n)$.)

Recall also that different Borel subsuperalgebras may correspond to different Dynkin diagrams and Cartan matrices. Let us then fix some Borel subsuperalgebra $\mathfrak{b}$, or equivalently some set of simple roots, $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, and the associated Dynkin diagram D.

3.3. The data for the theorem

In this setup, let $\Gamma_1, \Gamma_2 \subset \Gamma$ be two subsets and $\tau : \Gamma_1 \to \Gamma_2$ be a bijection. The triple $(\Gamma_1, \Gamma_2, \tau)$ will be called *admissible* if:

(1) for any $\alpha, \beta \in \Gamma_1$, $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$;
(2) for any $\alpha \in \Gamma_1$ there exists a $k \in \mathbb{N}$ such that $\tau^k(\alpha) \notin \Gamma_1$;
(3) τ preserves the grading of the root space.

Given an admissible triple $(\Gamma_1, \Gamma_2, \tau)$, let $\overline{\Gamma}_i$ for $i = 1, 2$ be the set of those roots that are non-negative integral linear combinations of the elements of Γ_i. Then τ extends linearly to a bijection $\bar{\tau} : \overline{\Gamma}_1 \to \overline{\Gamma}_2$, so we can define a partial order on Δ^+

$$\alpha \prec \beta \text{ if and only if there exists a } k \in \mathbb{N} \text{ such that } \beta = \bar{\tau}^k(\alpha).$$

For any $\alpha \in \Gamma$, pick a non-zero $e_\alpha \in \mathfrak{g}_\alpha$. Since each $\mathfrak{g}_\alpha$ is one dimensional, and the Killing form is a non-degenerate pairing of $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$, one can uniquely pick $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha, e_{-\alpha}) = 1$, so for each $\alpha \in \Gamma$

$$[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha}) h_\alpha$$

where $h_\alpha \in \mathfrak{h}$ is defined by $(h_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. The set $\{h_\alpha \mid \alpha \in \Gamma\}$ is a basis for $\mathfrak{h}$. Hence we can write Ω_0, the $\mathfrak{h}$-part of Ω, as follows

$$\Omega_0 = \sum_{i=1}^{r} h_{\alpha_i} \otimes h_{\alpha_i}^*,$$

where the set $\{h_\alpha^* \mid \alpha \in \Gamma\}$ is the basis in $\mathfrak{h}$ dual to $\{h_\alpha \mid \alpha \in \Gamma\}$.

Next, for each $\alpha \in \Delta^+ \backslash \Gamma$, choose a non-zero $e_\alpha \in \mathfrak{g}_\alpha$; this will uniquely determine $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ satisfying $(e_\alpha, e_{-\alpha}) = 1$. Then the duals with respect to the standard (Killing) form will be

$$e_\alpha^* = e_{-\alpha}, \qquad e_{-\alpha}^* = (-1)^{|\alpha|} e_\alpha$$

for all positive roots α, where $|\alpha|$ is the parity of the root α. Therefore, the quadratic Casimir element of $\mathfrak{g}$ will be

$$\begin{aligned}\Omega &= \sum_i (-1)^{|I_i|} I_i \otimes I_i^* = \sum_{i=1}^r h_{\alpha_i} \otimes h_{\alpha_i}^* + \sum_{\alpha\in\Delta} (-1)^{|e_\alpha|} e_\alpha \otimes e_\alpha^* \\ &= \Omega_0 + \sum_{\alpha\in\Delta^+} (-1)^{|\alpha|} e_\alpha \otimes e_{-\alpha} + \sum_{\alpha\in\Delta^+} e_{-\alpha} \otimes e_\alpha.\end{aligned}$$

Example (*continued*). Once again, let $\mathfrak{g} = gl(m,n)$. Let $\mathfrak{h}$ and $\mathfrak{b}$ be the diagonal matrices and the upper triangular matrices, respectively. Then the positive root spaces are spanned by $\{e_{ij} \mid i < j\}$. If for each positive root α, we let e_α be the unique $e_{ij} \in \mathfrak{g}_\alpha$, then $i < j$ and $e_{-\alpha} = (-1)^{[i]} e_{ji}$. We will have

$$\begin{aligned}e_\alpha^* = e_{i.j}^* &= (-1)^{[i]} e_{ji} = e_{-\alpha}, \\ e_{-\alpha}^* = (-1)^{[i]} e_{ji}^* &= (-1)^{[i]}(-1)^{[j]} e_{ij} = (-1)^{|\alpha|} e_\alpha\end{aligned}$$

and the above formula for Ω will agree with the Casimir element found earlier.

3.4. Statement of the theorem

We are now ready to state our main theorem. Its proof will be presented in the next three sections.

Theorem 2. *Let $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfy*

$$r_0^{12} + r_0^{21} = \Omega_0, \tag{1}$$

$$\big(\tau(\alpha) \otimes 1\big)(r_0) + (1 \otimes \alpha)(r_0) = 0 \quad \textit{for all } \alpha \in \Gamma_1. \tag{2}$$

Then the element r of $\mathfrak{g} \otimes \mathfrak{g}$ defined by

$$r = r_0 + \sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha,\beta>0,\ \alpha\prec\beta} \big(e_{-\alpha} \otimes e_\beta - (-1)^{|\alpha|} e_\beta \otimes e_{-\alpha}\big) \tag{$*$}$$

is a solution to the system

$$r^{12} + r^{21} = \Omega, \tag{3}$$

$$\big[r^{12}, r^{13}\big] + \big[r^{12}, r^{23}\big] + \big[r^{13}, r^{23}\big] = 0. \tag{4}$$

Remark. If $\mathfrak{g}$ is a simple Lie algebra, then $(*)$ reduces to the corresponding equation in Theorem 1.

4. Technical lemmas

Let $\mathfrak{g}$ be a simple Lie superalgebra with non-degenerate Killing form. Fix a homogeneous basis $\{I_\alpha\}$ for $\mathfrak{g}$ and denote by $\{I_\alpha^*\}$ the dual basis of $\mathfrak{g}$ with respect to the non-degenerate (Killing) form.

Lemma 1. *Let $f : \mathfrak{g} \to \mathfrak{g}$ be an even linear map, and set $r = (f \otimes 1)\Omega$. Then the system of equations*

$$r^{12} + r^{21} = \Omega, \tag{3}$$

$$\bigl[r^{12}, r^{13}\bigr] + \bigl[r^{12}, r^{23}\bigr] + \bigl[r^{13}, r^{23}\bigr] = 0 \tag{4}$$

is equivalent to the system

$$f + f^* = 1, \tag{5}$$

$$(f - 1)\bigl[f(x), f(y)\bigr] = f\bigl(\bigl[(f - 1)(x), (f - 1)(y)\bigr]\bigr) \tag{6}$$

where f^* stands for the adjoint of f with respect to the standard from $(\, ,)$.

