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Integral closures and weight functions over finite fields

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Abstract

Curves and surfaces of type I are generalized to *integral towers* of rank r. Weight functions with values in $\mathbf{N}^r$ and the corresponding weighted total-degree monomial orderings lift naturally from one domain R_{j-1} in the tower to the next, R_j, the integral closure of $R_{j-1}[x_j]/\langle \phi(x_j)\rangle$. The qth power algorithm is reworked in this more general setting to produce this integral closure over finite fields, though the application is primarily that of calculating the normalizations of curves related to one-point AG codes arising from towers of function fields. Every attempt has been made to couch all the theory in terms of multivariate polynomial rings and ideals instead of the terminology from algebraic geometry or function field theory, and to avoid the use of any type of series expansion.

Keywords: Integral closure; Normalization; Weight function; AG codes; Towers of function fields

1. Introduction

Type I curves were introduced by Feng and Rao [4] with defining equations of the form

$$x^a + y^b + g(x,y) = 0, \quad gcd(a,b) = 1, \quad a > b > \deg(g(x,y)).$$

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Some curves are described in terms of more than two variables, along the lines of Example 3.22 in [8]. Regardless of the number of variables involved, the proper view is that each defining function $\phi_j(x_j)$ determines a ring extension $R_{j-1}[x_j]/\langle \phi_j(x_j) \rangle$ of R_{j-1}. And what is sought here is to produce the integral closure of this extension in the corresponding function field extension $F_j := F_{j-1}(x_j)/\langle \phi_j(x_j) \rangle$.

The general form of the defining functions here will be

$$\phi_j(x_j) := x_j^{m_j} + u_j \prod_{i=1}^{j-1} x_i^{\alpha_{i,j}} + g_j(x_j, \dots, x_1) \in R_{j-1}[x_j],$$

(monic) irreducible, with $0 \neq u_j \in \mathbf{F}_q$, $gcd\{\phi_j(x_j), \phi_j{}'(x_j)\} \in \bar{R}$, and

$$wt(g_j(x_j, \dots, x_1)) < wt(x_j^{m_j}) = wt\left(\prod_{i=1}^{j-1} x_i^{\alpha_{i,j}}\right)$$

for wt a natural "weight" function to be described below, and some extra condition on m_j and the $\{\alpha_{i,j}\}_{i=1}^{j-1}$.

The concepts of *order functions* and *weight functions* are discussed in Geil and Pellikaan [7,8] as well. Here such functions will be viewed as maps from $\mathbf{F}_q[x_n, \dots, x_1]$ into $\mathbf{N}^r$ for some $0 < r \leqslant n$ that are weighted total orders that agree with the defining equations in the sense that $wt(\prod_{i=1}^{j-1} x_i^{\alpha_{i,j}}) = wt(x_j^{m_j})$; but will be used only when they satisfy the additional constraint $wt(g_j(x_j, \dots, x_1)) < wt(x_j^{m_j})$. It will be seen that these can be naturally extended to various *weighted total-degree monomial orderings* as well.

Finally it will be shown how to move from a ring R, integrally closed in its field of fractions F, to its integral closure $ic_{F'}(R)$ in an extension field $F' := F(y)/\langle \phi(y) \rangle$ defined by a monic polynomial $\phi(y)$, naturally lifting the weight function in the process (meaning that the weights of all elements in the integral closure have all non-negative entries).

More traditional methods such as Coates' algorithm [2,9], for calculating integral closures start with a basis for the ring R and adjoin new elements to produce larger and larger rings, culminating with the integral closure itself. But there are two more recent methods [10,16] using methods which start with a module containing the integral closure and delete elements not in the integral closure.

All, save the qth power algorithm require producing various series expansions, however. And, philosophically, expansion-driven algorithms are inherently point-wise algorithms; whereas polynomial-based algorithms are global in nature. So the computation of the integral closure will be done here by invoking the qth power algorithm introduced by Leonard [10], using the above monomial ordering to define normal forms, and using a variant of the *trace-dual basis* of the standard basis to define the initial set Δ_0^* in the algorithm. This algorithm is used to compute *missing functions* (see Pellikaan [13] and an example in Leonard [11]) for the AG codes from the towers of function fields introduced by Garcia and Stichtenoth [5,6]; that is (slightly more generally) if U^c is the set of points at which at least one element of R is not regular, then the algorithm computes the set of functions regular on U. But it can be viewed as an algorithm for producing the *integral closure* of a given ring or the *normalization* or *non-singular model* of a curve, particularly one in special position. (Technically it may give only an affine non-singular model, though the projective non-singular model is easily derived, by adding (dependent) variables so as to have functions with pole orders giving a complete set of non-gaps of size at most $2g+1$. See Porter [14] or Saints and Heegard [15].)

And the algorithm does this purely algebraically and globally, without reference to any local terms such as *places*, *valuations*, *points*, *singularities*, *blow-ups*, and other such usually found in discussions of normalization. In particular, as mentioned above, there are no series expansions of any sort involved, and no extensions of the ground field either.

2. Weight functions and monomial orderings

A *monomial ordering* of the multivariate polynomial ring $\mathbf{F}_q[x_n, \ldots, x_1]$ for the purposes of this paper is one that can be described by a non-singular matrix $M \in \mathcal{M}at_{n\times n}(\mathbf{N})$, with

$$\underline{x}^{\underline{\beta}} \succ_M \underline{x}^{\underline{\gamma}} \quad \text{iff } \underline{\beta} M \succ_{lex} \underline{\gamma} M.$$

Let J_n be the $n \times n$ $(0,1)$ matrix with $(J_n)_{i+j} = 1$ iff $i+j \leqslant n+1$, be the matrix defining a standard *total-degree monomial ordering*. A *weighted total-degree* monomial order is an order defined by M with $M_{i,1} \neq 0$ for all i and $M_{i,j} = 0$ for $i+j > n+1$. (The advantage of such orders is that there are only finitely many elements preceding any given element, unlike standard lexicographical orders.) This paper will deal *only* with weighted total-degree monomial orders. (Note that while the previous definition is really only a definition of a function with domain $Mon(\mathbf{F}_q[x_n, \ldots, x_1])$, the set of monomials $\underline{x}^{\underline{\alpha}}$ of $\mathbf{F}_q[x_n, \ldots, x_1]$ it is easily extended to the polynomial ring by choosing the maximum order of any monomial in a polynomial.)

NormalForm$(f, \mathcal{I})$, gotten by reducing f modulo a basis for the ideal $\mathcal{I}$, necessarily has a *leading monomial* not divisible by any leading monomial of any element of $\mathcal{I}$. The set of leading monomials of normal forms will be referred to as the *footprint* of the ideal $\mathcal{I}$ (or $R/\mathcal{I}$). If $LM(\mathcal{I}) := \{LM(f) : f \in \mathcal{I}\}$ is the ideal of leading monomials of $\mathcal{I}$, then this footprint is the complement of this ideal in $Mon(R)$.

Let W be the submatrix of M consisting of the first r columns. The function $\rho : \mathbf{F}[x_n, \ldots, x_1]/I \backslash \{0\} \to \mathbf{N}^r$, defined by $\rho(\underline{x}^{\underline{\alpha}}) := \underline{\alpha} W$ and $\rho(f) = \rho(LM_{\succ_M}(f))$, will be called a *weak weight function of rank r* on $\mathbf{F}[x_n, \ldots, x_1]/I \backslash \{0\}$.

The properties (numbered as in [4]) of such a weight function are:

(O.1) $\rho(\lambda f) = \rho(f)$ for $0 \neq \lambda \in \mathbf{F}_q$.

(O.2) If $\rho(g) \preceq \rho(f)$ and $f \neq g$, then $\rho(f-g) \preceq \rho(f)$, with equality when $\rho(g) \prec \rho(f)$.

(O.5) $\rho(fg) = \rho(f) + \rho(g)$.

Call ρ a *weight function* if it additionally satisfies

(O.4) If $\rho(f) = \rho(g)$, then there exists $0 \neq \lambda \in \mathbf{F}_q$ with either $f - \lambda g = 0$ or $\rho(f - \lambda g) \prec \rho(f)$.

The difference between a weak weight function and a weight function is that the former allows two monomials in the footprint to have the same weight, while the latter clearly does not.

Note that in terms of leading monomials (LM) of normal forms (NF) of elements, these conditions can be restated as:

(M.1) $LM(NF(\lambda f)) = LM(NF(f))$ for $0 \neq \lambda \in \mathbf{F}_q$.

(M.2) If $LM(NF(g)) \preceq LM(NF(f))$ and $f \neq g$, then $LM(NF(f-g)) \preceq LM(NF(f))$; and if $LM(NF(g)) \prec LM(NF(f))$, then $LM(NF(f-g)) = LM(NF(f))$.

(M.5) $LM(NF(fg)) = LM(NF(LM(f)LM(g)))$.

(M.4) If $LM(NF(f)) = LM(NF(g))$, then

$$LM\left(\frac{NF(f)}{LC(NF(f))} - \frac{NF(g)}{LC(NF(g))}\right) \prec LM(NF(f)),$$

(with LC denoting the leading coefficient). In particular, the λ in (O.4) is determined constructively.

Note also that a weight function ρ can be extended to a function on quotients by defining $\rho(f/g) := \rho(f) - \rho(g) \in \mathbf{Z}^r$. This is necessary in that the qth power algorithm [10], reworked below, acts on such elements.

Each type I defining equation for an ideal I of $\mathbf{F}_q[x_n, \ldots, x_1]$ can be viewed as determining a pair of monomials $\underline{x}^{\underline{\alpha}}$ and $\underline{x}^{\underline{\beta}}$ which should have the same "weight".

3. Integral closures, integral towers, canonical weight functions, and dual bases

Let S be a domain, and R a subdomain. An element $y \in S$ is said to be *integral over* R iff there exists a *monic* polynomial $\phi_y(T) \in R[T]$ such that $\phi_y(y) = 0$. The *integral closure* of R in S is defined to be $ic_S(R) := \{s \in S \mid s \text{ is integral over } R\}$. R is *integrally closed* in S iff $R = ic_S(R)$. And $ic_S(R)$ is a ring if S is.

Now define an *integral tower* as follows. Start with $\bar{R} = R_r := \mathbf{F}[x_r, \ldots, x_1]$ and its *field of fractions* $F_r := \mathbf{F}(x_r, \ldots, x_1) := \{a/b \mid a, b \in \bar{R},\ b \neq 0\}$. Then, for $r < j \leqslant n$, recursively define *simple field extensions* $F_j := F_{j-1}(x_j)$ with $\phi_j(x_j) = 0$ for $\phi_j(T) \in F_{j-1}[T]$ irreducible; and *subdomains* $R_j := ic_{F_j}(R_{j-1})$. Let $\mathscr{I}_j :=$ *ideal*$\langle \mathscr{I}_{j-1}, \phi_j(x_j) \rangle$. This sequence of domains $(R_j)_{j=r}^n$ (with each R_j integrally closed in the corresponding field of fractions F_j) will be called an *integral tower* (of rank r) iff

1. $$\phi_j(x_j) := x_j^{m_j} + u_j \prod_{i=1}^{j-1} x_i^{\alpha_{i,j}} + g_j(x_j, \dots, x_1) \in R_{j-1}[x_j],$$
   is (monic) irreducible, with $0 \neq u_j \in \mathbf{F}_q$;
2. $gcd(\phi_j(x_j), \phi_j{}'(x_j)) \in \bar{R} := R_r$;
3. The weight functions, given recursively by $W_r := J_r$, and $W_j := \binom{\underline{\alpha}_j W_{j-1}}{m_j W_{j-1}}$, with $\underline{\alpha}_j :$ $= (\alpha_{j-1,j}, \dots, \alpha_{1,j})$ satisfy
   $$wt(g_j(x_j, \dots, x_1)) < wt(x_j^{m_j}) = wt\left(\prod_{i=1}^{j-1} x_i^{\alpha_{i,j}}\right);$$
4. $gcd\{m_j, gcd_i\{(\underline{\alpha}_j W_{j-1})_i\}\} = 1.$

The weight function W_n can be easily extended to a weighted total-degree ordering, by completing W_n to a non-singular matrix, by appending $(J_{n-r} O_{(n-r)\times r})^T$.

