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Multivariate Distributions having Weibull Properties*

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Random variables $X_1, \ldots, X_n$ are said to have a joint distribution with Weibull minimums after arbitrary scaling if $\min_i(a_i X_i)$ has a one dimensional Weibull distribution for arbitrary constants $a_i > 0$, $i = 1, \ldots, n$. Some properties of this class are demonstrated, and some examples are given which show the existence of a number of distributions belonging to the class. One of the properties is found to be useful for computing component reliability importance. The class is seen to contain an absolutely continuous Weibull distribution which can be generated from independent uniform and gamma distributions.

1. INTRODUCTION

In the following $\bar{F}(\mathbf{x}) = P(X_1 > x_1, \ldots, X_n > x_n)$ is the joint survival function of nonnegative random variables $X_1, \ldots, X_n$ and $R = -\log \bar{F}$ is the hazard function which is nondecreasing and defined for nonnegative $\mathbf{x}$.

The Weibull distribution, $\bar{F}(x) = \exp(-kx^\alpha)$, $x \geqslant 0$, has become an important, often used, model for life length. Several multivariate extensions have been suggested [10, 11, 14]. However, the extensions appear to have little in common with the univariate Weibull distribution except that the marginal distributions are Weibull. An exception is the Weibull distribution mentioned by Marshall and Olkin [14], and also discussed in [12], which has the following form:

$$\bar{F}(\mathbf{x}) = \exp\left(-\sum_J \lambda_J \max_{i \in J} (x_i^\alpha)\right), \qquad \mathbf{x} > 0 \tag{1.1}$$

with $\alpha > 0$ and $\lambda_J > 0$ for $J \in \mathscr{J}$ where the sets J are elements of the class $\mathscr{J}$ of nonempty subsets of $\{1, \ldots, n\}$ having the property that for each i, $i \in J$ for

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some $J \in \mathscr{J}$. For $\alpha = 1$, (1.1) is the Marshall–Olkin [14] multivariate exponential distribution.

The purpose of this paper is to develop some properties of the class of multivariate distributions having Weibull minimums after arbitrary scaling. Random variables $X_1, \ldots, X_n$ have such a distribution if for arbitrary constants $a_i > 0$, $i = 1, \ldots, n$, $\min_i(a_i X_i)$ has a one dimensional Weibull distribution,

$$P(\min_i (a_i X_i) > t) = \exp(-k(\mathbf{a})\, t^\alpha), \qquad t \geqslant 0, \tag{1.2}$$

for some $\alpha > 0$ and constant $k(\mathbf{a}) > 0$. The Weibull distribution (1.1) belongs to this class, as do a number of other distributions which are presented in the next section.

2. Classes of Weibull Distributions

To clarify differences between distributions satisfying (1.2) and other classes of multivariate Weibull distributions it is helpful to consider a hierarchy of classes of multivariate Weibull distributions.

Consider random variables $X_1, \ldots, X_n$ having a joint distribution which satisfies one of the following conditions.

(a) $X_1, \ldots, X_n$ are independent and each X_i has a Weibull distribution of the form $\bar{F}_i(t) = \exp(-\lambda_i t^\alpha)$, $t \geqslant 0$, $i = 1, \ldots, n$.

(b) $X_1, \ldots, X_n$ have a multivariate Weibull distribution generated from independent Weibull distributions by letting

$$X_i = \min(Z_J : i \in J), \qquad i = 1, \ldots, n,$$

where the sets J are elements of a class $\mathscr{J}$ of nonempty subsets of $\{1, \ldots, n\}$ having the property that for each i, $i \in J$ for some $J \in \mathscr{J}$, and the random variables Z_J, $J \in \mathscr{J}$, are independent having Weibull distributions of the form $\bar{F}_J(t) = \exp(-\lambda_J t^\alpha)$.

(c) $X_1, \ldots, X_n$ have a joint distribution satisfying (1.2).

(d) $X_1, \ldots, X_n$ have a joint distribution with Weibull minimums, that is,

$$P(\min_{i \in S} (X_i) > t) = \exp(-\lambda_S t^\alpha)$$

for some $\lambda_S > 0$ and all nonempty subsets S of $\{1, \ldots, n\}$.

(e) Each X_i, $i = 1, \ldots, n$ has a Weibull distribution of the form $\bar{F}_i(t) = \exp(-\lambda_i t^{\alpha_i})$, with $\alpha_i > 0$, $i = 1, \ldots, n$.