Remark. This lemma is a basic step in the proof of Theorem 1, and our proof will follow the presentation in [4] with some modifications.

Proof. We have

$$r^{12} + r^{21} = (f \otimes 1)\Omega + (1 \otimes f)\Omega = (f \otimes 1)\Omega + \bigl(f^* \otimes 1\bigr)\Omega = \bigl(\bigl(f + f^*\bigr) \otimes 1\bigr)\Omega$$

which proves the equivalence of the statements

$$\Omega = r^{12} + r^{21} \quad \text{and} \quad 1 = \bigl(f + f^*\bigr).$$

Next we show that the CYBE for r (that is, Eq. (4)), translates to a nice expression in terms of the associated function f. We have

$$r = (f \otimes 1)\Omega = \sum_\alpha (-1)^{|I_\alpha|} f(I_\alpha) \otimes I_\alpha^*.$$

Let us write the three terms of the CYBE:

$$\bigl[r^{12}, r^{13}\bigr] = \sum_{\alpha,\beta} (-1)^{|I_\alpha|+|I_\beta|}(-1)^{|I_\alpha||I_\beta|}\bigl[f(I_\alpha), f(I_\beta)\bigr] \otimes I_\alpha^* \otimes I_\beta^*,$$

$$\bigl[r^{12}, r^{23}\bigr] = \sum_{\alpha,\beta} (-1)^{|I_\alpha|+|I_\beta|} f(I_\alpha) \otimes \bigl[I_\alpha^*, f(I_\beta)\bigr] \otimes I_\beta^*,$$

$$\bigl[r^{13}, r^{23}\bigr] = \sum_{\alpha,\beta} (-1)^{|I_\alpha|+|I_\beta|}(-1)^{|I_\alpha||I_\beta|} f(I_\alpha) \otimes f(I_\beta) \otimes \bigl[I_\alpha^*, I_\beta^*\bigr].$$

Here we use the consistency of the form, the evenness of f, and

$$[a\otimes b\otimes 1, c\otimes 1\otimes d] = (-1)^{|b||c|}[a,c]\otimes b\otimes d,$$
$$[a\otimes b\otimes 1, 1\otimes c\otimes d] = a\otimes [b,c]\otimes d,$$
$$[a\otimes 1\otimes b, 1\otimes c\otimes d] = (-1)^{|b||c|}a\otimes c\otimes [b,d].$$

We rewrite the last sum so that it ends with $\otimes I_\beta^*$

$$\sum_{\alpha,\beta}(-1)^{|I_\alpha|+|I_\beta|}(-1)^{|I_\alpha||I_\beta|}f(I_\alpha)\otimes f(I_\beta)\otimes\left[I_\alpha^*, I_\beta^*\right]$$
$$= -\sum_{\alpha,\beta}(-1)^{|I_\alpha|+|I_\beta|}f(I_\alpha)\otimes f\left(\left[I_\alpha^*, I_\beta\right]\right)\otimes {I_\beta}^*$$

where we use the invariance of the form, and the supersymmetry of the bracket.

Therefore we can rewrite the CYBE as

$$\sum_{\alpha,\beta}(-1)^{|I_\beta|}\begin{pmatrix}(-1)^{|I_\alpha|}(-1)^{|I_\alpha||I_\beta|}[f(I_\alpha), f(I_\beta)]\otimes I_\alpha^*\\ +(-1)^{|I_\alpha|}f(I_\alpha)\otimes[I_\alpha^*, f(I_\beta)]\\ -(-1)^{|I_\alpha|}f(I_\alpha)\otimes f([I_\alpha^*, I_\beta])\end{pmatrix}\otimes I_\beta^* = 0.$$

Since the $\{I_\beta^*\}$ form a basis for $\mathfrak{g}$, this last equation implies that, for any choice of β

$$\begin{pmatrix}\sum_\alpha(-1)^{|I_\alpha|}(-1)^{|I_\alpha||I_\beta|}[f(I_\alpha), f(I_\beta)]\otimes I_\alpha^*\\ +\sum_\alpha(-1)^{|I_\alpha|}f(I_\alpha)\otimes[I_\alpha^*, f(I_\beta)]\\ -\sum_\alpha(-1)^{|I_\alpha|}f(I_\alpha)\otimes f([I_\alpha^*, I_\beta])\end{pmatrix} = 0.$$

We want to rewrite the second and the third sums so that they end with $\otimes {I_\alpha}^*$. After some calculation, the second term becomes

$$\sum_\alpha(-1)^{|I_\alpha|}f(I_\alpha)\otimes\left[I_\alpha^*, f(I_\beta)\right] = \boxed{-\sum_\alpha(-1)^{|I_\alpha|}(-1)^{|I_\alpha||I_\beta|}f\left(\left[I_\alpha, f(I_\beta)\right]\right)\otimes I_\alpha^*}.$$

The third sum splits into two different sums when we use Eq. (3)

$$-\sum_\alpha(-1)^{|I_\alpha|}f(I_\alpha)\otimes f\left(\left[I_\alpha^*, I_\beta\right]\right)$$
$$= -\sum_\alpha(-1)^{|I_\alpha|}f(I_\alpha)\otimes\left[I_\alpha^*, I_\beta\right] + \sum_\alpha(-1)^{|I_\alpha|}f(I_\alpha)\otimes f^*\left(\left[I_\alpha^*, I_\beta\right]\right).$$

We calculate these terms separately

$$-\sum_\alpha (-1)^{|I_\alpha|} f(I_\alpha)\otimes \big[I_\alpha^*, I_\beta\big] = \boxed{\sum_\alpha (-1)^{|I_\alpha|}(-1)^{|I_\alpha||I_\beta|} f\big(\big[I_\alpha, I_\beta\big]\big)\otimes I_\alpha^*},$$

$$\sum_\alpha (-1)^{|I_\alpha|} f(I_\alpha)\otimes f^*\big(\big[I_\alpha^*, I_\beta\big]\big) = \boxed{-\sum_\alpha (-1)^{|I_\alpha|}(-1)^{|I_\alpha||I_\beta|} f\big(\big[f(I_\alpha), I_\beta\big]\big)\otimes I_\alpha^*}.$$