Proposition 3.1. *Each $W_j, j \geqslant r$ is a weighted total-degree monomial order on $R_{j-1}[x_j]$ which is injective on the footprint of $\mathcal{I}_j$ if and only if $gcd\{m_j, gcd_i\{(\alpha_j W_{j-1})_i\}\} = 1$. Hence it is also a weighted total-degree monomial order on $R_{j-1}[x_j]/\langle \phi_j(x_j) \rangle$.*

Proof. Since W_r is non-singular, it is trivially injective on $\bar{R}$. Assume that W_{j-1} is injective on the footprint of $\mathcal{I}_{j-1}$. Suppose that W_j were *not* injective on the footprint of $\mathcal{I}_j$, so that $(b, \underline{\beta}) W_j = (c, \underline{\gamma}) W_j$. If $b = c$, then $(\underline{\beta} - \underline{\gamma}) W_{j-1} = \underline{0}$, so $\underline{\beta} = \underline{\gamma}$ by recursion. And if $b \neq c$, then $\underline{\gamma} \neq \underline{\beta}$, so $m_j | (b - c)\underline{\alpha}_j W_{j-1}$. But since $gcd\{m_j, gcd_i\{(\alpha_j W_{j-1})_i\}\} = 1$, $m_j | b - c$. And clearly if $gcd\{m_j, gcd_i\{(\alpha_j W_{j-1})_i\}\} = d$, then $x_j^{m_j/d}$ has the same weight as an element of R_{j-1}.

Then apply the Factor Ring Theorem [7,12], to see that it is also a weighted total-degree monomial order on $R_{j-1}[x_j]/\langle \phi_j(x_j) \rangle$. □

Example 3.2. The *gcd* condition used here may not seem to be intuitive; so consider the following related examples, all starting with $R_1 := \mathbf{F}_2[x_1]$; and $R_2 := R_1[x_2]/\langle x_2^3 + x_1^2 + x_1 \rangle$, with $W_2 = (2,3)^T$. For the first extension, try using $\phi_3(x_3) := x_3^2 + x_2^3 + x_3$. This would give $W_3 = (6,4,6)^T$. On closer inspection $\phi_3(x_3) = (x_3 + x_1)(x_3 + x_1 + 1)$ is reducible, so this is not really an extension. For the second extension, try using $\phi_3(x_3) := x_3^2 + x_2(x_1^2 + x_1) + x_3 x_1 + x_2^2 x_1 + x_2^2$ and $W_3 = (8,4,6)^T$. Since $(1,0,0) W_3 = 8 = (0,2,0) W_3$, try $w := x_3 + x_2^2$ in place of x_3 to get $\phi(w) = w^2 + w x_1 + x_2^2$, which is not even of type I. And finally, for the third extension, try $\phi_3(x_3) := x_3^2 + x_3 x_1 + x_2(x_1^2 + x_1) + x_3 + x_2^2 + x_2$ and $W_3 = (8,4,6)^T$.

Since $(1,0,0)W_3 = 8 = (0,2,0)W_3$, again try $w := x_3 + x_2^2$ in place of x_3 to get $\phi_w(w) = w^2 + w + x_2$, and $W_3 = (2,4,6)^T$. Since $(1,1,0)W_3 = 6 = (0,0,1)W_3$, try $y := wx_2 + x_1$ in place of w to get $\phi_y(y) = y^2 + yx_2 + x_2^2x_1 + x_1^2 + x_2x_1$ and $W_3 = (7,4,6)^T$. This satisfies the hypotheses of the proposition, so would be an acceptable tower extension.

Consider the top level of such a tower by letting $R := R_{n-1}$, $F := F_{n-1}$, $y := x_n$, $f(y) := \phi_n(x_n)$, $R' := R_n$, $F' := F_n$, and $m := m_n$. It is easy to produce the subring $R[y]/\langle f(y)\rangle$ of R'. This can be viewed as an R-module with standard (ordered) basis $(1, y, \ldots, y^{m-1})$.

The following specialized version of Theorems III.3.4 and (the proof of) III.5.10 from Stichtenoth [17] is central to this paper:

Theorem 3.3. *Let $f(y) = \sum_{i=0}^{m} a_i y^{m-i}$ be a monic (irreducible) polynomial. Define $f_j(y) := \sum_{i=0}^{j} a_i y^{j-i}$ for $0 \leqslant j \leqslant m$. Then the standard ordered basis $(1, y, \ldots, y^{m-1})$ for the R-module $V := R[y]/\langle f(y)\rangle$ has trace-dual basis $(f_{m-1}(y), \ldots, f_0(y))/f'(y)$, (meaning that $Tr_{F'/F}(y^i f_j(y)/f'(y)) = \delta_{i,j}$). Further*

$$V := \sum_{i=0}^{m-1} Ry^i \subseteq ic_{F'}(R) \subseteq V^* := \sum_{i=0}^{m-1} R\frac{f_i(y)}{f'(y)}$$

as R-modules; and $ic_{F'}(R)$ is the largest *subring contained in the R-module V^*.*

It is useful to choose a slightly different dual basis in light of the following lemmas:

Lemma 3.4. *If $\sigma(y) := f^{q-1}(y) = \sum_{i=0}^{m(q-1)} \sigma_i y^{q(m-1)-i}$, then $f_j^q(y) = \sum_l \sigma_l f_{qj-l}(y)$.*

Proof. Since $f^q(y) = \sigma(y)f(y)$, $a_i^q = \sum_l \sigma_l a_{qi-l}$. So

$$\begin{aligned} f_j^q(y) &= \sum_{i=0}^{j} a_i^q y^{q(j-i)} = \sum_{i=0}^{j} \sum_l (\sigma_l a_{qi-l}) y^{qj-qi} \\ &= \sum_{s=0}^{qj} \left(\sum_l \sigma_l a_{s-l} \right) = \sum_l \left(\sigma_l \sum_{i=0}^{qj-l} a_i y^{qj-l-i} \right) = \sum_l \sigma_l f_{qj-l}(y). \qquad \square \end{aligned}$$

Lemma 3.5. *If $f(T)$, $f'(T) \in R[T]$ are relatively prime, then $f'(y)g(y) \equiv D(mod\ f(y))$ for some $g(T) \in R[T]$ and $D \in \bar{R}$.*

Proof. Since $f'(y)$ and $f(y)$ are relatively prime, there exist $h(T), l(T) \in R[T]$ such that $h(y)f'(y) - l(y)f(y) = E$ for some $E \in R$. But then there exists some $k(T) \in R[T]$ such that $k(y)E = D \in \bar{R}$. So $g(y)f'(y) \equiv D(\text{mod} f(y))$ for $g(y) := h(y)k(y)$. $\square$

Lemma 3.6. $\sum_{i=0}^{m-1} Rg(y)y^i \subseteq ic_{F'}(R) \subseteq \sum_{i=0}^{m-1} R(\frac{1}{D})f_i(y)$.

Proof. Rewrite $(f_{m-1}(y), \dots, f_0(y))/f'(y)$ as $\frac{1}{D}(f_{m-1}(y), \dots, f_0(y))g(y)$, to get an alternate basis, dual-basis pair

$$(g(y), yg(y), \dots, y^{m-1}g(y)), \quad \frac{1}{D}(f_{m-1}(y), \dots, f_0(y)). \qquad \square$$

The weight function ρ_j defined by W_j above was shown to be a weight function on $R_{j-1}[x_j]/\langle \phi_j(x_j) \rangle$. But it should also be a weight function on the integral closure of this ring in its field of fractions, $F_j' := F_{j-1}(x_j)/\langle \phi_j(x_j) \rangle$. This means that every element of the integral closure should have a weight with all coordinates non-negative.

Note that it is of practical importance to limit computations to $R[y]/\langle f(y) \rangle$. In particular, this allows the use of standard definitions (such as those used in symbolic manipulation packages) of leading monomials relative to the induced monomial ordering and normal forms relative to the ideal $\mathcal{I}_n$ of defining relations; though theoretically, these concepts can be extended in much the same manner as power series are extended to Laurent series.

Multiplying through by D^q will remove any denominators in the algorithm, meaning calculations will occur in the ring $R[y]/\langle f(y) \rangle$ rather than the function field $F' = F[y]/\langle f(y) \rangle$.

The qth power algorithm can now be used, starting with the induced monomial ordering, the alternative dual basis $\frac{1}{D}(f_{m-1}(y), \dots, f_0(y))$ for $R[y]/\langle f(y) \rangle$ over R, and a basis for R over $\bar{R}$.

4. Integral closures from the qth power algorithm

Though $R := R_{n-1}$ is being extended to R_n, the computations are all really done relative to $\bar{R} := R_r = \mathbf{F}_q[x_r, \dots, x_1]$. So the following is an $\bar{R}$-module version of the qth power algorithm from [10]. The idea of the algorithm is simple. If the integral closure $ic_{F'}(R)$ is contained in some $\bar{R}$-module V_0^* (such as the one gotten by multiplying the alternate dual R-module basis above by an $\bar{R}$-module basis for R), then only those elements whose qth powers are also in this module could possibly be in *any* subring (and in particular the integral closure) of V_0^*. So it is possible to define a sequence of $\bar{R}$-modules (V_k^*), $k \geqslant 0$, with

$$DV_{k+1}^* := \{Dv \in DV_k^* : \ NormalForm((Dv)^q, \mathcal{I}) \in DV_k^*\} \subseteq DV_k^*.$$

It may be helpful to view each recursive step of the qth power map as a function from V_k^* to, say, S/V_k^* (if S is viewed as an $\bar{R}$-module), in order to view V_{k+1}^* as the kernel of this mapping, and hence as an $\bar{R}$-module.

This is an FGLM-type reduction algorithm [3], in that it can be viewed as a reduction algorithm on pairs of the forms

$$(\tilde{f}^{(k)}_{\underline{\beta}} \,|\, f^{(k)}_{\underline{\beta}}) \quad \text{and} \quad (D^{q-1} f^{(k)}_{\underline{\beta}} \,|\, 0)$$

with $f^{(k)}_{\underline{\beta}} \in \Delta^*_k$ and $\tilde{f}^{(k)}_{\underline{\beta}} := NF((f^{(k)}_{\underline{\beta}})^q, \mathscr{I})$ to obtain pairs of the form

$$\left(\sum_{\underline{\alpha}} \bar{s}_{\underline{\alpha},\underline{\beta}} D^{q-1} f^{(k)}_{\underline{\alpha}} + \sum_{\underline{\alpha} \prec \underline{\beta}} \bar{r}^q_{\underline{\alpha},\underline{\beta}} \tilde{f}^{(k)}_{\underline{\alpha}} + \tilde{f}^{(k)}_{\underline{\beta}} \,\middle|\, \sum_{\underline{\alpha} \prec \underline{\beta}} \bar{r}_{\underline{\alpha},\underline{\beta}} f^{(k)}_{\underline{\alpha}} + f^{(k)}_{\underline{\beta}} \right).$$

If the first entry is 0, then the second entry should be in Δ^*_{k+1}; and if it is not, then it is a *leading entry* in the sense that $lm(f^{(k)}_{\underline{\beta}}) \notin LM(DV^*_{k+1})$. So certain $\bar{R}$-multiples of this second entry should be considered, if they could conceivably be reduced further.

(This works in much the same way that row-reduction of a matrix over a ring does, and is not far removed from the Berlekamp–Massey–Sakata decoding algorithms for one-point AG codes or the change of order methods that employ FGLM.)

If $g = \sum \{ c_{\underline{\alpha}} f^{(k)}_{\underline{\alpha}} : f^{(k)}_{\underline{\alpha}} \in \Delta^*_k \}$, then define the leading term *of this representation* as $lt(g) := c_{\underline{\gamma}} f^{(k)}_{\underline{\gamma}}$ iff $wt(c_{\underline{\gamma}} f^{(k)}_{\underline{\gamma}}) = \max_{\underline{\alpha}} \{ wt(c_{\underline{\alpha}} f^{(k)}_{\underline{\alpha}}) : c_{\underline{\alpha}} \neq 0 \}$ iff $LM(c_{\underline{\gamma}} f^{(k)}_{\underline{\gamma}}) = LM(g)$.