The class of Weibull distributions (e) has been described in a slightly different, although equivalent, way be Johnson and Kotz [9, p. 269]. They mention that by specifying only that $Y_1, \ldots, Y_n$ have a multivariate distribution with exponential marginals, the transformation $X_i = Y_i^{1/\alpha_i}$, $i = 1, \ldots, n$ produces a multivariate distribution having Weibull marginals.

The classes (a)–(e) contain the corresponding classes of multivariate exponential distributions constructed by Esary and Marshall [7]. Each class satisfies certain multivariate closure properties similar to those that they describe. See their properties P_1, P_2, P_3, P_4. Also each class a–e is a subclass of the one which follows it.

The condition (b) is an alternative and equivalent way to describe the distributions of (1.1). The representation of (1.1) in terms of independent random variables is discussed in [12].

The examples which follow show the classes a–e are distinct since each class is seen to contain distributions not belonging to the class preceeding it.

Example 2.1. The bivariate Weibull distribution, $\bar{F}(x, x_2) = \exp[-(\lambda_1 x_1^{\alpha_1} + \lambda_2 x_2^{\alpha_2} + \lambda_{12} \max(x_1^{\alpha_1}, x_2^{\alpha_2})]$, with $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_{12} \geqslant 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ is mentioned by Marshall and Olkin [14], and several of its properties are discussed by Moeschberger [15]. If $\alpha_1 \neq \alpha_2$, then $\bar{F}$ satisfies (e) but not (d). If, however, $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_{12} > 0$, then $\bar{F}$ satisfies (b) but not (a), and in this case arises from independent Weibull distributions, $P(Z_1 > t) = \exp(-\lambda_1 t^\alpha)$, $P(Z_2 > t) = \exp(-\lambda_2 t^\alpha)$ and $P(Z_{12} > t) = \exp(-\lambda_{12} t^\alpha)$ by the representation $X_1 = \min(Z_1, Z_{12})$ and $X_2 = \min(Z_2, Z_{12})$ as specified by the condition (b).

Example 2.2. Let X_1 and X_2 have the joint distribution of example 2.1 with $\alpha_1 = \alpha_2 = \alpha$ and let $Y_i = c_i^{-1} X_i$ with $c_i > 0$, $i = 1, 2$. Then $\bar{F}(y_1, y_2) = \exp[-(\lambda_1 c_1^\alpha y_1^\alpha + \lambda_2 c_2^\alpha y_2^\alpha + \lambda_{12} \max(c_1^\alpha y_1^\alpha, c_2^\alpha y_2^\alpha))]$. The distribution of Y_1 and Y_2 has a singular component on the line $c_1 y_1 = c_2 y_2$. Thus it differs from distributions satisfying (b). If $c_1 \neq c_2$ and $\lambda_{12} > 0$, the joint distribution of Y_1 and Y_2 satisfies (c) but not (b).

Example 2.3. $\bar{G}(x_1, x_2) = \exp[-(x_1^4 + x_2^4)^{1/2}]$ satisfies (c) but not (b). In a later section it is shown that this distribution can be generated by a transformation of independent random variables. $\bar{G}(x_1, x_2)$ is absolutely continuous and therefore cannot satisfy (b). That it satisfies (c) can be verified by computing $P(\min_i(a_i X_i) > t) = \exp[-t^2(a_1^{-4} + a_2^{-4})^{1/2}]$, $t \geqslant 0$, for $a_i > 0$, $i = 1, 2$.

Example 2.4. Let $\bar{H}(x_1, x_2) = \bar{G}(x_1, x_2)\bar{F}(x_1, x_2)$ where $\bar{F}$ is the distribution of example 2.1 with $\alpha_1 = \alpha_2 = 2$ and $\bar{G}(x_1, x_2)$ is the distribution of example 2.3. $\bar{H}(x_1, x_2)$ is not absolutely continuous and satisfies (c) but not (b).