Hence we get

$$\sum_\alpha (-1)^{|I_\alpha|}(-1)^{|I_\alpha||I_\beta|}\begin{pmatrix}[f(I_\alpha), f(I_\beta)] - f([I_\alpha, f(I_\beta)]) \\ +f([I_\alpha, I_\beta]) - f([f(I_\alpha), I_\beta])\end{pmatrix}\otimes I_\alpha^* = 0.$$

Again using the fact that the $\{I_\alpha^*\}$ form a basis for $\mathfrak{g}$, we obtain, for all α, β

$$\big[f(I_\alpha), f(I_\beta)\big] - f\big(\big[I_\alpha, f(I_\beta)\big]\big) + f\big([I_\alpha, I_\beta]\big) - f\big(\big[f(I_\alpha), I_\beta\big]\big) = 0$$

which can be rewritten as

$$(f-1)\big[f(I_\alpha), f(I_\beta)\big] = f\big(\big[(f-1)(I_\alpha), (f-1)(I_\beta)\big]\big),$$

which is equivalent to

$$\boxed{(f-1)\big[f(x), f(y)\big] = f\big(\big[(f-1)(x), (f-1)(y)\big]\big) \quad \text{for all } x, y \in \mathfrak{g}}. \tag{6}$$

This proves one direction of the lemma. To see the other direction, we need only trace the steps above backwards. Hence one can easily see that a function f satisfying Eqs. (5) and (6) will correspond to an r-matrix $r \in \mathfrak{g}\otimes\mathfrak{g}$ that satisfies Eqs. (3) and (4). This completes the proof. □

Lemma 2. *Let f_0 be a linear map on $\mathfrak{h}$, and set $r_0 = (f_0 \otimes 1)\Omega_0$. Then the system*

$$r_0^{12} + r_0^{21} = \Omega_0, \tag{1}$$

$$\big(\tau(\alpha)\otimes 1\big)(r_0) + (1\otimes\alpha)(r_0) = 0 \quad \textit{for all } \alpha \in \Gamma_1 \tag{2}$$

is equivalent to the system

$$f_0 + f_0^* = 1, \tag{7}$$

$$f_0(h_\alpha) = (f_0 - 1)(h_{\tau(\alpha)}) \quad \textit{for all } \alpha \in \Gamma_1. \tag{8}$$

Proof. We will prove a stronger result, namely, that, for any $1 \leqslant s, t \leqslant r$, the system of equations

$$r_0^{12} + r_0^{21} = \Omega_0, \qquad (\alpha_t \otimes 1)(r_0) + (1\otimes\alpha_s)(r_0) = 0$$

is equivalent to the following system of equations

$$f_0 + f_0^* = 1, \qquad f_0(h_{\alpha_s}) = (f_0 - 1)(h_{\alpha_t}).$$

It is easy to see the equivalence of the first equations

$$r_0^{12} + r_0^{21} = (f_0 \otimes 1 + 1 \otimes f_0)\Omega_0 = \big((f_0 + f_0^*) \otimes 1\big)\Omega_0 = \Omega_0$$
$$\text{if and only if} \quad f_0 + {f_0}^* = 1.$$

Next we look at $(\alpha_t \otimes 1)r_0 + (1 \otimes \alpha_s)r_0$. This is equal to

$$\begin{aligned}
&(\alpha_t \otimes 1)\left(\sum_i f_0(h_{\alpha_i}) \otimes h_{\alpha_i}^*\right) + (1 \otimes \alpha_s)\left(\sum_i f_0(h_{\alpha_i}) \otimes h_{\alpha_i}^*\right)\\
&\quad = \sum_i \alpha_t\big(f_0(h_{\alpha_i})\big) \cdot h_{\alpha_i}^* + \sum_i \alpha_s\big(f_0^*\big(h_{\alpha_i}^*\big)\big) \cdot h_{\alpha_i}\\
&\quad = \sum_i \alpha_t\left(\sum_k \big(f_0(h_{\alpha_i}), h_{\alpha_k}\big)h_{\alpha_k}^*\right) \cdot h_{\alpha_i}^* + \sum_i \alpha_s\left(\sum_k \big(f_0^*\big(h_{\alpha_i}^*\big), h_{\alpha_k}\big)h_{\alpha_k}^*\right) \cdot h_{\alpha_i}\\
&\quad = \sum_{i,k} \big(f_0(h_{\alpha_i}), h_{\alpha_k}\big)\alpha_t\big(h_{\alpha_k}^*\big) \cdot h_{\alpha_i}^* + \sum_{i,k} \big(f_0^*\big(h_{\alpha_i}^*\big), h_{\alpha_k}\big)\alpha_s\big(h_{\alpha_k}^*\big) \cdot h_{\alpha_i}.
\end{aligned}$$

We have

$$\alpha_s\big(h_{\alpha_k}^*\big) = \big(h_{\alpha_s}, h_{\alpha_k}^*\big) = \delta_{sk} \quad \text{and} \quad \alpha_t\big(h_{\alpha_k}^*\big) = \big(h_{\alpha_t}, h_{\alpha_k}^*\big) = \delta_{tk}.$$

Therefore the above expression becomes

$$\begin{aligned}
&\sum_i \big(f_0(h_{\alpha_i}), h_{\alpha_t}\big)h_{\alpha_i}^* + \sum_i \big(f_0^*\big(h_{\alpha_i}^*\big), h_{\alpha_s}\big)h_{\alpha_i}\\
&\quad = \sum_i \big(h_{\alpha_i}, f_0^*(h_{\alpha_t})\big)h_{\alpha_i}^* + \sum_i \big(h_{\alpha_i}^*, f_0(h_{\alpha_s})\big)h_{\alpha_i}\\
&\quad = f_0^*(h_{\alpha_t}) + f_0(h_{\alpha_s}) = (1 - f_0)(h_{\alpha_t}) + f_0(h_{\alpha_s}).
\end{aligned}$$

This shows that $(\alpha_t \otimes 1)r_0 + (1 \otimes \alpha_s)r_0 = (1 - f_0)(h_{\alpha_t}) + f_0(h_{\alpha_s})$. Clearly, one side is equal to zero if and only if the other side is. This proves the lemma. □

We also need the consistency of the system of equations

$$r_0^{12} + r_0^{21} = \Omega_0, \tag{1}$$

$$\big(\tau(\alpha) \otimes 1\big)(r_0) + (1 \otimes \alpha)(r_0) = 0 \quad \text{for all } \alpha \in \Gamma_1. \tag{2}$$

However, the arguments used to prove this are the same as for the Lie algebra case (see [2] for details), and hence will not be included here.