Recursively, Δ^*_{k+1} is an $\bar{R}$-module basis for $DV^*_{k+1} \subseteq DV^*_k$, $\tilde{f}^{(k+1)}_{\underline{\beta}} = NF(f^{(k+1)}_{\underline{\beta}}, \mathscr{I})$, $h^{(k+1)}_{\underline{\beta}} = \tilde{f}^{(k+1)}_{\underline{\beta}} - D^{q-1}(u^{(k+1)}_{\underline{\beta}} + v^{(k+1)}_{\underline{\beta}})$, and $L^{(k+1)}_{\underline{\beta}} = lt(h^{(k+1)}_{\underline{\beta}})$. The tables in the examples contain only $f^{(k+1)}_{\underline{\beta}}$, $v^{(k+1)}_{\underline{\beta}}$, $L^{(k+1)}_{\underline{\beta}}$, and the updating actions taken, as $\tilde{f}^{(k+1)}_{\underline{\beta}}$ $h^{(k+1)}_{\underline{\beta}}$, and $u^{(k+1)}_{\underline{\beta}}$ are derivatives of them.

Because there is an upper bound on the weights of elements in this algorithm, namely the maximum weight of any basis element of V_0, the whole algorithm is necessarily finite.

The important properties of the algorithm alluded to here will be summarized in the theorem that immediately follows the statement of the algorithm.

Algorithm 4.1. *Use the notation above, but let* lc *denote the leading coefficient relative to* $\mathbf{F}_q$, *and* LM *the leading monomial relative to* $\bar{R}$.

Let B^* *be an* R*-module basis for* V^* *and* B *an* $\bar{R}$*-module basis for* R (*all made monic by dividing by the appropriate element of* $\mathbf{F}_q$). *Let* Δ^*_0 *be the set of the products* $f^* f D$ *with* $f^* \in B^*$, *and* $f \in B$. *Let* $\mathscr{I}$ *be the ideal generated by the polynomials* $\phi_{r+1}(x_{r+1}), \ldots, \phi_n(x_n)$.

Recursively, *starting with* $k = 0$, (*stopping when* $\Delta^*_{k+1} = \Delta^*_k$),

1. (*Initialization*) *Set* $B_k := \Delta^*_k$ *and* $\Delta^*_{k+1} := \emptyset$. *For each* $f^{(k)}_{\underline{\beta}} \in \Delta^*_k$, set $l^{(k)}_{\underline{\beta}} := LM(D^{q-1}) f^{(k)}_{\underline{\beta}}$, $f^{(k+1)}_{\underline{\beta}} := f^{(k)}_{\underline{\beta}}$, $u^{(k)}_{\underline{\beta}} := 0$; $v^{(k)}_{\underline{\beta}} := 0$; $\tilde{f}^{(k+1)}_{\underline{\beta}} := \tilde{f}^{(k)}_{\underline{\beta}} = NF((f^{(k)}_{\underline{\beta}})^q, \mathscr{I})$, $h^{(k+1)}_{\underline{\beta}} := \tilde{f}^{(k+1)}_{\underline{\beta}}$, $L^{(k+1)}_{\underline{\beta}} := lt(h^{(k+1)}_{\underline{\beta}})$.

2. *For* $\underline{\beta}$ *the smallest weight of any unscanned element of* B_k, *scan* $f_{\underline{\beta}}^{(k+1)}$.
3. (*Reduction*) *While* $L_{\underline{\beta}}^{(k+1)} \neq 0$, *try the following two reductions as long as they apply*:
   (a) (*Reduction* mod $D^{q-1}\Delta_k^*$) *If* $L_{\underline{\beta}}^{(k+1)} = \bar{r} l_{\underline{\alpha}}^{(k)}$ *for some* $\bar{r} \in \bar{R}$ *and some* $f_{\underline{\alpha}}^{(k)} \in \Delta_k^*$, *then* $v_{\underline{\beta}}^{(k)} := v_{\underline{\beta}}^{(k)} - lc(h_{\underline{\beta}}^{(k+1)})\bar{r} f_{\underline{\alpha}}^{(k)}$, $h_{\underline{\beta}}^{(k+1)} := h_{\underline{\beta}}^{(k+1)} - lc(h_{\underline{\beta}}^{(k+1)})\bar{r} D^{q-1} f_{\underline{\alpha}}^{(k)}$, $L_{\underline{\beta}}^{(k+1)} := lt(h_{\underline{\beta}}^{(k+1)})$.
   (b) (*Reduction using* $\bar{R}$-*linear combinations*) *If* $L_{\underline{\beta}}^{(k+1)} = \bar{r}^q L_{\underline{\alpha}}^{(k+1)}$ *for some* $\bar{r} \in \bar{R}$ *and* $f_{\underline{\alpha}}^{(k+1)} \in B_k$ (*and* $\underline{\alpha} \neq \underline{\beta}$), *then* $h_{\underline{\beta}}^{(k+1)} := h_{\underline{\beta}}^{(k+1)} - \bar{r}^q h_{\underline{\alpha}}^{(k+1)}$, $f_{\underline{\beta}}^{(k+1)} := f_{\underline{\beta}}^{(k)} - \bar{r} f_{\underline{\alpha}}^{(k+1)}$, $\tilde{f}_{\underline{\beta}}^{(k+1)} := \tilde{f}_{\underline{\beta}}^{(k+1)} - \bar{r}^q \tilde{f}_{\underline{\alpha}}^{(k+1)}$, $u_{\underline{\beta}}^{(k)} := u_{\underline{\beta}}^{(k)} - \bar{r}^q u_{\underline{\alpha}}^{(k)}$, $L_{\underline{\beta}}^{(k+1)} := lt(h_{\underline{\beta}}^{(k+1)})$.
4. (*Updating* Δ_{k+1} *and* B_k)
   (a) (*Finding elements of* Δ_{k+1}^*) *If* $L_{\underline{\beta}}^{(k+1)} = 0$, *then remove* $f_{\underline{\beta}}^{(k+1)}$ *from* B_k *and place it in* Δ_{k+1}^*.
   (b) (*S-polynomial calculations*) *If* $L_{\underline{\beta}}^{(k+1)} \neq 0$, *but* $\bar{r}^q L_{\underline{\beta}}^{(k+1)} = \bar{s}^q L_{\underline{\alpha}}^{(k+1)}$ *for some* $f_{\underline{\alpha}}^{(k+1)} \in B_k$ *and some* $\bar{r}, \bar{s} \in \bar{R}$, *with* $\bar{r}$ *minimal*, *then place* $f_{\underline{\gamma}}^{(k+1)} := \bar{r} f_{\underline{\beta}}^{(k+1)} - \bar{s} f_{\underline{\alpha}}^{(k+1)}$ *in* B_k, *and set* $u_{\underline{\gamma}}^{(k)} := \bar{r}^q u_{\underline{\beta}}^{(k)} - \bar{s}^q u_{\underline{\alpha}}^{(k)}$, $v_{\underline{\gamma}}^{(k)} := 0$ $\tilde{f}_{\underline{\gamma}}^{(k+1)} := \bar{r}^q \tilde{f}_{\underline{\beta}}^{(k+1)} - \bar{s}^q \tilde{f}_{\underline{\alpha}}^{(k+1)}$, $h_{\underline{\gamma}}^{(k+1)} := \bar{r}^q h_{\underline{\beta}}^{(k+1)} - \bar{s}^q h_{\underline{\alpha}}^{(k+1)}$, $L_{\underline{\gamma}}^{(k+1)} := lt(h_{\underline{\gamma}}^{(k+1)})$.
   (c) (*Multiplication by an element of* $Mon(\bar{R})$) *If* $L_{\underline{\beta}}^{(k+1)} \neq 0$, *but* $\bar{r}^q L_{\underline{\beta}}^{(k+1)} = \bar{s}^q l_{\underline{\alpha}}^{(k)}$ *for some* $f_{\underline{\alpha}}^{(k)} \in B_k$ *and some* $\bar{r}, \bar{s} \in \bar{R}$, *with* $\bar{r}$ *minimal*, *then place* $f_{\underline{\gamma}}^{(k+1)} := \bar{r} f_{\underline{\beta}}^{(k+1)}$ *in* B_k, *and set* $u_{\underline{\gamma}}^{(k)} := \bar{r}^q u_{\underline{\beta}}^{(k)}$, $v_{\underline{\gamma}}^{(k)} := 0$, $\tilde{f}_{\underline{\gamma}}^{(k+1)} := \bar{r}^q \tilde{f}_{\underline{\beta}}^{(k+1)}$, $h_{\underline{\gamma}}^{(k+1)} := \bar{r}^q h_{\underline{\beta}}^{(k+1)}$, $L_{\underline{\gamma}}^{(k+1)} := \bar{r}^q L_{\underline{\beta}}^{(k+1)}$.
   (d) (*Removing redundant elements of* B_k) *If* $L_{\underline{\delta}}^{(k+1)} = \bar{r} L_{\underline{\varepsilon}}^{(k+1)}$ *for some* $f_{\underline{\delta}}^{(k+1)}, f_{\underline{\varepsilon}}^{(k+1)} \in B_k$ *and* $\bar{r} \in \bar{R}$, *then remove* $f_{\underline{\delta}}^{(k+1)}$ *from* B_k.

Theorem 4.2. 1. *Each* DV_k^*, *generated by the Gröbner basis* Δ_k^* *in the algorithm*, *is an* $\bar{R}$-*module*, *and hence finitely-generated.*
2. $DV_{k+1}^* = \{Dv \in DV_k^* \colon NormalForm((Dv)^q, \mathscr{I}) \in DV_k^*\}$.
3. Δ_{k+1}^* *is produced from* Δ_k^* *in a finite number of steps.*
4. $DV_l^* = DV_{l+1}^*$ *for some* (*smallest*) *non-negative integer* l.

Proof. The first assertion is clear, but necessary.

From step 5, it is clear that $DV_{k+1}^* \subseteq \{Dv \in DV_k^* \colon NormalForm((Dv)^q, \mathscr{I}) \in DV_k^*\}$. Suppose $Dv \in DV_{k+1}^*$ were not in the module generated by $\Delta_{k+1}^*$and that Dw has minimal weight, $\underline{\alpha}$, relative to this condition. Then $Dw = \sum\{\bar{s}_{\underline{\beta}}(Dv_{\underline{\beta}})\colon Dv_{\underline{\beta}} \in B_k,\ \underline{\beta} \preceq \underline{\alpha}\}$, with $\bar{s}_{\underline{\alpha}} \neq 0$. But then (the monic version of) $\bar{s}_{\underline{\alpha}}(Dv_{\underline{\alpha}})$ would have been scanned *and reduced* by the algorithm, a contradiction.

The other two claims follow from the fact that there are only finitely many leading monomials to consider in the whole algorithm, since their weights (less the weight of D) all are between $-\rho(D)$ and $\max\{\rho(v)\colon Dv \in \Delta_0^*\}$ in the weighted *total-degree* ordering; and $V_0 \subseteq V_k^* \subseteq V_0^*$. □

The following *constructive* algorithm actually produces a monic *affine polynomial* satisfied by h_i for each basis element h_i in the final $\Delta_l^*/D (= \Delta_{l+1}^*/D)$ above.

Algorithm 4.3. *Let $h_0 = 1, h_1, \ldots, h_s$ be an $\bar{R}$ -module basis for $V_l^* = V_{l+1}^*$, such as the one produced by the preceding algorithm. Fix $i > 0$ and show that h_i is integral over $\bar{R}$. Because $V_l^* = V_{l+1}^*$, it is possible to write*

$$h_i^{q^m} = \sum_{j=0}^{s} \alpha_{i,m,j} h_j$$

for some $\alpha_{i,m,j} \in \bar{R}$ for any $m \geqslant 1$. Initialize $g_m := h_i^{q^m} - \alpha_{i,m,j} h_i - \alpha_{i,m,0} h_0$. Then apply the following FGLM-type reduction algorithm to these g_m, $m \geqslant 1$.