Example 2.5. Let X_1, X_2 have the distribution $\bar{G}(x_1, x_2)$ of example (2.3) and let Y_1, Y_2 have the distribution $\bar{F}(y_1, y_2) = \exp[-(2y_1^8 + 2y_2^8)^{1/4}]$. Let $(T_1, T_2) = (X_1, X_2)$ with probability p and $(T_1, T_2) = (Y_1, Y_2)$ with probability $1 - p$. Then T_1, T_2 have the distribution of the mixture $\bar{H}(t_1, t_2) = p\bar{G}(t_1, t_2) + (1 - p)\bar{F}(t_1, t_2)$. If $0 < p < 1$, then $\bar{H}$ satisfies (d) but not (c).

EXAMPLE 2.6. Let $\bar{F}(x_1, x_2) = \bar{F}_1(x_1)\bar{F}_2(x_2)[1 + \gamma(1 - \bar{F}_1(x_1))(1 - \bar{F}_2(x_2))]$ where $\bar{F}_j(x_j) = \exp(-x_j^{c_j})$, $c_j > 0$, $x_j \geqslant 0$, $j = 1, 2$ are univariate Weibull distributions. This bivariate Weibull distribution is mentioned in [10] as a special case of the Morgenstern–Gumbel–Farlie system of distributions. It satisfies (e) but it $\gamma > 0$ does not satisfy (d).

EXAMPLE 2.7. The bivariate distribution, $\bar{F}(x_1, x_2) = (1 - \rho^2)\sum_{j=0}^{\infty} \rho^{2j} I(c_1 x_1^{\alpha_1}, j, \rho) I(c_2, x_2^{\alpha_2}, j, \rho)$ with $-1 < \rho < 1$ representing a correlation index and $I(z, j, \rho) = [j!]^{-1}[1 - \rho^2]^{-j-1} \int_z^{\infty} y^j \exp[-y(1 - \rho^2)^{-1}]\, dy$ denoting the upper tail integral of the gamma density is discussed by Krishnaiah [11]. Actually, he presents a more general form of the distribution, but in the present form $\bar{F}$ satisfies (e). If $\rho = 0$ and $\alpha_1 = \alpha_2 = \alpha$, then $\bar{F}$ satisfies (a), but if $\rho^2 > 0$ $\bar{F}$ does not satisfy (d). To show (d) is not satisfied when $\rho^2 > 0$ and $\alpha_1 = \alpha_2 = \alpha$, note that the first term of the series defining $\bar{F}$ provides a lower bound on $\bar{F}(t, t)$ which can be used to compute $\lim_{t\to\infty} \bar{F}(t, t)^{t^{-\alpha}} > 0$. Then, using the Poisson expansion for $I(c_i t^{\alpha}, j, \rho)$, an upper bound on $\bar{F}(t, t)$ with t sufficiently small can be found, and used to show that $\lim_{t\to 0} \bar{F}(t, t)^{t^{-\alpha}} = 0$. Thus $\bar{F}$ cannot satisfy (d) when $\rho^2 > 0$.

EXAMPLE 2.8. The following distribution, $\bar{F}(x_1, x_2) = \exp\{-c_1 x_1^{\alpha_1} - c_2 x_2^{\alpha_2} - c_3[\max(x_1, x_2)]^{\alpha_3}\}$ arises from independent variables Z_1, Z_2 and Z_3 having Weibull distributions, $P(Z_i > t) = \exp(-c_i t^{\alpha_i})$, $i = 1, 2, 3$ by the transformation $X_1 = \min(Z_1, Z_2)$ and $X_2 = \min(Z_2, Z_3)$, and is mentioned by David [6] and Lee and Thompson [12]. If $\alpha_1 \neq \alpha_2$, $\bar{F}$ does not satisfy (e), however, if $\alpha_1 = \alpha_2$ and $c_3 > 0$, then $\bar{F}$ satisfies (b) but not (a).

In summary, the class of Weibull distributions (1.2) contains independent Weibull distributions satisfying (a) and the class of Weibull distributions (b) arising from the Marshall–Olkin [14] models. Examples (2.2), (2.3) and (2.4) show the existence of other Weibull distributions satisfying (1.2) which are distinct from the classes (a) and (b).

3. PROPERTIES OF DISTRIBUTIONS HAVING WEIBULL MINIMUMS AFTER ARBITRARY SCALING

A distribution $\bar{F}$ satisfies (1.2) if and only if the hazard function satisfies the following functional equation:

$$R(t\mathbf{x}) = t^{\alpha} R(\mathbf{x}) \text{ for some } \alpha > 0 \text{ whenever } t \geqslant 0 \text{ and } \mathbf{x} \geqslant 0. \tag{3.1}$$

Equation (3.1) is the basis in this section for developing properties of distributions having Weibull minimums after arbitrary scaling.