The results of this section allow us to translate the conditions on the continuous parameter of the main theorem into conditions on a linear map $f_0:\mathfrak{h}\to\mathfrak{h}$, and the CYBE and Eq. (3) become conditions on the associated linear map $f:\mathfrak{g}\to\mathfrak{g}$. Thus we can restate our problem as follows: given an admissible triple (Γ_1,Γ_2,τ) with a linear map $f_0:\mathfrak{h}\to\mathfrak{h}$ satisfying Eqs. (7) and (8), construct a linear map $f:\mathfrak{g}\to\mathfrak{g}$ satisfying Eqs. (5) and (6).

5. The Cayley transform

Following [2], we will now introduce a variation on the theme of Cayley transforms. For a linear function $f:\mathfrak{g}\to\mathfrak{g}$ with $(f-1)$ invertible, the *Cayley transform* of f is $\Theta = f/(f-1)$. If f satisfies Eq. (5), then $\Theta^* = f^*/(f-1)^* = (1-f)/-f$. Then we can see that $\Theta\Theta^*=1$, so Θ preserves the invariant form. If f also satisfies Eq. (6), then we have $[\Theta(x),\Theta(y)] = \Theta([x,y])$, so Θ is a Lie superalgebra automorphism.

However, this does not work for simple Lie algebras, and in fact it does not work for simple Lie superalgebras, either. To see this, assume that f is a linear map satisfying Eqs. (5) and (6), $f-1$ is invertible, and Θ is defined as above. Then $\Theta-1$ is the inverse of $f-1$, so $\det(\Theta-1)\neq 0$. But we have:

Lemma 3. *If Θ is an automorphism of a finite dimensional* (*classical*) *simple Lie superalgebra* $\mathfrak{g}$, *then* $\det(\Theta-1)=0$.

Proof. The automorphism Θ restricts to a (Lie algebra) automorphism θ on $\mathfrak{g}_{\bar{0}}$, the even part of $\mathfrak{g}$. $\mathfrak{g}_{\bar{0}}$ is reductive with non-trivial $\mathfrak{g}_{\bar{0}}' = [\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{0}}]$. $\mathfrak{g}_{\bar{0}}'$ is semisimple and θ restricts to an automorphism φ on $\mathfrak{g}_{\bar{0}}'$. Using Theorem 9.2 of [2] we can find some non-zero $x\in\mathfrak{g}_{\bar{0}}'$ with $\varphi(x)=x$. Then $\Theta(x)=x$ and hence $x\in \operatorname{Ker}(\Theta-1)$. Thus $\det(\Theta-1)=0$. □

Thus Eqs. (5) and (6) imply that $f-1$ is not invertible. Therefore, we cannot define the Cayley transform as above for the functions we are interested in.

However it turns out that we can modify our definition and still get a lot of what we want. First note that for any linear operator f, $\operatorname{Ker}(f)\subset \operatorname{Im}(f-1)$ and $\operatorname{Ker}(f-1)\subset\operatorname{Im}(f)$. We will define the *Cayley transform of f* to be the function $\Theta:\operatorname{Im}(f-1)/\operatorname{Ker}(f)\to \operatorname{Im}(f)/\operatorname{Ker}(f-1)$ that maps $\overline{(f-1)(x)}$ to $\overline{f(x)}$. (It is easy to check that this is well defined.) This version of the Cayley transform will be sufficient for our purposes. We have:

Lemma 4. *Let $f:\mathfrak{g}\to\mathfrak{g}$ be a linear map satisfying*

$$f+f^*=1. \tag{5}$$

Then $\operatorname{Ker}(f)=\operatorname{Im}(f-1)^\perp$, $\operatorname{Ker}(f-1)=\operatorname{Im}(f)^\perp$, *and the map Θ preserves the invariant form. Furthermore, f satisfies*

$$(f-1)\big[f(x),f(y)\big] = f\big(\big[(f-1)(x),(f-1)(y)\big]\big), \tag{6}$$

if and only if $\mathrm{Im}(f)$ *and* $\mathrm{Im}(f-1)$ *are Lie subsuperalgebras of* $\mathfrak{g}$, *and* Θ *is a Lie superalgebra isomorphism.*

Remark. The proof of this lemma is exactly the same as the proof of the analogous result in the Lie algebra case. See [2].

6. The construction—end of the proof of the theorem

For a given admissible triple $(\Gamma_1, \Gamma_2, \tau)$, and a linear map $f_0 : \mathfrak{h} \to \mathfrak{h}$ satisfying Eqs. (7) and (8), we want to construct a function $f : \mathfrak{g} \to \mathfrak{g}$ that will satisfy Eqs. (5) and (6). Here is how we proceed.

Define $\overline{\Gamma}_i$ and $\bar{\tau}$ as above. Also define the following Lie subsuperalgebras of $\mathfrak{g}$

$$\mathfrak{h}_i = \bigoplus_{\alpha \in \Gamma_i} \mathbb{C} h_\alpha, \qquad \mathfrak{g}_i = \mathfrak{h}_i \oplus \sum_{\alpha \in \overline{\Gamma}_i} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

$$\mathfrak{n}_i^+ = \sum_{\alpha \in \Delta^+ / \overline{\Gamma}_i} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_i^+ = \mathfrak{g}_i + \mathfrak{n}_i{}^+,$$

$$\mathfrak{n}_i^- = \sum_{\alpha \in \Delta^+ / \overline{\Gamma}_i} \mathfrak{g}_{-\alpha}, \qquad \mathfrak{p}_i^- = \mathfrak{g}_i + \mathfrak{n}_i^-.$$

We can see that the $\mathfrak{n}_i^{+/-}$ are ideals in $\mathfrak{p}_i^{+/-}$.

Let $f_0 : \mathfrak{h} \to \mathfrak{h}$ satisfy Eq. (8). Then

$$h_\alpha = (f_0 - 1)(h_{\tau(\alpha)} - h_\alpha), \qquad h_{\tau(\alpha)} = f_0(h_{\tau(\alpha)} - h_\alpha)$$

for all $\alpha \in \Gamma_1$. This implies that $h_\alpha \in \mathrm{Im}(f_0 - 1)$ and $h_{\tau(\alpha)} \in \mathrm{Im}(f_0)$. Therefore $\mathfrak{h}_1 \subset \mathrm{Im}(f_0 - 1)$, and $\mathfrak{h}_2 \subset \mathrm{Im}(f_0)$.