Start with $m = 1$. For $1 \leqslant j \leqslant s$, do

1. *if $j = i$ then increase j by 1;*
2. *if $coef(g_m, h_j) = 0$, then increase j by 1, and either stop if $j > s$ or repeat this step;*
3. *if $LM(coef(g_l, h_j)) | LM(coef(g_m, h_j))$ for some $l < m$ with (g_l, h_j) already marked, then replace g_m by $g_m - g_l LT(coef(g_m, h_j))/LT(coef(g_l, h_j))$, and return to the previous step;*
4. *otherwise mark the pair (g_m, h_j), increase m by 1 and start over.*

Theorem 4.4. *The algorithm above actually produces a monic affine polynomial g_m satisfied by h_i in a finite number of steps.*

Proof. Clearly, the algorithm can only produce monic affine polynomials (evaluated at h_i) at any step, as can easily be seen from the initialization and the replacement step. If the algorithm stops, it is because $g_m(h_i) = 0$. So the real question is whether the algorithm stops or not. For any fixed (i, j), the set $\{LM(coef(g_l, h_j)) : (g_l, h_j) \text{ is marked}\}$ is a basis for the monomial ideal generated by them. But by Dickson's lemma, this ideal is generated by a finite subset of its elements. Since there are only $s - 1$ choices for j, it is clear that this is a finite algorithm. □

Corollary 4.5. *The qth power algorithm actually produces $ic_{F'}(R)$.*

Proof. From the above theorem $V_l^* \subseteq ic_{F'}(\bar{R})$. But $ic_{F'}(\bar{R}) \subseteq V_0^*$. Recursively, if $f \in V_k^* \cap ic_{F'}(\bar{R})$ then because $ic_{F'}(\bar{R})$ is a ring, $f^q \in V_k^*$ so $NF(f^q, I) \in V_k^*$. But then $f \in V_{k+1}^*$. □

Note that, in fact, this proves that any ring contained in V_0^* is contained in each V_k^* and hence in $ic_{F'}(\bar{R})$; which is equivalent to saying that $ic_{F'}(\bar{R})$ is the largest subring of V_0^*.

Theorem 4.6. *The weight function ρ_j defined by W_j on $R_{j-1}[x_j]/\langle \phi_j(x_j)\rangle$ is a weight function on the integral closure R_j.*

Proof. Suppose that $0 \neq z \in R_j$, but that $(\rho_j(z))_k < 0$ for some coordinate k. But $zD \in R$, so $(\rho_j(z))_k \geqslant -(\rho_j(D))_k$. But $z^e \in R_j$ for all e, since R_j is a ring. So there is an e with $(\rho_j(z^e))_k = e(\rho_j(z))_k < -(\rho_j(D))_k$. This is a contradiction. □

5. Examples

Example 5.1. Consider the type II curve [4] $\mathscr{X}$ over $\bar{\mathbf{F}}_2$, defined by

$$X^2Y^5 + (X^3+1)Y^2 + Y + X^9 = 0.$$

Trying to apply the algorithm directly to this would produce functions with poles where X has poles *or* where Y has poles. Instead it is possible to view this as defining a one-point AG code by considering the rational functions $x_1 = h_5 := X$ and $x_2 = h_{12} := XY$, regular except at a single point P_∞, at which the pole orders are 5 and 12, respectively. (This is an example of a general method of changing a type II curve into one of type I, usually at the expense of introducing further singularities.) To produce the missing functions for this one-point AG code, start with the domain $R = \bar{R} = R_1 := \bar{\mathbf{F}}_2[h_5]$, the field $F = \bar{F} = F_1 := \bar{\mathbf{F}}_2(h_5)$, and the extension $F' = F_2 := F(h_{12})/\langle \phi_2(h_{12})\rangle$ with

$$f(h_{12}) = f_5(h_{12}) = \phi_2(h_{12}) := h_{12}^5 + ah_{12}^2 + bh_{12} + c \in R_1[h_{12}],$$

with $a := h_5(h_5^3+1)$, $b := h_5^2$, $c := h_5^{12}$ (gotten by multiplying the original equation above by x^3 and substituting). $W = (12,5)^T$ defines the canonic weight function and $M = \begin{pmatrix} 12 & 1 \\ 5 & 0 \end{pmatrix}$, the corresponding monomial order. Then consider the subring $V := R_1[h_{12}]/\langle f(h_{12})\rangle$ of F_2. As an R_1-module, V has basis $(1, h_{12}, h_{12}^2, h_{12}^3, h_{12}^4)$ and trace-dual basis

$$(f_4(h_{12}), f_3(h_{12}), f_2(h_{12}), f_1(h_{12}), f_0(h_{12}))/f'(h_{12}).$$

Since $f'(h_{12}) = h_{12}^4 + h_5^2$, $1/f'(h_{12}) = g(h_{12})/D$ for $g(h_{12}) := h_{12}^3 + h_5h_{12} + (h_5^4 + h_5) \in V$ and $D := h_5^{24} + h_5^{10} + h_5^4 \in R$.

Choose as elements of $\Delta_0^*, f_{-120}^{(0)} := 1, f_{-108}^{(0)} := h_{12}, f_{-96}^{(0)} := h_{12}^2, f_{-84}^{(0)} := h_{12}^3$, and $f_{-72}^{(0)} := h_{12}^4$; with $f_b^{(0)}/D$ having weight b, corresponding to its pole-size at P_∞. Then apply the qth power algorithm to produce the integral closure R_2. Let $x := h_5$ to save space.

$f_b^{(1)}$	$v_b^{(0)}$	$L_b^{(1)}$	
$f_{-120}^{(0)}$	0	$f_{-120}^{(0)}$	$x^{12}f_{-120}^{(1)} \to B_0$
$f_{-108}^{(0)}$	0	$f_{-96}^{(0)}$	$x^{12}f_{-108}^{(1)} \to B_0$
$f_{-96}^{(0)}$	0	$f_{-72}^{(0)}$	$x^{12}f_{-96}^{(1)} \to B_0$
$f_{-84}^{(0)}$	0	$x^{12}f_{-108}^{(0)}$	$x^{6}f_{-84}^{(1)} \to B_0$
$f_{-72}^{(0)}$	0	$x^{12}f_{-84}^{(0)}$	$x^{6}f_{-72}^{(1)} \to B_0$
$(x^{12}+x^5+x^2)f_{-120}^{(1)}$	$f_{-120}^{(0)}$	0	$f_{-60}^{(1)} \to \Delta_1$
$x^6f_{-84}^{(1)} + x^2f_{-72}^{(1)}$	$f_{-108}^{(0)}$	$x^{13}f_{-84}^{(0)}$	$x^{6}f_{-54}^{(1)} \to B_0$
$(x^{12}+x^5+x^2)f_{-108}^{(1)}$	$f_{-96}^{(0)}$	0	$f_{-48}^{(1)} \to \Delta_1$
$x^6f_{-72}^{(1)} + x^{10}f_{-108}^{(1)}$	$f_{-84}^{(0)} + (x^4+x)f_{-120}^{(0)}$	$x^{15}f_{-108}^{(0)}$	$x^{5}f_{-42}^{(1)} \to B_0$
$+x^7f_{-96}^{(1)} + x^3f_{-84}^{(1)} + x^7f_{-108}^{(1)}$			
$(x^{12}+x^5+x^2)f_{-96}^{(1)}$	$f_{-72}^{(0)}$	0	$f_{-36}^{(1)} \to \Delta_1$
$x^6f_{-54}^{(1)} + x^9f_{-96}^{(1)} + x^9f_{-108}^{(1)}$	$xf_{-84}^{(0)} + x^8f_{-120}^{(0)}$	0	$f_{-24}^{(1)} \to \Delta_1$
$+x^2f_{-42}^{(1)} + x^2f_{-84}^{(1)} + (x^5+x^2)f_{-108}^{(1)}$	$+x^2f_{-96}^{(0)} + x^5f_{-120}^{(0)} + f_{-96}^{(0)}$		
$x^5f_{-42}^{(1)} + x^2f_{-54}^{(1)} + xf_{-72}^{(1)} + x^8f_{-108}^{(1)}$	$xf_{-108}^{(0)}$	0	$f_{-17}^{(1)} \to \Delta_1$
$+(x^5+x^2)f_{-96}^{(1)} + x^2f_{-108}^{(1)}$			

$f_b^{(2)}$	$v_b^{(1)}$	$L_b^{(2)}$	
$f_{-60}^{(1)}$	0	$x^{12}f_{-60}^{(1)}$	$x^6f_{-60}^{(1)} \to B_1$
$f_{-48}^{(1)}$	0	$x^{12}f_{-36}^{(1)}$	$x^6f_{-48}^{(2)} \to B_1$
$f_{-36}^{(1)}$	0	$x^{13}f_{-17}^{(1)}$	$x^6f_{-36}^{(2)} \to B_1$
$x^6f_{-60}^{(2)}$	$f_{-60}^{(1)}$	$x^{17}f_{-60}^{(1)}$	$x^4f_{-30}^{(2)} \to B_1$
$f_{-24}^{(1)}$	$f_{-48}^{(1)}$	$x^{16}f_{-24}^{(1)}$	$x^4f_{-24}^{(2)} \to B_1$
$x^6f_{-48}^{(2)}$	$f_{-36}^{(1)}$	$x^{17}f_{-36}^{(1)}$	$x^4f_{-18}^{(2)} \to B_1$
$f_{-17}^{(1)}$	0	$x^{22}f_{-24}^{(1)}$	$xf_{-17}^{(2)} \to B_1$
			$x^3f_{-24}^{(2)} + f_{-17}^{(2)} \to B_1$
			$x^4f_{-24}^{(2)} \leftarrow B_1$

$xf^{(2)}_{-17}$	$f^{(1)}_{-24} + x^4 f^{(1)}_{-60}$	$x^{17} f^{(1)}_{-24}$	$x^4 f^{(2)}_{-12} \to B_1$
	$+f^{(1)}_{-48} + x f^{(1)}_{-60}$		
$x^4 f^{(2)}_{-30} + x^5 f^{(2)}_{-60} + x^3 f^{(2)}_{-60}$	$xf^{(1)}_{-60}$	$x^{15} f^{(1)}_{-60}$	$xf^{(2)}_{-10} + f^{(2)}_{-30} \to B_1$
$x^3 f^{(2)}_{-24} + f^{(2)}_{-17}$	0	$x^{19} f^{(1)}_{-24}$	$x^3 f^{(2)}_{-9} \to B_1$
			$xf^{(2)}_{-12} + f^{(2)}_{-9} \to B_1$
			$x^4 f^{(2)}_{-12} \leftarrow B_1$
$xf^{(2)}_{-12} + f^{(2)}_{-9}$	$x^2 f^{(1)}_{-60}$	$x^{23} f^{(1)}_{-48}$	$xf^{(2)}_{-7} \to B_1$
$x^6 f^{(2)}_{-36}$	$xf^{(1)}_{-17} + x^4 f^{(1)}_{-48}$	$x^{18} f^{(1)}_{-17}$	$x^3 f^{(2)}_{-6} \to B_1$
	$+xf^{(1)}_{-36} + xf^{(1)}_{-48}$		
$xf^{(2)}_{-10} + f^{(2)}_{-30} + xf^{(2)}_{-60}$	0	0	$f^{(2)}_{-5} \to \Delta_2$
$xf^{(2)}_{-7} + x^5 f^{(2)}_{-48} + xf^{(2)}_{-24} + x^4 f^{(2)}_{-48}$	$xf^{(1)}_{-48}$	$x^{15} f^{(1)}_{-24}$	$xf^{(2)}_{-2} + f^{(2)}_{-12} \to B_1$
$x^4 f^{(2)}_{-18} + (x^5 + x^3) f^{(2)}_{-48}$	$xf^{(1)}_{-36}$	$x^{15} f^{(1)}_{-36}$	$xf^{(2)}_{2} + f^{(2)}_{18} \to B_1$
$xf^{(2)}_{-2} + f^{(2)}_{-12} + x^3 f^{(2)}_{-30} + x^3 f^{(2)}_{-48}$	0	$x^{14} f^{(1)}_{-24}$	$xf^{(2)}_{3} + f^{(2)}_{-24} \to B_1$
$x^3 f^{(2)}_{-9} + xf^{(2)}_{-24} + (x+1) f^{(2)}_{-18}$	$xf^{(1)}_{-24} + x^8 f^{(1)}_{-60} + x^5 f^{(1)}_{-48}$	0	$f^{(2)}_{6} \to \Delta_2$
$+x^2 f^{(2)}_{-48} + xf^{(2)}_{-30} + xf^{(2)}_{-48}$	$+x^2 f^{(1)}_{-36} + x^5 f^{(1)}_{-60} + f^{(1)}_{-36}$		
$+(x^3 + x^2) f^{(2)}_{-60} + f^{(2)}_{-10}$	$+(x^2 + x) f^{(1)}_{-48} + xf^{(1)}_{-60}$		
$xf^{(2)}_{2} + f^{(2)}_{-18} + xf^{(2)}_{-48}$	0	0	$f^{(2)}_{7} \to \Delta_2$
$xf^{(2)}_{3} + f^{(2)}_{-24} + x^3 f^{(2)}_{-48}$	0	$x^{13} f^{(1)}_{-24}$	$xf^{(2)}_{8} + f^{(2)}_{-2} \to B_1$
$x^3 f^{(2)}_{-6} + (x^4 + x^2) f^{(2)}_{-36}$	$f^{(1)}_{-17} + (x^3 + 1) f^{(1)}_{-48} + f^{(1)}_{-36}$	$x^{14} f^{(1)}_{-17}$	$x^2 f^{(2)}_{9} + f^{(2)}_{-6} \to B_1$
$xf^{(2)}_{8} + f^{(2)}_{-2} + x^2 f^{(2)}_{-30}$	0	0	$f^{(2)}_{13} \to \Delta_2$
$x^2 f^{(2)}_{9} + f^{(2)}_{-6} + xf^{(2)}_{-36}$	0	0	$f^{(2)}_{19} \to \Delta_2$