To show that distributions satisfying (1.2) are continuous, let $\mathbf{x}' \leqslant \mathbf{x}$ and consider $\lim_{x_1' \uparrow x_1} R(\mathbf{x}') = \lim_{t \uparrow 1} t^{\alpha} R(x_1, t^{-1}x_2', ..., t^{-1}x_n') = R(x_1, x_2', ..., x_n')$ by (3.1) and the right continuity of R. Repeated use of (3.1) and the right continuity of R proves that R is left continuous, and thus is continuous.

In the present section it is assumed that the hazard gradient, $r_j(\mathbf{x}) = (\partial/\partial x_j) R(\mathbf{x})$, $j = 1,..., n$ exists except possibly on a finite set of values of x_j. Further, it is assumed that $r_j(\mathbf{x})$ is a continuous function of x_j with the exception of the points where it fails to exist. Let $r_j(\mathbf{x})$ represent the right hand derivative at the exceptional points, which is assumed to exist for all $\mathbf{x}$.

Absolutely continuous distributions satisfy such conditions as do also the multivariate Weibull distributions satisfying (1.1). For the distribution (1.1), $\bar{F}(\mathbf{x})$ is a continuous function of $\mathbf{x}$ and $r_j(\mathbf{x}) = \sum_J \lambda_J \alpha x_j^{\alpha-1} I_J(\mathbf{x})$ where $I_J(\mathbf{x}) = 1$ (and zero otherwise) if $j \in J$ and $x_j > \max\{x_i: i \in J \text{ and } i \neq j\}$, with $\max \phi = 0$ whenever the null set occurs. It is seen that $r_j(\mathbf{x})$ is continuous in x_j except on a finite set of values and can be defined at the exceptional values by the right hand derivative.

The hazard gradient is useful for describing failure rate properties of multivariate distributions. In [10] it is shown that $r_j(\mathbf{x})$ can be interpreted as the failure rate of the conditional distributions of X_j given that $X_i > x_i$, $i \neq j$, $i = 1,..., n$. It reduces to the usual concept of failure rate when the distribution involves independent random variables. In [4] the hazard gradient is used to describe certain multivariate monotone failure rate concepts, and to characterize the loss of memory property of the Marshall–Olkin distribution. Further discussion of the hazard gradient is given in [13].

THEOREM 1. *Let $X_1,..., X_n$ have a joint distribution satisfying* (1.2) *with* $\alpha > 0$ *given by* (1.2) *and having the hazard gradient* $r_j(\mathbf{x})$, $j = 1,..., n$. *Then*

(a) $r_j(t\mathbf{x}) = t^{\alpha-1} r_j(\mathbf{x})$, $j = 1,..., n$ *for all vectors* $\mathbf{x} \geqslant 0$ *and scalar* $t > 0$.

(b) $r_j(\mathbf{x})$ *is nonincreasing in* x_i *for* $i \neq j$, $i = 1,..., n$.

(c) $r_j(\mathbf{x})$ *is nondecreasing in* x_j, $j = 1,..., n$ *providing* $\alpha \geqslant 1$.

Proof. (a) Using (3.1) write $R(\mathbf{x}) = x_j^{\alpha} R(1_j, x_j^{-1}\mathbf{x})$ where the notation $(1_j, x_j^{-1}\mathbf{x})$ represents a vector with a one in the jth position and the remaining elements have been multiplied by the scalar x_j^{-1}. For $i \neq j$, $r_i(\mathbf{x}) = x_j^{\alpha}(\partial/\partial x_i) R(1_j, x_j^{-1}\mathbf{x}) = x_j^{\alpha-1} r_i(1_j, x_j^{-1}\mathbf{x})$. Therefore, $r_i(t\mathbf{x}) = (tx_j)^{\alpha-1} r_i(1_j, x_j^{-1}\mathbf{x}) = t^{\alpha-1} r_i(\mathbf{x})$, for any $\mathbf{x} \geqslant \mathbf{0}$ and $t > 0$.