Fix a Weyl–Chevalley basis $\{X_{\alpha_i}, Y_{\alpha_i}, H_{\alpha_i} \mid \alpha_i \in \Gamma\}$. It is known that such a set of generators exists and satisfies the usual Serre-type relations (see [5] and [7] for details). Define a map φ by

$$\varphi(X_\alpha) = X_{\tau(\alpha)}, \qquad \varphi(Y_\alpha) = Y_{\tau(\alpha)}, \qquad \varphi(H_\alpha) = H_{\tau(\alpha)}$$

for all $\alpha \in \Gamma_1$. Then this can be extended to an isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ because the relations between X_α, Y_α, H_α for $\alpha \in \Gamma_1$ will be the same as the relations between $X_{\tau(\alpha)}, Y_{\tau(\alpha)}, H_{\tau(\alpha)}$ for $\alpha \in \Gamma_1$. (Here we are using the fact that τ is an isometry preserving grading.) Note that φ^{-1} is a map from $\mathfrak{g}_2$ onto $\mathfrak{g}_1$. Since τ is an isometry, $(\varphi(x), y)_{\mathfrak{g}_2} = (x, \varphi^{-1}(y))_{\mathfrak{g}_1}$ for all $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_2$. But φ^* should map $\mathfrak{g}_2$ into $\mathfrak{g}_1$ and satisfy exactly the same conditions; hence $\varphi^* = \varphi^{-1}$.

For each $\alpha \in \Delta$, choose an element $e_\alpha \in \mathfrak{g}_\alpha$ such that $(e_\alpha, e_{-\alpha}) = 1$ for any α, and $\varphi(e_\alpha) = e_{\overline{\tau}(\alpha)}$ for all $\alpha \in \overline{\Gamma}_1$. The conditions on τ ensure that this is possible. Next define a linear map as follows

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } x \in \mathfrak{g}_1, \\ 0 & \text{if } x \in \mathfrak{n}_1^+. \end{cases}$$

This restricts to a map on $\mathfrak{n}_+ = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$, since $\mathfrak{n}_+ = (\mathfrak{g}_1 \cap \mathfrak{n}_+) \oplus \mathfrak{n}_1^+$. The proof of the following lemma is exactly the same as in the Lie algebra case (see [2]):

Lemma 5. $\det(\psi - 1)$ *is non-zero if and only if* τ *satisfies the second condition in the definition of an admissible triple.*

Therefore we can define a function on $\mathfrak{n}_+$ by

$$f_+ = \frac{\psi}{\psi - 1} = -\bigl(\psi + \psi^2 + \cdots\bigr).$$

Clearly the sum on the right-hand side is finite as ψ is nilpotent. Notice that ψ^* and so f_+^* are maps on $\mathfrak{n}_- = \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$, since the Killing form induces a non-degenerate pairing of $\mathfrak{n}_+$ with $\mathfrak{n}_-$.

Now define a linear map on $\mathfrak{n}_-$ by

$$f_- = 1 - f_+^* = 1 + \psi^* + \psi^{*2} + \cdots.$$

Then define f to be the function whose restriction to $\mathfrak{h}$, $\mathfrak{n}_+$, $\mathfrak{n}_-$ is f_0, f_+, f_-, respectively. (Note that f is even.) We have

$$\begin{aligned} f + f^* &= (f_0 + f_+ + f_-) + (f_0 + f_+ + f_-)^* \\ &= \bigl(f_0 + f_0^*\bigr) + \bigl(f_+ + f_-^*\bigr) + \bigl(f_+^* + f_-\bigr) \\ &= 1_{\mathfrak{h}} + 1_{\mathfrak{n}_+} + 1_{\mathfrak{n}_-} = 1_{\mathfrak{g}}. \end{aligned}$$

Lemma 4 implies that, to show that f satisfies Eq. (6), one only needs to show that $C_1 = \mathrm{Im}(f - 1)$ and $C_2 = \mathrm{Im}(f)$ are Lie subsuperalgebras of $\mathfrak{g}$, and the Cayley transform Θ of f is a Lie superalgebra isomorphism. We have

$$\begin{aligned} C_1 &= \mathrm{Im}(f - 1) = \mathrm{Im}(f_0 - 1) \oplus \mathrm{Im}(f_+ - 1) \oplus \mathrm{Im}(f_- - 1), \\ C_2 &= \mathrm{Im}(f) = \mathrm{Im}(f_0) \oplus \mathrm{Im}(f_+) \oplus \mathrm{Im}(f_-). \end{aligned}$$

We have seen that $\mathrm{Im}(f_0 - 1) \supset \mathfrak{h}_1$ and $\mathrm{Im}(f_0) \supset \mathfrak{h}_2$. We will therefore define V_1, V_2 as (vector) subspaces of $\mathfrak{h}$ such that $\mathrm{Im}(f_0 - 1) = \mathfrak{h}_1 \oplus V_1$ and $\mathrm{Im}(f_0) = \mathfrak{h}_2 \oplus V_2$.

In the Lie algebra case, the Killing form restricts to a positive definite non-degenerate form on (the real subspace generated by $\{H_\alpha \mid \alpha \in \Gamma\}$ of) $\mathfrak{h}$. So we can define the orthogonal complements of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ with respect to this form; call these $\mathfrak{h}_1^c$ and $\mathfrak{h}_2^c$; then we have:

$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_1^c = \mathfrak{h}_2 \oplus \mathfrak{h}_2^c$. Then for a fixed f_0, the two subspaces V_1 and V_2 are uniquely determined if we add the condition that $V_i \subset \mathfrak{h}_i^c$. In the super case, this is no longer possible; the real Cartan subalgebra $\mathfrak{h}$ may have isotropic elements and subspaces of $\mathfrak{h}$ may intersect their orthogonal complements non-trivially. However in our case we still can define $\mathfrak{h}_i^c$ as follows

$$\mathfrak{h}_i^c = \bigoplus_{\alpha \in \Gamma \setminus \Gamma_i} \mathbb{C} h_\alpha.$$

Thus we still can write $\mathfrak{h} = \mathfrak{h}_i \oplus \mathfrak{h}_i^c$, and still can demand that $V_i \subset \mathfrak{h}_i^c$. In this way the V_i are then well defined, but clearly depend on the choice of Γ.

Next we compute

$$\begin{aligned}
\operatorname{Im}(f_+ - 1) &= \operatorname{Im}\left(\frac{1}{\psi - 1}\right) = \mathfrak{n}_+,\\
\operatorname{Im}(f_- - 1) &= \operatorname{Im}\left(\frac{\psi^*}{1 - \psi^*}\right) = \operatorname{Im}\big(\psi^*\big) = \mathfrak{g}_1 \cap \mathfrak{n}_-,\\
\operatorname{Im}(f_+) &= \operatorname{Im}\left(\frac{\psi}{1 - \psi}\right) = \operatorname{Im}(\psi) = \mathfrak{g}_2 \cap \mathfrak{n}_+,\\
\operatorname{Im}(f_-) &= \operatorname{Im}\left(\frac{1}{1 - \psi^*}\right) = \mathfrak{n}_-
\end{aligned}$$

where we use the fact that $\psi - 1$ is invertible. The above then yields

$$C_1 = \mathfrak{p}_1^+ \oplus V_1, \qquad C_2 = \mathfrak{p}_2^- \oplus V_2.$$

It is now easy to check that C_1 and C_2 are both closed under the bracket and hence are Lie subsuperalgebras of $\mathfrak{g}$.