$f^{(3)}_b$	$v^{(2)}_b$	$L^{(3)}_b$	
$f^{(2)}_{-5}$	0	$x^{23} f^{(2)}_{-5}$	$xf^{(3)}_{-5} \to B_2$
$xf^{(3)}_{-5}$	$xf^{(2)}_{-5}$	0	$f^{(3)}_{0} \to \Delta_3$
$f^{(2)}_{6}$	$xf^{(2)}_{7}$	$x^{19} f^{(2)}_{6}$	$x^3 f^{(3)}_{6} \to B_2$
$f^{(2)}_{7}$	0	$x^{23} f^{(2)}_{19}$	$xf^{(3)}_{7} \to B_2$
$xf^{(3)}_{7}$	$xf^{(2)}_{19}$	0	$f^{(3)}_{12} \to \Delta_3$
$f^{(2)}_{13}$	$x^4 f^{(2)}_{6} + x^3 f^{(2)}_{7} + xf^{(2)}_{13}$	$x^{23} f^{(2)}_{19}$	$xf^{(3)}_{13} \to B_2$
	$+x^2 f^{(2)}_{6} + f^{(2)}_{7}$		$x^2 f^{(3)}_{6} + f^{(3)}_{13} \to B_2$

			$x^3 f_6^{(3)} \leftarrow B_2$
$x^2 f_6^{(3)} + f_{13}^{(3)}$	0	$x^{20} f_{13}^{(2)}$	$x^2 f_{16}^{(3)} \rightarrow B_2$
$(x+1) f_{13}^{(3)} + f_{16}^{(3)}$	$f_6^{(2)}$	$x^{20} f_6^{(2)}$	$x^2 f_{18}^{(3)} \rightarrow B_2$
$f_{19}^{(2)}$	$x^5 f_{13}^{(2)} + x^6 f_6^{(2)}$	0	$f_{19}^{(3)} \rightarrow \Delta_3$
	$+x^3 f_7^{(2)} + f_{19}^{(2)} + f_{13}^{(2)} + f_7^{(2)}$		
$(x^2+x) f_{16}^{(3)} + f_{13}^{(3)} + f_{16}^{(3)}$	$f_{13}^{(2)}$	$x^{19} f_{13}^{(2)}$	$x^3 f_{26}^{(3)} \rightarrow B_2$
$x^2 f_{18}^{(3)}$	$f_6^{(2)}$	$x^{21} f_{13}^{(2)}$	$x^2 f_{28}^{(3)} \rightarrow B_2$
			$x f_{26}^{(3)} + f_{28}^{(3)} \rightarrow B_2$
			$x^3 f_{26}^{(3)} \leftarrow B_2$
$x f_{26}^{(3)} + f_{28}^{(3)} + f_6^{(3)}$	0	0	$f_{31}^{(3)} \rightarrow \Delta_3$
$x^2 f_{28}^{(3)} + f_{13}^{(3)}$	$x f_{13}^{(2)} + x^2 f_6^{(2)}$	0	$f_{38}^{(3)} \rightarrow \Delta_3$

$f_b^{(4)}$	$v_b^{(3)}$	$L_b^{(4)}$	
$f_0^{(3)}$	$f_0^{(3)}$	0	$f_0^{(4)} \rightarrow \Delta_4$
$f_{12}^{(3)}$	$x f_{19}^{(3)}$	0	$f_{12}^{(4)} \rightarrow \Delta_4$
$f_{19}^{(3)}$	$f_{38}^{(3)} + x^2 f_{12}^{(3)} + f_{19}^{(3)}$	$x^{23} f_{12}^{(3)}$	$x f_{19}^{(4)} \rightarrow B_3$
$x f_{19}^{(4)}$	$x f_{12}^{(3)}$	0	$f_{24}^{(4)} \rightarrow \Delta_4$
$f_{31}^{(3)}$	$x^{10} f_{12}^{(3)} + (x^3+1) f_{31}^{(3)} + (x^3+1) f_{12}^{(3)}$	0	$f_{31}^{(4)} \rightarrow \Delta_4$
$f_{38}^{(3)} + f_{19}^{(4)}$	$x^9 f_{31}^{(3)} + (x^9+x^2) f_{12}^{(3)}$	0	$f_{38}^{(4)} \rightarrow \Delta_4$

$f_b^{(5)}$	$v_b^{(4)}$	$L_b^{(5)}$	
$f_0^{(4)}$	$f_0^{(4)}$	0	$f_0^{(5)} \rightarrow \Delta_5$
$f_{12}^{(4)}$	$f_{24}^{(4)}$	0	$f_{12}^{(5)} \rightarrow \Delta_5$
$f_{24}^{(4)}$	$x f_{12}^{(4)}$	0	$f_{24}^{(5)} \rightarrow \Delta_5$
$f_{31}^{(4)}$	$x^{10} f_{12}^{(4)} + (x^3+1) f_{31}^{(4)} + (x^3+1) f_{12}^{(4)}$	0	$f_{31}^{(5)} \rightarrow \Delta_5$
$f_{38}^{(4)}$	$x^9 f_{31}^{(4)} + x^9 f_{12}^{(4)} + f_{38}^{(4)}$	0	$f_{38}^{(5)} \rightarrow \Delta_5$

So

$$h_{31} := \frac{f_{31}^{(4)} + f_{12}^{(4)}}{D} = \frac{(h_{12}^3 + a)}{h_5}$$

and

$$h_{38} := \frac{f_{38}^{(4)} + f_0^{(4)}}{D} = \frac{(h_{12}^4 + ah_{12} + b)}{h_5^2}$$

are the missing functions. (This happens to be a curve that fits the Newton polygon theory in [1]. The particular choices of h_{31} and h_{38} above were made to match the said theory.)

The affine normalization of the original curve is then described by (a Gröbner basis for) the ideal of relations among h_5, h_{12}, h_{31}, and h_{38}:

$$h_{12}^3 + h_{31}h_5 + h_5^4 + h_5,$$

$$h_{31}h_{12} + h_{38}h_5 + h_5,$$

$$h_{31}^2 + h_{12}h_5^{10} + h_{31}(h_5^3 + 1) + h_{12}^2,$$

$$h_{38}h_{12} + h_5^{10},$$

$$h_{38}h_{31} + h_{12}^2 h_5^9 + h_{38}(h_5^3 + 1),$$

$$h_{38}^2 + h_{31}h_5^9 + h_{38}.$$

The projective normalization would require homogenization and the use of the dependent variables $h_{12i+5j} := h_{12}^i h_5^j$, $0 \leqslant i,j$, $12i + 5j \leqslant 2g + 1 = 39$.

Example 5.2. The function field with $n = 2$ and $q = 2$ from the second tower of Garcia and Stichtenoth [6] could be given [10] by

$$x_1^2 x_2 + x_1 x_2 + x_2^2 + 1 = 0 \quad \text{and} \quad x_2^2 x_4 + x_2 x_4 + x_4^2 + 1 = 0.$$

But, instead, let $h_4 := x_4$, $h_6 := x_2 x_4$, $h_7 := x_1 x_2 x_4$, $\bar{R} := R_1 := \mathbf{F}_2[x_4]$, $R := R_2 := \bar{R}[h_6]/\langle h_6^2 + h_6 h_4 + h_4(h_4+1)^2 \rangle$, and $V := R_2[h_7]/\langle h_7^2 + h_7 h_6 + (h_6 + h_4)(h_4+1)^2 \rangle$.

$$W_2 := (3 \quad 2)^T, \quad W_3 := \begin{pmatrix} (1,2)W_2 \\ 2W_2 \end{pmatrix} = (7 \quad 6 \quad 4)^T.$$

R has $\bar{R}$-module basis $(1, h_6)$; and V^* has R-module basis $(1/h_6, h_7/h_6)$. Rewriting $1/h_6$ as g/D with $g := h_6 + h_4 \in R$ and $D := h_4(h_4+1)^2 \in \bar{R}$, gives a Δ_0 with elements $f_{-12}^{(0)} := 1, f_{-6}^{(0)} := h_6, f_{-5}^{(0)} := h_7$ and $f_1^{(0)} = h_7 h_6$, with $f_b^{(0)}/D$ having weight b equal to its pole size at P_∞.

b	$f_b^{(1)}$	$v_b^{(0)}$	$L_b^{(1)}$	
-12	$f_{-12}^{(0)}$	0	$f_{-12}^{(0)}$	$h_4^2 f_{-12}^{(1)} \to B_0$
-6	$f_{-6}^{(0)}$	$f_{-12}^{(0)}$	$h_4 f_{-6}^{(0)}$	$h_4 f_{-6}^{(1)} \to B_0$
-5	$f_{-5}^{(0)}$	0	$h_4^2 f_{-6}^{(0)}$	$h_4 f_{-5}^{(1)} \to B_0$

-4	$(h_4^2+h_4)f_{-12}^{(1)}$	$xf_{-12}^{(0)}$	0	$f_{-4}^{(1)} \to \Delta_1$
-2	$(h_4+1)f_{-6}^{(1)}$	$f_{-6}^{(0)}$	0	$f_{-2}^{(1)} \to \Delta_1$
-1	$h_4 f_{-5}^{(1)}$	0	$h_4^2 f_1^{(0)}$	$h_4 f_{-1}^{(1)} \to B_0$
1	$f_1^{(0)} + f_{-1}^{(1)}$	$h_4^2 f_{-6}^{(0)} + f_1^{(0)} + h_4 f_{-5}^{(0)} + h_4^2 f_{-12}^{(0)} + f_{-6}^{(0)}$	0	$f_1^{(1)} \to \Delta_1$
3	$(h_4+1)f_{-1}^{(1)}$	$h_4 f_1^{(0)} + (h_4^4 + h_4^2)f_{-12}^{(0)}$	0	$f_3^{(1)} \to \Delta_1$

b	$f_b^{(2)}$	$v_b^{(1)}$	$L_b^{(2)}$	
-4	$f_{-4}^{(1)}$	0	$h_4^2 f_{-4}^{(1)}$	$h_4 f_{-4}^{(2)} \to B_1$
-2	$f_{-2}^{(1)}$	$f_{-4}^{(1)}$	$h_4^2 f_{-2}^{(1)}$	$h_4 f_{-2}^{(2)} \to B_1$
0	$(h_4+1)f_{-4}^{(2)}$	$h_4 f_{-4}^{(1)}$	0	$f_0^{(2)} \to \Delta_2$
1	$f_1^{(1)} + f_{-4}^{(2)}$	$h_4 f_{-2}^{(1)} + f_1^{(1)} + f_{-4}^{(1)}$	$h_4 f_{-4}^{(1)}$	$h_4 f_1^{(2)} \to B_1$
2	$(h_4+1)f_{-2}^{(2)} + f_{-4}^{(2)} + f_1^{(2)}$	$h_4 f_{-2}^{(1)} + h_4 f_{-4}^{(1)}$	0	$f_2^{(2)} \to \Delta_2$
3	$f_3^{(1)}$	$h_4^2 f_{-2}^{(1)} + f_3^{(1)} + h_4 f_{-2}^{(1)} + h_4 f_{-4}^{(1)}$	$h_4^2 f_3^{(1)}$	$h_4 f_3^{(2)} \to B_1$
5	$h_4 f_1^{(2)} + f_{-4}^{(2)}$	$h_4^2 f_1^{(1)} + f_{-4}^{(1)}$	0	$f_5^{(2)} \to \Delta_2$
7	$(h_4+1)f_3^{(2)}$	$(h_4+1)f_3^{(1)}$	0	$f_7^{(2)} \to \Delta_2$