(b) First observe from (3.1) that $(-\partial/\partial x_j)\bar{F}(t\mathbf{x}) = t^{\alpha} r_j(\mathbf{x}) \bar{F}(t\mathbf{x})$ for $t > 0$. Since $-t^{-\alpha}(\partial/\partial x_j)\bar{F}(t\mathbf{x})$ is nonincreasing in x_i, for $i \neq j$, and all $t > 0$, and since $\lim_{t \to 0^+} -t^{-\alpha}(\partial/\partial x_j)\bar{F}(t\mathbf{x}) = r_j(\mathbf{x})$, we have that $r_j(\mathbf{x})$ is nonincreasing in x_i for $i \neq j$.

(c) From part a, $r_j(\mathbf{x}) = x_j^{\alpha-1} r_j(1_j, x_j^{-1}\mathbf{x})$. Also from part b, $r_j(1_j, x_j^{-1}\mathbf{x})$ is nondecreasing in x_j, and since by assumption $\alpha \geqslant 1$, it follows that $r_j(\mathbf{x})$ is nondecreasing in x_j.

As pointed out in references [2] and [5] a form of positive dependence is likely to be a reasonable assumption for many reliability problems. For random variables $X_1, \ldots, X_n$ satisfying (1.2), part b of the theorem can be used to show that each subset S of the variables is right tail increasing (See [5] for a discussion of right tail increasing) in the remaining set $\bar{S}$. That is, the conditional probability

$$P(X_i > x_i, i \in S \mid X_j > y_j, j \in \bar{S}) = \exp[-R(\mathbf{x}, \mathbf{y}) + R(\mathbf{0}, \mathbf{y})]$$

is nondecreasing in y_j, $j \in \bar{S}$. From part b we have $(\partial/\partial y_j) R(\mathbf{x}, \mathbf{y})$ is nonincreasing in x_i. Therefore, $(\partial/\partial y_j) R(\mathbf{x}, \mathbf{y}) \leqslant (\partial/\partial y_j) R(\mathbf{0}, \mathbf{y})$, which says that $R(\mathbf{x}, \mathbf{y}) - R(\mathbf{0}, \mathbf{y})$ is nonincreasing in y_j, $j \in \bar{S}$. This proves right tail increasing for distributions (1.2).

For a second application consider $X_1, \ldots, X_n$ satisfying (1.2) with $\alpha \geqslant 1$. This corresponds to $\min_i(a_i X_i)$ having a one dimensional IFR (increasing failure rate) Weibull distribution for each choice of constants $a_i > 0$, $i = 1, \ldots, n$. Part c of the theorem shows that the distributions (1.2) have the property that Johnson and Kotz [10] call multivariate IHR (increasing hazard rate).

Next consider $V = \min(X_i)$ and define the event that X_j coincides with V by

$$X_j = V \Leftrightarrow X_j \leqslant \min_{i \neq j} (X_i). \tag{3.2}$$

Since for distributions satisfying (b) of section 2 there is positive probability of tied values, it is important to note when computing $P(X_j = V)$ that equality is allowed in (3.2).

To develop a special property of distributions satisfying (1.2), let $Y_j = \min_{i \neq j}(X_i)$ and write

$$P(X_j = V \text{ and } V > x) = \int_x^\infty P(Y_j \geqslant t \mid X_j = t) f_j(t)\, dt \tag{3.3}$$

since the density $f_j(t)$ of X_j exists for the distributions of (1.2).

The integrand of (3.3) is equal to $\lim_{\Delta \to 0^+} \Delta^{-1} P(Y_j \geqslant X_j \geqslant t, t \leqslant X_j < t + \Delta)$, which is equal to the difference of the limits, $\lim_{\Delta \to 0^+} \Delta^{-1} P(Y_j \geqslant t, t \leqslant X_j < t + \Delta) - \lim_{\Delta \to 0^+} \Delta^{-1} P(t \leqslant Y_j < X_j, t \leqslant X_j < t + \Delta)$. Both terms of the difference exist, and the second term is zero as can be seen by representing the triangular region $(t \leqslant Y_j < X_j, t \leqslant X_j < t + \Delta)$ of two dimensional Euclidean space as the limit of a union of rectangles, and examining $\lim_{\Delta \to 0^+} \lim_{n \to \infty} \Delta^{-1} \sum_{j=0}^{n-1} P(t + j/n \leqslant Y_j < t + (j+1)/n \leqslant X_j < t + \Delta)$, then changing the order of taking limits.