Finally we need to see that the Cayley transform Θ is a Lie superalgebra isomorphism. We note that by the last lemma above, $C_i \supset C_i^\perp$. So we have

$$C_1^\perp = \big(\mathfrak{p}_1^+ \oplus V_1\big)^\perp = \mathfrak{n}_1^+ \oplus (\mathfrak{h}_1 \oplus V_1)^\perp = \mathfrak{n}_1^+ \oplus \big(\mathfrak{h}_1^\perp \cap V_1^\perp\big) \subset \mathfrak{p}_1^+ \oplus V_1$$

and similarly

$$C_2^\perp = \big(\mathfrak{p}_2^- \oplus V_2\big)^\perp = \mathfrak{n}_2^- \oplus (\mathfrak{h}_2 \oplus V_2)^\perp = \mathfrak{n}_2^- \oplus \big(\mathfrak{h}_2^\perp \cap V_2^\perp\big) \subset \mathfrak{p}_2^- \oplus V_2.$$

Hence $\mathfrak{h}_i^\perp \cap V_i^\perp \subset \mathfrak{h}_i \oplus V_i$, and so

$$C_1/C_1^\perp = \frac{\mathfrak{p}_1^+ \oplus V_1}{(\mathfrak{p}_1^+ \oplus V_1)^\perp} = \left(\bigoplus_{\alpha \in \overline{\Gamma}_1} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\right) \oplus \frac{\mathfrak{h}_1 \oplus V_1}{\mathfrak{h}_1^\perp \cap V_1^\perp}$$

and similarly

$$C_2/C_2^{\perp} = \frac{\mathfrak{p}_2^- \oplus V_2}{(\mathfrak{p}_2^- \oplus V_2)^{\perp}} = \left(\bigoplus_{\alpha\in\overline{\Gamma}_2} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\right) \oplus \frac{\mathfrak{h}_2 \oplus V_2}{\mathfrak{h}_2^{\perp} \cap V_2^{\perp}}.$$

Since C_i is a Lie subsuperalgebra and $C_i^{\perp}$ is an ideal, there is a Lie superalgebra structure on $C_i/C_i^{\perp}$. But $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}H_\alpha$, therefore there is a complete copy of $\mathfrak{h}_i$ and a copy of $\mathfrak{g}_i$ in $C_i/C_i^{\perp}$. Thus $\mathfrak{h}_i^{\perp} \cap V_i^{\perp} \subset V_i$, and

$$C_i/C_i^{\perp} = \mathfrak{g}_i \oplus \frac{V_i}{\mathfrak{h}_i^{\perp} \cap V_i^{\perp}}.$$

So we need to show that

$$\Theta : \mathfrak{g}_1 \oplus \frac{V_1}{\mathfrak{h}_1^{\perp} \cap V_1^{\perp}} \to \mathfrak{g}_2 \oplus \frac{V_2}{\mathfrak{h}_2^{\perp} \cap V_2^{\perp}}$$

is a Lie superalgebra isomorphism.

We first note that $\Theta(x) = \varphi(x)$ for all $x \in \mathfrak{g}_1$. Indeed if $\alpha \in \Gamma_1$

$$X_\alpha = (f_+ - 1)(X_{\tau(\alpha)} - X_\alpha)$$

and so is mapped via Θ to

$$f_+(X_{\tau(\alpha)} - X_\alpha) = X_{\tau(\alpha)}$$

and similarly

$$Y_\alpha = (f_- - 1)(Y_{\tau(\alpha)} - Y_\alpha)$$

is mapped via Θ to

$$f_-(Y_{\tau(\alpha)} - Y_\alpha) = Y_{\tau(\alpha)}.$$

Also it is easy to see that since $H_\alpha = (f_0 - 1)(H_{\tau(\alpha)} - H_\alpha)$ for each $\alpha \in \Gamma_1$, Θ sends H_α to $f_0(H_{\tau(\alpha)} - H_\alpha) = H_{\tau(\alpha)}$. Hence the restriction of Θ to $\mathfrak{g}_1$ is exactly the Lie superalgebra isomorphism φ.

Next we look at how Θ acts on the Cartan part of the $C_i/{C_i}^{\perp}$. We have

$$C_i/C_i^{\perp} = \left(\bigoplus_{\alpha\in\overline{\Gamma}_i} \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\right) \oplus \frac{\mathfrak{h}_i \oplus V_i}{\mathfrak{h}_i^{\perp} \cap V_i^{\perp}},$$

i.e., $C_i/C_i^{\perp}$ is the direct sum of a Cartan part and a non-Cartan part. Then the arguments above show that Θ, like φ, maps the non-Cartan part of $C_1/C_1^{\perp}$ into the non-Cartan part of $C_2/C_2^{\perp}$. Also, since Θ preserves the invariant form, it maps the Cartan part of $C_1/C_1^{\perp}$ to the Cartan part of $C_2/C_2^{\perp}$

$$\Theta\bigl(\bigl(\text{non-Cartan of } C_1/C_1^{\perp}\bigr)^{\perp}\bigr) = \bigl(\text{non-Cartan of } C_2/C_2^{\perp}\bigr)^{\perp}.$$

In other words

$$\Theta\left(\frac{\mathfrak{h}_1 \oplus V_1}{\mathfrak{h}_1^{\perp} \cap V_1^{\perp}}\right) = \left(\frac{\mathfrak{h}_2 \oplus V_2}{\mathfrak{h}_2^{\perp} \cap V_2^{\perp}}\right).$$

Since $\mathfrak{h}_i \oplus V_i/(\mathfrak{h}_i^{\perp} \cap V_i^{\perp})$ is abelian, Θ restricts to an isomorphism there as well. Therefore Θ is an isomorphism. Therefore the associated linear map f satisfies Eqs. (5) and (6) and so corresponds to an r-matrix satisfying Eqs. (3) and (4).

Checking that the function f constructed in this way yields the tensor r of Eq. $(*)$ is straightforward. This completes the proof of the theorem.