b	$f_b^{(3)}$	$v_b^{(2)}$	$L_b^{(3)}$	
0	$f_0^{(2)}$	$f_0^{(2)}$	0	$f_0^{(3)} \to \Delta_3$
2	$f_2^{(2)}$	$h_4 f_0^{(2)}$	$h_4^2 f_5^{(2)}$	$h_4 f_2^{(3)} \to B_2$
5	$f_5^{(2)}$	$h_4^2 f_2^{(2)} + (h_4+1)f_0^{(2)}$	0	$f_5^{(3)} \to \Delta_3$
6	$h_4 f_2^{(3)}$	$h_4 f_5^{(2)} + h_4 f_0^{(2)}$	0	$f_6^{(3)} \to \Delta_3$
7	$f_7^{(2)}$	$(h_4+1)^3 f_0^{(2)} + h_4 f_7^{(2)} + h_4^2 f_2^{(2)} + (h_4+1)f_5^{(2)}$	0	$f_7^{(3)} \to \Delta_3$

b	$f_b^{(4)}$	$v_b^{(3)}$	$L_b^{(4)}$	
0	$f_0^{(3)}$	$f_0^{(3)}$	0	$f_0^{(4)} \to \Delta_4$
5	$f_5^{(3)}$	$h_4 f_6^{(3)} + (h_4+1)f_0^{(3)}$	0	$f_5^{(4)} \to \Delta_4$
6	$f_6^{(3)}$	$h_4 f_5^{(3)} + h_4 f_0^{(3)}$	0	$f_6^{(4)} \to \Delta_4$
7	$f_7^{(3)}$	$(h_4+1)^3 f_0^{(3)} + h_4 f_7^{(3)} + h_4 f_6^{(3)} + h_4 f_5^{(3)} + f_5^{(3)}$	0	$f_7^{(4)} \to \Delta_4$

So

$$\frac{f_5^{(4)} + f_0^{(4)}}{D} = \frac{(h_7 + h_4 + 1)(h_6 + h_4)}{(h_4 + 1)^2}$$

is the missing function.

Example 5.3. Start with the surface defined by $x^3y + y^2z + z^2x = 0$ in characteristic 2. Let $x_1 := z$, $x_2 := y$, $x_3 := xy$, $R = \bar{R} = R_2 := \mathbf{F}_2[x_2, x_1]$, and $f(x_3) = f_3(x_3) = \phi_3(x_3) := x_3^3 + x_3(x_2x_1^2) + x_2^4x_1$. $F := \mathbf{F}_2(x_2, x_1)$ $F' := F(x_3)/(f_3(x_3))$. $W = \begin{pmatrix} 5 & 3 & 3 \\ 4 & 3 & 0 \end{pmatrix}^T$. Let $D := x_2^4x_1$, and start with $\Delta_0^* := \{f_{-15,-12}^{(0)} = 1,\ f_{-10,-8}^{(0)} = x_3,\ f_{-5,-4}^{(0)} = x_3^2\}$ with $f_{\underline{\beta}}^{(0)}/D$ having weight $\underline{\beta}$.

$\underline{\beta}$	$f_{\underline{\beta}}^{(1)}$	$v_{\underline{\beta}}^{(0)}$	$L_{\underline{\beta}}^{(1)}$	
$-15,-12$	$f_{-15,-12}^{(0)}$	0	$f_{-15,-12}^{(0)}$	$x_2^2x_1f_{-15,-12}^{(1)} \to B_0$
$-10,-8$	$f_{-10,-8}^{(0)}$	0	$f_{-5,-4}^{(0)}$	$x_2^2x_1f_{-10,-8}^{(1)} \to B_0$
$-6,-6$	$x_2^2x_1f_{-15,-12}^{(1)}$	$x_1f_{-15,-12}^{(0)}$	0	$f_{-6,-6}^{(1)} \to \Delta_1$
$-5,-4$	$f_{-5,-4}^{(0)}$	$f_{-10,-8}^{(0)}$	$x_2x_1^2f_{-5,-4}^{(0)}$	$x_2^2f_{-5,-4}^{(1)} \to B_0$
$-1,-2$	$x_2^2x_1f_{-10,-8}^{(1)}$	$x_1f_{-5,-4}^{(0)}$	0	$f_{-1,-2}^{(1)} \to \Delta_1$
$1,2$	$x_2^2f_{-5,-4}^{(1)}$	$x_2^4f_{-10,-8}^{(0)} + x_2x_1f_{-5,-4}^{(0)}$	0	$f_{1,2}^{(1)} \to \Delta_1$

$\underline{\beta}$	$f_{\underline{\beta}}^{(2)}$	$v_{\underline{\beta}}^{(1)}$	$L_{\underline{\beta}}^{(2)}$	
$-6,-6$	$f_{-6,-6}^{(1)}$	0	$x_2^2x_1f_{-6,-6}^{(1)}$	$x_2f_{-6,-6}^{(2)} \to B_1$
$-3,-3$	$x_2f_{-6,-6}^{(2)}$	$f_{-6,-6}^{(1)}$	0	$f_{-3,-3}^{(2)} \to \Delta_2$
$-1,-2$	$f_{-1,-2}^{(1)}$	0	$x_2^2x_1^2f_{1,2}^{(1)}$	$x_2f_{-1,-2}^{(2)} \to B_1$
$1,2$	$f_{1,2}^{(1)}$	0	$x_2^6f_{-1,-2}^{(1)}$	$x_1f_{1,2}^{(2)} \to B_1$
$2,1$	$x_2f_{-1,-2}^{(2)}$	$x_1f_{1,2}^{(1)}$	0	$f_{2,1}^{(2)} \to \Delta_2$
$4,2$	$x_1f_{1,2}^{(2)}$	$x_2^2x_1f_{-1,-2}^{(1)}$	$x_2^3x_1^4f_{1,2}^{(1)}$	$x_2f_{4,2}^{(2)} \to B_1$
$7,5$	$x_2f_{4,2}^{(2)}$	$x_2^4x_1f_{-1,-2}^{(1)} + x_2x_1^3f_{1,2}^{(1)}$	0	$f_{7,5}^{(2)} \to \Delta_2$

$\underline{\beta}$	$f^{(3)}_{\underline{\beta}}$	$v^{(2)}_{\underline{\beta}}$	$L^{(3)}_{\underline{\beta}}$	
$-3,-3$	$f^{(2)}_{-3,-3}$	0	$x_2^3x_1f^{(2)}_{-3,-3}$	$x_2f^{(3)}_{-3,-3}\to B_2$
$0,0$	$x_2f^{(2)}_{-3,-3}$	$x_2f^{(2)}_{-3,-3}$	0	$f^{(3)}_{0,0}\to\Delta_3$
$2,1$	$f^{(2)}_{2,1}$	0	$x_2^3x_1f^{(2)}_{7,5}$	$x_2f^{(3)}_{2,1}\to B_2$
$5,4$	$x_2f^{(2)}_{2,1}$	$x_2f^{(2)}_{7,5}$	0	$f^{(3)}_{5,4}\to\Delta_3$
$7,5$	$f^{(2)}_{7,5}$	$x_2^3x_1f^{(2)}_{2,1}+x_1^2f^{(2)}_{7,5}$	0	$f^{(3)}_{7,5}\to\Delta_3$

$\underline{\beta}$	$f^{(4)}_{\underline{\beta}}$	$v^{(3)}_{\underline{\beta}}$	$L^{(4)}_{\underline{\beta}}$	
$0,0$	$f^{(3)}_{0,0}$	$f^{(3)}_{0,0}$	0	$f^{(4)}_{0,0}\to\Delta_4$
$5,4$	$f^{(3)}_{5,4}$	$Dx_2f^{(3)}_{7,5}$	0	$f^{(4)}_{5,4}\to\Delta_4$
$7,5$	$f^{(3)}_{7,5}$	$x_2^2x_1f^{(3)}_{4,1}+x_1^2f^{(3)}_{5,2}$	0	$f^{(4)}_{7,5}\to\Delta_4$

This means that the missing function is

$$\frac{f^{(4)}_{5,4}}{D}=\frac{x_3^2}{x_2}.$$

Example 5.4. This is an extension of Example 5.2. h_8, h_{10}, h_{12}, and h_{14} can be gotten by doubling all the subscripts in that example; adding the extra condition that $x_1^2x_2+x_1x_2+x_2^2+1=0$, and setting $h_{15}:=x_1x_2x_4x_8$; which gives rise to the defining polynomial of the extension:

$$f(h_{15}):=h_{15}^2+h_{15}h_{14}+(h_{14}+h_{12}+h_8+1)(h_8+1)^2+h_{14}(h_{12}+h_8).$$

The weight function is given by $W_4:=\big((1,0,2)W_3\quad 2W_3\big)^T=\big(15\quad 14\quad 12\quad 8\big)^T$ with $W_3:=\big(7\quad 6\quad 4\big)^T$.

$1/h_{14}=(h_{14}+h_{12})/(h_8+1)^2=(h_{14}+h_{12})h_{12}/(h_8(h_8+1)^4)$; so $D=h_8(h_8+1)^4$.

Δ_0^* has elements $f^{(0)}_{-40}:=1$, $f^{(0)}_{-30}:=h_{10}$, $f^{(0)}_{-28}:=h_{12}$, $f^{(0)}_{-26}:=h_{14}$, $f^{(0)}_{-25}:=h_{15}$, $f^{(0)}_{-15}:=h_{15}h_{10}, f^{(0)}_{-13}:=h_{15}h_{12}, f^{(0)}_{-11}:=h_{15}h_{14}$; with $f^{(0)}_b/D$ having weight b, corresponding to its pole size at P_∞. Then apply the qth power algorithm to produce the integral closure R_4.

b	$f^{(1)}_b$	$v^{(0)}_b$	$L^{(1)}_b$	
-40	$f^{(0)}_{-40}$	0	$f^{(0)}_{-40}$	$h_8^3f^{(1)}_{-40}\to B_0$
-30	$f^{(0)}_{-30}$	0	$h_8f^{(0)}_{-28}$	$h_8^2f^{(1)}_{-30}\to B_0$
-28	$f^{(0)}_{-28}$	0	$h_8^3f^{(0)}_{-40}$	$h_8f^{(1)}_{-28}\to B_0$