Thus the integrand of (3.3) is equal to

$$\lim_{\Delta \to 0^+} \Delta^{-1} P(Y_j \geqslant t, t \leqslant X_j < t + \Delta) = r_j(t,..., t) \bar{F}(t,..., t). \tag{3.4}$$

The integrand is also equal to

$$P(X_j = V \mid V = t) g(t) \tag{3.5}$$

where $g(t) = -(d/dt) \bar{F}(t,..., t)$ is the density function of V. Equating (3.4) and (3.5) gives the conditional probability,

$$P(X_j = V \mid V = t) = r_j(t,..., t) \bar{F}(t,..., t)[g(t)]^{-1}. \tag{3.6}$$

The following theorem extends a property of the Marshall–Olkin [14] distribution (see [2]) to the class of distributions having Weibull minimums after arbitrary scaling.

THEOREM 2. *Let $X_1 ,..., X_n$ have a joint distribution satisfying* (1.2) *with hazard gradient $r_j(x)$ computed as the right hand derivative. Then V is independent of the events $X_j = V, j = 1,..., n$. Also, $P(X_j = V) = r_j(1,..., 1)/\alpha R(1,..., 1)$.*

Proof. Since for distributions satisfying (1.2), $g(t) = \alpha t^{\alpha-1} R(1,..., 1) \bar{F}(t,..., t)$, and from theorem 1, part a, $r_j(t,..., t) = t^{\alpha-1} r_j(1,..., 1)$ it is seen that (3.6) simplifies to $r_j(1,..., 1)/\alpha R(1,..., 1)$. Therefore, $P(X_j = V \mid V = t)$ is constant in t which proves the independence of V and $X_j = V$.

4. APPLICATION—COMPUTING COMPONENT RELIABILITY IMPORTANCE

One problem arising in reliability practice is that ot assessing the importance of the individual components of a system. Some components are more likely to cause system failure than others.

Importance measures have been suggested by Barlow and Proschan [3] and others. Of course the result of applying importance measures depends on assumptions made concerning component life lengths. Suppose it is desired to allow for some form of dependence in the joint distribution of life lengths, and further that the life lengths follow the Marshall–Olkin distribution, or some alternative distribution satisfying (1.2). Two properties of such distributions were mentioned earlier following theorem 1 which help make their use appealing as models of the distribution of life lengths.

Let $\tau(\mathbf{X})$ represent system life length and suppose the system is coherent having minimal path sets $P_1 ,..., P_p$. Suppose $X_1, X_2 ,..., X_n$ represent component life lengths. Then $\tau(\mathbf{X}) = \max_{j=1,...,p} (\tau_j)$ where $\tau_j = \min_{m \in P_j} (X_m)$, $j = 1,..., p$. This representation of system life length in terms of minimal path sets is discussed in [2].

Barlow and Proschan [3] define their measure of a component's reliability importance as the probability that component life length coincides with system life length. If the two coincide, the component is said to cause the system to fail. Since

$$P(X_i = \tau(\mathbf{X})) = P(\max_{j=1,\ldots,p} (\tau_j) = X_i)$$

is the probability of the union of p events, the importance measure can be expressed as follows:

$$P(X_i = \tau(\mathbf{X})) = \sum_{j=1}^{p} P(\tau_j = X_i) - \sum_{j,k=1;j\neq k}^{p} P(\min(\tau_j, \tau_k) = X_i)$$
$$+ \cdots \pm P(\min(\tau_1, \ldots, \tau_p) = X_i). \tag{4.1}$$

Barlow and Proschan [3] express their formulas for the importance measure in terms of the system reliability function for the case of independent component life lengths, and thus (4.1) is not mentioned by them.

Noting that $\min(\tau_j, \tau_k) = \min_{m\in P_j\cup P_k}(X_m)$, and so on, it is seen that each term of the various sums reduces to computing probabilities like those expressed in theorem 2. Note also that if $i \notin P_j \cup P_k$ and if $X_1, X_2, \ldots, X_n$ have an absolutely continuous distribution then $P(\min(\tau_j, \tau_k) = X_i) = P(\min_{m\in P_j\cup P_k} (X_m) = X_i) = 0$. Other terms may equal zero for the same reason.