7. Examples: r-matrices on $sl(2,1)$

Recall that two Dynkin diagrams of a given Lie superalgebra may be non-isomorphic, but one can be obtained from another via a chain of odd reflections (see [10] for information about odd reflections and more on root systems of graded Lie algebras). Then we may wish to know how r-matrices obtained from two non-isomorphic Dynkin diagrams are related, if at all. This question in all its generality needs to be addressed systematically. However, we will see that at least in the case of $sl(2,1)$, if r and r' are the standard r-matrices associated to the Dynkin diagrams D and D', respectively, and D' is obtained from D by the odd reflection σ_α associated to the root α, then r' is the image of r under σ_α.

7.1. Dynkin diagrams of $sl(2,1)$

The roots of $sl(2,1)$ are

$$\Delta_{\bar{0}} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_1\}, \qquad \Delta_{\bar{1}} = \{\varepsilon_1 - \lambda_1, \varepsilon_2 - \lambda_1, \lambda_1 - \varepsilon_1, \lambda_1 - \varepsilon_2\}$$

where ε_i is the (restriction to the Cartan subalgebra of $sl(2,1)$ of the) standard basis: $\varepsilon_i(E_{jk}) = \delta_{ij}\delta_{ik}$, and $\lambda_1 = \varepsilon_3$. Denote the set of simple roots by Γ.

There are six possible Dynkin diagrams:

(1) $\Gamma(D_1) = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \lambda_1\}$. We will set $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \lambda_1$. α_1 is even; α_2 is odd. The third positive root is $\alpha_1 + \alpha_2$ and is odd.
(2) $\Gamma(D_2) = \{\varepsilon_1 - \lambda_1, \lambda_1 - \varepsilon_2\} = \{\alpha_1 + \alpha_2, -\alpha_2\}$. D_2 is obtained from D_1 via the odd reflection σ_{α_2}. The third positive root is α_1 and is even.
(3) $\Gamma(D_3) = \{\lambda_1 - \varepsilon_1, \varepsilon_1 - \varepsilon_2\} = \{-\alpha_1 - \alpha_2, \alpha_1\}$. D_3 is obtained from D_2 via the odd reflection $\sigma_{\alpha_1+\alpha_2}$. The third positive root is $-\alpha_2$ and is odd.
(4) $\Gamma(D_4) = -\Gamma(D_1) = \{-\alpha_1, -\alpha_2\}$. The third positive root is $-\alpha_1 - \alpha_2$ and is odd.

(5) $\Gamma(D_5) = -\Gamma(D_2) = \{-\alpha_1 - \alpha_2, \alpha_2\}$. The third positive root is $-\alpha_1$ and is even. D_5 is obtained from D_4 via $\sigma_{-\alpha_2}$, as expected.
(6) $\Gamma(D_6) = -\Gamma(D_3) = \{\alpha_1 + \alpha_2, -\alpha_1\}$. The third positive root is α_2 and is odd. D_6 is obtained from D_5 via the odd reflection $\sigma_{-\alpha_1-\alpha_2}$.

Hence, up to sign, there are three Dynkin diagrams, and these can be obtained from one another via a chain of odd reflections (which change the signs of some of the odd roots but a positive even root stays positive).

7.2. The standard r-matrices

Given $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfying $r_0 + r_0^{21} = \Omega_0$, the standard r-matrix for a fixed Dynkin diagram is

$$r = r_0 + \sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha.$$

So fixing r_0 we write down the standard r-matrices for the above diagrams:

(1) D_1: let $e_{\alpha_1} = E_{12}$, $e_{\alpha_2} = E_{23}$, $e_{\alpha_1+\alpha_2} = E_{13}$. This determines $e_{-\alpha}$ by $(e_\alpha, e_{-\alpha}) = 1$: $e_{-\alpha_1} = E_{21}$, $e_{-\alpha_2} = E_{32}$, $e_{-\alpha_1-\alpha_2} = E_{31}$. Therefore we get

$$r_{st}(D_1) = r_0 + (E_{21} \otimes E_{12}) + (E_{32} \otimes E_{23}) + (E_{31} \otimes E_{13}).$$

(2) D_2: let $e_{\alpha_1} = E_{12}$, $e_{-\alpha_2} = E_{32}$, $e_{\alpha_1+\alpha_2} = E_{13}$. This determines $e_{-\alpha}$ by $(e_\alpha, e_{-\alpha}) = 1$: $e_{-\alpha_1} = E_{21}$, $e_{\alpha_2} = -E_{23}$, $e_{-\alpha_1-\alpha_2} = E_{31}$. Therefore we get

$$r_{st}(D_2) = r_0 + (E_{21} \otimes E_{12}) - (E_{23} \otimes E_{32}) + (E_{31} \otimes E_{13}).$$

(3) D_3: let $e_{\alpha_1} = E_{12}$, $e_{-\alpha_2} = E_{32}$, $e_{-\alpha_1-\alpha_2} = E_{31}$. This determines $e_{-\alpha}$ by $(e_\alpha, e_{-\alpha}) = 1$: $e_{-\alpha_1} = E_{21}$, $e_{\alpha_2} = -E_{23}$, $e_{\alpha_1+\alpha_2} = -E_{13}$. Therefore we get

$$r_{st}(D_3) = r_0 + (E_{21} \otimes E_{12}) - (E_{23} \otimes E_{32}) - (E_{13} \otimes E_{31}).$$

(4) D_4: let $e_{-\alpha_1} = E_{21}$, $e_{-\alpha_2} = E_{32}$, $e_{-\alpha_1-\alpha_2} = E_{31}$. This determines $e_{-\alpha}$ by $(e_\alpha, e_{-\alpha}) = 1$: $e_{\alpha_1} = E_{12}$, $e_{\alpha_2} = -E_{23}$, $e_{\alpha_1+\alpha_2} = -E_{13}$. Therefore we get

$$r_{st}(D_4) = r_0 + (E_{12} \otimes E_{21}) - (E_{23} \otimes E_{32}) - (E_{13} \otimes E_{31}).$$

(5) D_5: let $e_{-\alpha_1} = E_{21}$, $e_{\alpha_2} = E_{23}$, $e_{-\alpha_1-\alpha_2} = E_{31}$. This determines $e_{-\alpha}$ by $(e_\alpha, e_{-\alpha}) = 1$: $e_{\alpha_1} = E_{12}$, $e_{-\alpha_2} = E_{32}$, $e_{\alpha_1+\alpha_2} = -E_{13}$. Therefore we get

$$r_{st}(D_5) = r_0 + (E_{12} \otimes E_{21}) + (E_{32} \otimes E_{23}) - (E_{13} \otimes E_{31}).$$

(6) D_6: let $e_{-\alpha_1} = E_{21}$, $e_{\alpha_2} = E_{23}$, $e_{\alpha_1+\alpha_2} = E_{13}$. This determines $e_{-\alpha}$ by $(e_\alpha, e_{-\alpha}) = 1$: $e_{\alpha_1} = E_{12}$, $e_{-\alpha_2} = E_{32}$, $e_{-\alpha_1-\alpha_2} = E_{31}$. Therefore we get

$$r_{st}(D_6) = r_0 + (E_{12} \otimes E_{21}) + (E_{32} \otimes E_{23}) + (E_{31} \otimes E_{13}).$$

We note that the first three of the r-matrices constructed above (and similarly the last three) are connected via odd reflections which correspond to the odd reflections that connect the associated Dynkin diagrams. The even reflection which changes the signs of the even roots will connect the first three to the last three. Hence all these r-matrices are related to one another via (even or odd) reflections.