-26	$f_{-26}^{(0)}$	0	$h_8^2 f_{-28}^{(0)}$	$h_8^2 f_{-26}^{(1)} \to B_0$
-25	$f_{-25}^{(0)}$	0	$h_8^2 f_{-26}^{(0)}$	$h_8^2 f_{-25}^{(1)} \to B_0$
-20	$h_8 f_{-28}^{(1)} + h_8 f_{-30}^{(1)}$	0	$h_8^3 f_{-30}^{(0)}$	$h_8 f_{-20}^{(1)} \to B_0$
-16	$(h_8^3 + h_8) f_{-40}^{(1)}$	$h_8 f_{-40}^{(0)}$	0	$f_{-16}^{(1)} \to \Delta_1$
-15	$f_{-15}^{(0)}$	$f_{-30}^{(0)}$	$h_8^3 f_{-15}^{(0)}$	$h_8 f_{-15}^{(1)} \to B_0$
-14	$(h_8^2 + 1) f_{-30}^{(1)}$	$f_{-28}^{(0)} + f_{-30}^{(0)} + f_{-40}^{(0)}$	0	$f_{-14}^{(1)} \to \Delta_1$
-13	$f_{-13}^{(0)}$	$f_{-26}^{(0)}$	$h_8^3 f_{-11}^{(0)}$	$h_8 f_{-13}^{(1)} \to B_0$
-12	$h_8 f_{-20}^{(1)} + f_{-28}^{(1)} + f_{-30}^{(1)}$	$f_{-30}^{(0)}$	0	$f_{-12}^{(1)} \to \Delta_1$
-11	$f_{-11}^{(0)}$	$h_8 f_{-30}^{(0)}$	$h_8^4 f_{-15}^{(0)}$	$h_8 f_{-11}^{(1)} \to B_0$
-10	$h_8^2 f_{-26}^{(1)}$	$h_8 f_{-28}^{(0)} + h_8 f_{-30}^{(0)} + h_8^2 f_{-40}^{(0)} + f_{-26}^{(0)}$ $+ f_{-28}^{(0)} + h_8 f_{-40}^{(0)}$	$h_8^4 f_{-30}^{(0)}$	$h_8 f_{-10}^{(1)} \to B_0$
-9	$h_8^2 f_{-25}^{(1)}$	$h_8 f_{-26}^{(0)}$	$h_8^4 f_{-11}^{(0)}$	$h_8 f_{-9}^{(1)} \to B_0$
-7	$h_8 f_{-15}^{(1)} + f_{-11}^{(1)}$	$f_{-15}^{(0)} + h_8 f_{-25}^{(0)} + h_8 f_{-28}^{(0)} + h_8 f_{-30}^{(0)}$ $+ h_8^2 f_{-40}^{(0)} + f_{-25}^{(0)} + f_{-26}^{(0)}$	$h_8^3 f_{-13}^{(0)}$	$h_8 f_{-7}^{(1)} \to B_0$
-5	$h_8 f_{-13}^{(1)} + f_{-9}^{(1)}$	$f_{-11}^{(0)} + h_8^2 f_{-28}^{(0)} + f_{-15}^{(0)} + h_8 f_{-28}^{(0)}$	$h_8^4 f_{-13}^{(0)}$	$h_8 f_{-5}^{(1)} \to B_0$
-3	$h_8 f_{-11}^{(1)} + f_{-9}^{(1)} + f_{-13}^{(1)}$ $+ h_8 f_{-25}^{(1)}$	$h_8 f_{-15}^{(0)} + h_8^2 f_{-25}^{(0)} + h_8^2 f_{-26}^{(0)} + h_8 f_{-11}^{(0)}$ $+ h_8^2 f_{-28}^{(0)} + f_{-13}^{(0)} + h_8^2 f_{-30}^{(0)} + f_{-15}^{(0)}$ $+ h_8^3 f_{-40}^{(0)} + h_8 f_{-25}^{(0)} + h_8 f_{-28}^{(0)}$ $+ h_8^2 f_{-40}^{(0)} + f_{-28}^{(0)}$	0	$f_{-3}^{(1)} \to \Delta_1$
-2	$h_8 f_{-10}^{(1)} + h_8 f_{-26}^{(1)}$	$h_8 f_{-30}^{(0)}$	0	$f_{-2}^{(1)} \to \Delta_1$
-1	$h_8 f_{-9}^{(1)} + h_8 f_{-25}^{(1)}$	$h_8 f_{-11}^{(0)} + h_8^3 f_{-28}^{(0)} + h_8^3 f_{-30}^{(0)} + h_8^4 f_{-40}^{(0)}$ $+ h_8^2 f_{-28}^{(0)} + h_8^3 f_{-40}^{(0)} + h_8 f_{-26}^{(0)} + h_8 f_{-30}^{(0)}$	0	$f_{-1}^{(1)} \to \Delta_1$
1	$h_8 f_{-7}^{(1)} + f_{-9}^{(1)} + f_{-13}^{(1)}$ $+ f_{-15}^{(1)} + h_8 f_{-25}^{(1)}$	$f_{-13}^{(0)} + f_{-15}^{(0)} + h_8^3 f_{-40}^{(0)} + h_8 f_{-26}^{(0)}$ $+ h_8 f_{-26}^{(0)} + h_8 f_{-28}^{(0)} + h_8^2 f_{-40}^{(0)}$ $+ f_{-26}^{(0)} + f_{-28}^{(0)} + h_8 f_{-40}^{(0)}$	0	$f_{1}^{(1)} \to \Delta_1$
3	$h_8 f_{-5}^{(1)} + f_{-13}^{(1)} +$ $h_8 f_{-25}^{(1)}$	$h_8 f_{-13}^{(0)} + h_8^3 f_{-30}^{(0)} + h_8^4 f_{-40}^{(0)}$ $+ h_8^2 f_{-25}^{(0)} + f_{-11}^{(0)} + h_8^2 f_{-28}^{(0)}$ $+ f_{-13}^{(0)} + f_{-15}^{(0)} + h_8^3 f_{-40}^{(0)}$ $+ h_8 f_{-25}^{(0)} + h_8 f_{-26}^{(0)} + h_8 f_{-30}^{(0)} + f_{-28}^{(0)}$	0	$f_{3}^{(1)} \to \Delta_1$

b	$f_b^{(2)}$	$v_b^{(1)}$	$L_b^{(2)}$	
-16	$f_{-16}^{(1)}$	0	$h_8^3 f_{-16}^{(1)}$	$h_8 f_{-16}^{(2)} \to B_1$
-14	$f_{-14}^{(1)}$	0	$h_8^3 f_{-12}^{(1)}$	$h_8 f_{-14}^{(2)} \to B_1$
-12	$f_{-12}^{(1)}$	0	$h_8^4 f_{-16}^{(1)}$	$h_8 f_{-12}^{(2)} \to B_1$
-8	$(h_8+1) f_{-16}^{(2)}$	$f_{-16}^{(1)}$	0	$f_{-8}^{(2)} \to \Delta_2$
-6	$h_8 f_{-14}^{(2)} + f_{-12}^{(2)} + f_{-14}^{(2)}$	$f_{-12}^{(1)}$	$h_8^3 f_{-14}^{(1)}$	$h_8 f_{-6}^{(2)} \to B_1$
-4	$(h_8+1) f_{-12}^{(2)}$	$h_8 f_{-16}^{(1)} + f_{-14}^{(1)}$	0	$f_{-4}^{(2)} \to \Delta_2$
-3	$f_{-3}^{(1)}$	$h_8 f_{-14}^{(1)}$	$h_8^4 f_1^{(1)}$	$h_8 f_{-3}^{(2)} \to B_1$
-2	$f_{-2}^{(1)}$	$h_8 f_{-12}^{(1)} + h_8 f_{-16}^{(1)}$	$h_8^4 f_{-2}^{(1)}$	$h_8 f_{-2}^{(2)} \to B_1$
-1	$f_{-1}^{(1)}$	$f_{-2}^{(1)} + f_{-3}^{(1)} + h_8 f_{-12}^{(1)} + h_8 f_{-16}^{(1)}$	$h_8^4 f_{-1}^{(1)}$	$h_8 f_{-1}^{(2)} \to B_1$
1	$f_1^{(1)} + f_{-1}^{(2)} + f_{-2}^{(2)} +$ $f_{-14}^{(2)} + f_{-6}^{(2)}$	$h_8^2 f_{-14}^{(1)} + f_1^{(1)} + f_{-1}^{(1)} + f_{-3}^{(1)}$ $+ h_8 f_{-12}^{(1)} + h_8 f_{-14}^{(1)} + h_8 f_{-16}^{(1)}$ $+ f_{-14}^{(1)} + f_{-16}^{(1)}$	0	$f_1^{(2)} \to \Delta_2$
2	$h_8 f_{-6}^{(2)} + f_{-12}^{(2)}$	$f_{-14}^{(1)}$	0	$f_2^{(2)} \to \Delta_2$
3	$f_3^{(1)} + f_{-2}^{(2)}$	$h_8 f_{-2}^{(1)} + h_8 f_{-3}^{(1)} + h_8^2 f_{-12}^{(1)} + h_8^2 f_{-14}^{(1)}$ $+ f_1^{(1)} + f_{-1}^{(1)} + f_{-2}^{(1)}$ $+ f_{-3}^{(1)} + h_8 f_{-12}^{(1)} + h_8 f_{-16}^{(1)}$	$h_8^4 f_{-3}^{(1)}$	$h_8 f_3^{(2)} \to B_1$
5	$h_8 f_{-3}^{(2)}$	$h_8 f_1^{(1)} + h_8 f_{-1}^{(1)} + h_8 f_{-2}^{(1)} + h_8^2 f_{-12}^{(1)}$ $+ f_3^{(1)} + f_1^{(1)} + f_{-1}^{(1)} + h_8^2 f_{-16}^{(1)}$ $+ f_{-2}^{(1)} + f_{-3}^{(1)} + h_8 f_{-12}^{(1)}$	$h_8^4 f_3^{(1)}$	$h_8 f_5^{(2)} \to B_1$
6	$(h_8+1) f_{-2}^{(2)}$	$h_8 f_{-2}^{(1)} + h_8^2 f_{-12}^{(1)} + h_8^2 f_{-14}^{(1)} + h_8^2 f_{-16}^{(1)}$ $+ h_8 f_{-12}^{(1)} + h_8 f_{-14}^{(1)} + h_8 f_{-16}^{(1)}$	0	$f_6^{(2)} \to \Delta_2$
7	$(h_8+1) f_{-1}^{(2)}$	$h_8 f_{-1}^{(1)} + h_8^2 f_{-12}^{(1)} + f_3^{(1)}$ $+ h_8^2 f_{-14}^{(1)} + h_8^2 f_{-16}^{(1)} + h_8 f_{-12}^{(1)}$ $+ h_8 f_{-14}^{(1)} + h_8 f_{-16}^{(1)}$	0	$f_7^{(2)} \to \Delta_2$
11	$h_8 f_3^{(2)} + f_5^{(2)} + f_{-3}^{(2)} + f_3^{(2)}$	$h_8 f_{-3}^{(1)} + f_3^{(1)} + h_8^2 f_{-12}^{(1)}$ $+ h_8^2 f_{-14}^{(1)} + h_8 f_{-16}^{(1)}$	0	$f_{11}^{(2)} \to \Delta_2$
13	$(h_8+1) f_5^{(2)}$	$h_8 f_3^{(1)} + h_8 f_1^{(1)} + h_8^3 f_{-16}^{(1)} + h_8 f_{-1}^{(1)}$ $+ h_8 f_{-3}^{(1)} + f_3^{(1)} + h_8^2 f_{-14}^{(1)} + f_1^{(1)}$ $+ h_8^2 f_{-16}^{(1)} + f_{-1}^{(1)} + f_{-2}^{(1)}$ $+ f_{-3}^{(1)} + h_8 f_{-12}^{(1)}$	0	$f_{13}^{(2)} \to \Delta_2$