To illustrate the application of theorem 2 for a two out of three system, let X_1, X_2, X_3 represent component life lengths having the joint distribution $\bar{F}(\mathbf{x}) = \exp[-(x_1^2 + 2x_2^2 + 3x_3^2)^{1/2}]$. A two out of three system fails when any two of its components fail. System life length is $\tau(\mathbf{X}) = \max[\min(X_1, X_2), \min(X_2, X_3), \min(X_1, X_3)]$. Using (4.1) it is seen that $P(X_1 = \tau(\mathbf{X})) = P(X_1 = \min(X_1, X_2)) + P(X_1 = \min(X_1, X_3)) - 2P(X_1 = \min(X_1, X_2, X_3))$, since the remaining terms become zero for the reason mentioned above. From theorem 2 we have $P(X_1 = \min(X_1, X_2)) = r_1(1, 1, 0)/R(1, 1, 0) = \frac{1}{3}$, $P(X_1 = \min(X_1, X_3)) = r_1(1, 0, 1)/R(1, 0, 1) = \frac{1}{4}$, and $P(X_1 = \min(X_1, X_2, X_3)) = r_1(1, 1, 1)/R(1, 1, 1) = \frac{1}{6}$. Thus the probability that component 1 causes the system to fail is $\frac{1}{3} + \frac{1}{4} - \frac{1}{3} = \frac{1}{4}$. Similar computations would show $P(X_2 = \tau(\mathbf{X})) = \frac{2}{5}$ and $P(X_3 = \tau(\mathbf{X})) = \frac{7}{20}$.

5. An Absolutely Continuous Weibull Distribution

Consider the following bivariate Weibull distribution:

$$\bar{F}(x_1, x_2) = \exp[-(\lambda_1 x_1^\beta + \lambda_2 x_2^\beta)^\gamma] \tag{5.1}$$

with $\lambda_i > 0$, $x_i \geqslant 0$, $i = 1, 2$, $\beta > 0$ and $0 < \gamma \leqslant 1$. This distribution has the properties discussed in section 3. For $\beta\gamma = 1$ it reduces to the third of the three distributions studied by Gumbel [8], and has several properties in common with the Marshall–Olkin distribution, e.g., exponential marginals, exponential minimums after arbitrary scaling and the independence property discussed in theorem 2. The distribution easily extends to n variables.

Let us show that random variables X_1, X_2 having distribution (5.1) can be represented in terms of independent random variables. Such a representation can be useful for analyzing properties of the distribution and generating random samples.

Consider the random variables

$$Z_i = \lambda_i X_i^{\beta}, \qquad i = 1, 2 \tag{5.2}$$

and their joint distribution given by

$$\bar{G}(z_1, z_2) = \exp[-(z_1 + z_2)^{\gamma}]. \tag{5.3}$$

The joint density function is of the form

$$g(z_1, z_2) = [\gamma(1-\gamma)(z_1 + z_2)^{\gamma-2} + \gamma^2(z_1 + z_2)^{2\gamma-2}] \exp[-(z_1 + z_2)^{\gamma}]. \tag{5.4}$$

Consider next the transformation

$$\begin{aligned} U &= Z_1/(Z_1 + Z_2), \\ S &= (Z_1 + Z_2)^{\gamma} \end{aligned} \tag{5.5}$$

having the jacobian $(1/\gamma)S^{2/\gamma-1}$.

The joint density of U and S is given by

$$h(u, s) = [(1-\gamma) + \gamma s]e^{-s},$$

$0 < u < 1$, $0 < s < \infty$. Thus U and S are independent random variables with U having a uniform distribution on the interval (0, 1) and the distribution of S is a mixture of gamma distributions having the density

$$h(s) = [1 - \gamma + \gamma s]e^{-s}, \qquad s > 0.$$

In summary we have from (5.5) that

$$\begin{aligned} Z_1 &= US^{1/\gamma}, \\ Z_2 &= (1-U)\, S^{1/\gamma} \end{aligned} \tag{5.6}$$

are represented in terms of independent random variables U and S.

It is an easy exercise to compute the covariance from the distributions of U and S:

$$\mathrm{COV}(Z_1, Z_2) = (1/\gamma)\,\Gamma(2/\gamma) - (1/\gamma^2)\,\Gamma^2(1/\gamma)$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ is the gamma function.

Using formulas 6.1.2 and 6.1.18 for the gamma function given in [1], it is possible to show that the covariance is decreasing in γ. Since for $\gamma = 1$, Z_1 and Z_2 are independent, it follows that the covariance must be nonnegative for all $0 < \gamma < 1$.

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