7.3. Constructing non-standard r-matrices

The non-standard r-matrices that we can construct with our theorem come from the two diagrams D_2 and D_5.

For D_2 let $\Gamma_1 = \{\alpha_1 + \alpha_2\}$ and $\Gamma_2 = \{-\alpha_2\}$. Define $\tau(\alpha_1 + \alpha_2) = -\alpha_2$. The partial order on positive roots will be: $\alpha_1 + \alpha_2 \prec -\alpha_2$. Given that r_0 satisfies

$$(-\alpha_2 \otimes 1)(r_0) + \bigl(1 \otimes (\alpha_1 + \alpha_2)\bigr)(r_0) = 0,$$

the associated r-matrix will be

$$\begin{aligned} r_{ns_1} = r_0 &+ (E_{21} \otimes E_{12}) - (E_{23} \otimes E_{32}) + (E_{31} \otimes E_{13}) \\ &+ \bigl((E_{31} \otimes E_{32}) + (E_{32} \otimes E_{31})\bigr). \end{aligned}$$

The first few terms will actually make up $r_{st}(D_2)$ for the chosen r_0, so we can rewrite the above as

$$r_{ns_1} = r_{st}(D_2) + (E_{31} \otimes E_{32}) + (E_{32} \otimes E_{31}).$$

For D_5 let $\Gamma_1 = \{\alpha_2\}$ and $\Gamma_2 = \{-\alpha_1 - \alpha_2\}$. Define $\tau(\alpha_2) = -\alpha_1 - \alpha_2$. The partial order on positive roots will be: $\alpha_2 \prec -\alpha_1 - \alpha_2$. Given that r_0 satisfies

$$\bigl((-\alpha_1 - \alpha_2) \otimes 1\bigr)(r_0) + (1 \otimes \alpha_2)(r_0) = 0,$$

the associated r-matrix will be

$$\begin{aligned} r_{ns_2} = r_0 &+ (E_{12} \otimes E_{21}) + (E_{32} \otimes E_{23}) - (E_{13} \otimes E_{31}) \\ &+ \bigl((E_{32} \otimes E_{31}) + (E_{31} \otimes E_{32})\bigr). \end{aligned}$$

The first few terms will actually make up $r_{st}(D_5)$ for the chosen r_0, so we can rewrite the above as

$$r_{ns_2} = r_{st}(D_5) + (E_{32} \otimes E_{31}) + (E_{31} \otimes E_{32}).$$

Note that if for D_2, we redefine τ so that $\tau(-\alpha_2) = \alpha_1 + \alpha_2$, then $-\alpha_2 \prec \alpha_1 + \alpha_2$, and we get

$$r_{ns_3} = r_{st}(D_2) + (-E_{23} \otimes E_{13}) + (-E_{13} \otimes E_{23});$$

and if for D_5, we redefine τ by $\tau(-\alpha_1 - \alpha_2) = \alpha_2$, then the order becomes: $-\alpha_1 - \alpha_2 \prec \alpha_2$, and we get

$$r_{ns_4} = r_{st}(D_5) + (-E_{13} \otimes E_{23}) + (-E_{23} \otimes E_{13}).$$

8. Conclusion

In the Lie algebra case, the main classification theorem comes in two parts. The constructive part that gives an r-matrix for a given admissible triple is accompanied with the assertion that any given r-matrix that satisfies $r + r^{21} = \Omega$ can be obtained by the same construction for a suitable choice of an admissible triple. We would like to prove such an assertion for Lie superalgebras, or come up with a counterexample.

We consider once again the simple Lie superalgebra $sl(2, 1)$. We define

$$\begin{aligned} f(E_{11} + E_{33}) = 0, \qquad & f(E_{22} + E_{33}) = E_{22} + E_{33}, \\ f(E_{21}) = 0, \qquad & f(E_{12}) = E_{12}, \\ f(E_{23}) = 0, \qquad & f(E_{13}) = E_{13}, \\ f(E_{31}) = -E_{13}, \qquad & f(E_{32}) = E_{23} + E_{32} \end{aligned}$$

and extend f to a linear map on $\mathfrak{g}$. We can easily check that this function satisfies Eq. (6) which is equivalent to the associated 2-tensor being an r-matrix.

We write the quadratic Casimir element

$$\begin{aligned} \Omega = \Omega_0 &+ (E_{12} \otimes E_{21} + E_{21} \otimes E_{12}) + (-E_{13} \otimes E_{31} + E_{31} \otimes E_{13}) \\ &+ (-E_{23} \otimes E_{32} + E_{32} \otimes E_{23}) \end{aligned}$$

where $\Omega_0 = (E_{11} + E_{33}) \otimes (-E_{22} - E_{33}) + (-E_{22} - E_{33}) \otimes (E_{11} + E_{33})$. Then if we define $r(f)$ to be the 2-tensor $(f \otimes 1)\Omega$, we get

$$r(f) = r_0 + E_{12} \otimes E_{21} - E_{13} \otimes E_{31} + E_{32} \otimes E_{23} - E_{13} \otimes E_{13} + E_{23} \otimes E_{23}$$

where $r_0 = (-E_{22} - E_{33}) \otimes (E_{11} + E_{33})$. Clearly $r(f)$ satisfies Eq. (3).

This r-matrix is not among those constructed using Theorem 2. In fact we can prove that the two subsuperalgebras $\mathrm{Im}(f)$ and $\mathrm{Im}(f-1)$ will never be simultaneously isomorphic to root subsuperalgebras. The corresponding subsuperalgebras for functions constructed by the theorem will always be root subsuperalgebras. Thus the Belavin–Drinfeld type data we used is not enough to classify all solutions to the system of Eqs. (3) and (4). A full classification result should also explain how r-matrices obtained from non-isomorphic Dynkin diagrams are related to one another. We hope to address these problems in a separate paper.

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