b	$f_b^{(3)}$	$v_b^{(2)}$	$L_b^{(3)}$	
-8	$f_{-8}^{(2)}$	0	$h_8^4 f_{-8}^{(2)}$	$h_8 f_{-8}^{(3)} \to B_2$
-4	$f_{-4}^{(2)}$	$f_{-8}^{(2)}$	$h_8^3 f_2^{(2)}$	$h_8 f_{-4}^{(3)} \to B_2$
0	$(h_8+1)f_{-8}^{(3)}$	$(h_8+1)f_{-8}^{(2)}$	0	$f_0^{(3)} \to \Delta_3$
1	$f_1^{(2)}$	$f_2^{(2)} + f_1^{(2)}$	$h_8^4 f_2^{(2)}$	$h_8 f_1^{(3)} \to B_2$
2	$f_2^{(2)}$	$(h_8+1)f_{-4}^{(2)}$	$h_8^4 f_{-4}^{(2)}$	$h_8 f_2^{(3)} \to B_2$
4	$h_8 f_{-4}^{(3)} + f_1^{(3)} + f_2^{(3)} + f_{-8}^{(3)}$	$f_2^{(2)} + h_8 f_{-8}^{(2)} + f_{-4}^{(2)}$	0	$f_4^{(3)} \to \Delta_3$
6	$f_6^{(2)}$	$h_8^2 f_{-4}^{(2)} + h_8^2 f_{-8}^{(2)} + f_6^{(2)} + f_2^{(2)}$	$h_8^4 f_6^{(2)}$	$h_8 f_6^{(3)} \to B_2$
7	$f_7^{(2)}$	$h_8 f_6^{(2)} + f_{13}^{(2)} + h_8^2 f_{-4}^{(2)}$ $+h_8^2 f_{-8}^{(2)} + f_7^{(2)} + f_6^{(2)}$	$h_8^4 f_{13}^{(2)}$	$h_8 f_7^{(3)} \to B_2$
9	$h_8 f_1^{(3)} + f_2^{(3)} + f_{-4}^{(3)} + f_{-8}^{(3)}$	$h_8 f_2^{(2)} + h_8^2 f_{-8}^{(2)} + f_{-4}^{(2)}$	0	$f_9^{(3)} \to \Delta_3$
10	$h_8 f_2^{(3)} + f_1^{(3)} + f_{-4}^{(3)} + f_{-8}^{(3)}$	$h_8 f_{-4}^{(2)} + f_2^{(2)} + f_{-8}^{(2)}$	0	$f_{10}^{(3)} \to \Delta_3$
11	$f_{11}^{(2)} + f_1^{(3)} + f_{-4}^{(3)} + f_{-8}^{(3)}$	$h_8^2 f_6^{(2)} + h_8 f_{13}^{(2)} + h_8^3 f_{-4}^{(2)} + h_8^2 f_2^{(2)}$ $+h_8^2 f_1^{(2)} + h_8 f_6^{(2)} + h_8 f_2^{(2)} + h_8 f_1^{(2)}$ $+h_8 f_{-8}^{(2)} + f_{-4}^{(2)} + f_{-8}^{(2)}$	0	$f_{11}^{(3)} \to \Delta_3$
13	$f_{13}^{(2)} + f_6^{(3)} + f_1^{(3)}$ $+f_{-4}^{(3)} + f_{-8}^{(3)}$	$h_8^3 f_2^{(2)} + h_8^3 f_1^{(2)} + h_8 f_{11}^{(2)} + h_8^2 f_1^{(2)}$ $+h_8^3 f_{-8}^{(2)} + h_8 f_6^{(2)} + h_8^2 f_{-4}^{(2)} + f_6^{(2)}$ $+h_8 f_{-4}^{(2)} + f_2^{(2)} + f_{-4}^{(2)} + f_{-8}^{(2)}$	0	$f_{13}^{(3)} \to \Delta_3$
14	$(h_8+1)f_6^{(3)}$	$h_8 f_6^{(2)} + h_8^2 f_{-4}^{(2)} + f_6^{(2)} + f_{-4}^{(2)}$	0	$f_{14}^{(3)} \to \Delta_3$
15	$(h_8+1)f_7^{(3)}$	$h_8 f_{13}^{(2)} + h_8 f_{11}^{(2)} + h_8^2 f_2^{(2)} + h_8 f_7^{(2)}$ $+h_8 f_6^{(2)} + f_{13}^{(2)} + f_{11}^{(2)} + h_8^2 f_{-8}^{(2)}$ $+f_7^{(2)} + h_8^2 f_{-4}^{(2)} + h_8^2 f_{-8}^{(2)}$ $+f_6^{(2)} + f_2^{(2)} + f_{-4}^{(2)}$	0	$f_{15}^{(3)} \to \Delta_3$

b	$f_b^{(4)}$	$v_b^{(3)}$	$L_b^{(4)}$	
0	$f_0^{(3)}$	$f_0^{(3)}$	0	$f_0^{(4)} \to \Delta_4$
4	$f_4^{(3)}$	$h_8 f_0^{(3)} + f_4^{(3)}$	$h_8^4 f_{10}^{(3)}$	$h_8 f_4^{(4)} \to B_3$
9	$f_9^{(3)}$	$h_8 f_{10}^{(3)} + (h_8+1)f_9^{(3)} + h_8 f_0^{(3)} + f_4^{(3)}$	$h_8^3 f_{11}^{(3)}$	$h_8 f_9^{(4)} \to B_3$
10	$f_{10}^{(3)} + f_4^{(4)}$	$h_8^2 f_4^{(3)} + h_8 f_{10}^{(3)} + h_8 f_9^{(3)}$ $+h_8 f_4^{(3)} + h_8 f_0^{(3)} + f_4^{(3)}$	0	$f_{10}^{(4)} \to \Delta_4$

11	$f_{11}^{(3)}$	$h_8 f_{14}^{(3)} + h_8 f_{13}^{(3)} + h_8^2 f_4^{(3)}$ $+f_{14}^{(3)} + f_{10}^{(3)} + h_8 f_0^{(3)}$	$h_8^4 f_{14}^{(3)}$	$h_8 f_{11}^{(4)} \to B_3$
12	$h_8 f_4^{(4)} + f_9^{(4)}$	$h_8 f_{10}^{(3)} + f_9^{(3)} + h_8 f_0^{(3)} + f_4^{(3)} + f_0^{(3)}$	0	$f_{12}^{(4)} \to \Delta_4$
13	$f_{13}^{(3)} + f_{14}^{(4)} + f_9^{(4)}$	$h_8^2 f_{10}^{(3)} + h_8^2 f_9^{(3)} + h_8 f_{11}^{(3)}$ $+h_8 f_{10}^{(3)} + h_8^2 f_0^{(3)} + f_{14}^{(3)}$	0	$f_{13}^{(4)} \to \Delta_4$
14	$f_{14}^{(3)}$	$h_8^3 f_4^{(3)} + h_8^2 f_{10}^{(3)} + h_8^2 f_9^{(3)} + h_8^3 f_0^{(3)}$ $+h_8 f_{14}^{(3)} + h_8 f_9^{(3)} + f_{10}^{(3)} + h_8 f_0^{(3)} + f_4^{(3)}$	0	$f_{14}^{(4)} \to \Delta_4$
15	$f_{15}^{(3)}$	$h_8^2 f_{14}^{(3)} + h_8^2 f_{13}^{(3)} + h_8^3 f_4^{(3)} + h_8^2 f_{10}^{(3)}$ $+h_8^2 f_9^{(3)} + h_8^3 f_0^{(3)} + h_8 f_{15}^{(3)} + h_8 f_{14}^{(3)}$ $+h_8 f_{11}^{(3)} + h_8 f_4^{(3)} + f_{11}^{(3)}$ $+f_{10}^{(3)} + f_9^{(3)} + f_0^{(3)}$	0	$f_{15}^{(4)} \to \Delta_4$
17	$(h_8 + 1) f_9^{(4)}$	$h_8 f_9^{(3)} + h_8^2 f_0^{(3)} + h_8 f_4^{(3)}$ $+f_9^{(3)} + f_4^{(3)} + f_0^{(3)}$	0	$f_{17}^{(4)} \to \Delta_4$
19	$(h_8 + 1) f_{11}^{(4)} + f_4^{(4)} + f_9^{(4)}$	$h_8 f_{14}^{(3)} + h_8^2 f_4^{(3)} + h_8 f_9^{(3)} + h_8^2 f_0^{(3)}$ $+f_{14}^{(3)} + f_{10}^{(3)} + h_8 f_0^{(3)}$	0	$f_{19}^{(4)} \to \Delta_4$

b	$f_b^{(5)}$	$v_b^{(4)}$	$L_b^{(5)}$	
0	$f_0^{(4)}$	$f_0^{(4)}$	0	$f_0^{(5)} \to \Delta_5$
10	$f_{10}^{(4)}$	$h_8 f_{12}^{(4)} + h_8 f_{10}^{(4)}$	0	$f_{10}^{(5)} \to \Delta_5$
12	$f_{12}^{(4)}$	$h_8^3 f_0^{(4)} + h_8 f_{12}^{(4)} + h_8 f_0^{(4)}$	0	$f_{12}^{(5)} \to \Delta_5$
13	$f_{13}^{(4)}$	$h_8^2 f_{10}^{(4)} + h_8 f_{17}^{(4)} + h_8 f_{14}^{(4)} + h_8 f_{13}^{(4)}$ $+f_{17}^{(4)} + h_8^2 f_0^{(4)} + f_{10}^{(4)}$	0	$f_{13}^{(5)} \to \Delta_5$
14	$f_{14}^{(4)}$	$h_8^2 f_{12}^{(4)} + h_8^2 f_{10}^{(4)} + h_8^3 f_0^{(4)} + h_8 f_{14}^{(4)}$ $+h_8 f_{12}^{(4)} + f_{10}^{(4)} + h_8 f_0^{(4)}$	0	$f_{14}^{(5)} \to \Delta_5$
15	$f_{15}^{(4)}$	$h_8^2 f_{14}^{(4)} + h_8^2 f_{13}^{(4)} + h_8^2 f_{12}^{(4)} + h_8 f_{19}^{(4)}$ $+h_8^2 f_{10}^{(4)} + h_8 f_{17}^{(4)} + h_8^3 f_0^{(4)} + h_8 f_{15}^{(4)}$ $+h_8 f_{14}^{(4)} + h_8 f_{13}^{(4)} + h_8 f_{12}^{(4)}$ $+f_{19}^{(4)} + f_{10}^{(4)} + f_0^{(4)}$	0	$f_{15}^{(5)} \to \Delta_5$
17	$f_{17}^{(4)}$	$h_8^3 f_{10}^{(4)} + h_8^2 f_{17}^{(4)} + h_8^2 f_{12}^{(4)}$ $+h_8 f_{17}^{(4)} + h_8^3 f_0^{(4)} + h_8 f_{12}^{(4)}$ $+h_8 f_{10}^{(4)} + (h_8^2 + h_8 + 1) f_0^{(4)}$	0	$f_{17}^{(5)} \to \Delta_5$

19	$f_{19}^{(4)}$	$h_8^3 f_{14}^{(4)} + h_8^3 f_{13}^{(4)} + h_8^3 f_{12}^{(4)} + h_8^2 f_{19}^{(4)}$ $+h_8^2 f_{14}^{(4)} + h_8 f_{19}^{(4)} + h_8^2 f_{10}^{(4)} + h_8 f_{17}^{(4)}$ $+h_8^3 f_0^{(4)} + h_8 f_{13}^{(4)} + h_8 f_{10}^{(4)} + f_{17}^{(4)} + h_8^2 f_0^{(4)}$	0	$f_{19}^{(5)} \to \Delta_5$

Example 5.5. Start with the surface defined by $x_3^3 + x_2^2 x_1^3 + x_3 x_2 x_1 = 0$ in characteristic 2. $W_3 = \begin{pmatrix} 5 & 3 & 3 \\ 2 & 3 & 0 \end{pmatrix}^T$. Let $D := x_2^2 x_1^3$, and start with $\Delta_0^* := \{f_{-15,-6}^{(0)} = 1,\ f_{-10,-4}^{(0)} = x_3,\ f_{-5,-2}^{(0)} = x_3^2\}$ with $f_{\underline{\beta}}^{(0)}/D$ having weight $\underline{\beta}$.

$\underline{\beta}$	$f_{\underline{\beta}}^{(1)}$	$v_{\underline{\beta}}^{(0)}$	$L_{\underline{\beta}}^{(1)}$	
$-15,-6$	$f_{-15,-6}^{(0)}$	0	$f_{-15,-6}^{(0)}$	$x_2 x_1^2 f_{-15,-6}^{(1)} \to B_0$
$-10,-4$	$f_{-10,-4}^{(0)}$	0	$f_{-5,-2}^{(0)}$	$x_2 x_1^2 f_{-10,-4}^{(1)} \to B_0$
$-6,-3$	$x_2 x_1^2 f_{-15,-6}^{(1)}$	$x_1 f_{-15,-6}^{(0)}$	0	$f_{-6,-3}^{(1)} \to \Delta_1$
$-5,-2$	$f_{-5,-2}^{(0)}$	$f_{-10,-4}^{(0)}$	$x_2 x_1 f_{-5,-2}^{(0)}$	$x_2 x_1 f_{-5,-2}^{(1)} \to B_0$
$-1,-1$	$x_2 x_1^2 f_{-10,-4}^{(1)}$	$x_1 f_{-5,-2}^{(0)}$	0	$f_{-1,-1}^{(1)} \to \Delta_1$
$1,1$	$x_2 x_1 f_{-5,-2}^{(1)}$	$x_2^2 x_1^2 f_{-10,-4}^{(0)} + x_2 f_{-5,-2}^{(0)}$	0	$f_1^{(1)} \to \Delta_1$

$\underline{\beta}$	$f_{\underline{\beta}}^{(2)}$	$v_{\underline{\beta}}^{(1)}$	$L_{\underline{\beta}}^{(2)}$	
$-6,-3$	$f_{-6,-3}^{(1)}$	0	$x_2 x_1^2 f_{-6,-3}^{(1)}$	$x_2 x_1 f_{-6,-3}^{(2)} \to B_1$
$-1,-1$	$f_{-1,-1}^{(1)}$	0	$x_2 x_1^3 f_{1,1}^{(1)}$	$x_2 f_{-1,-1}^{(2)} \to B_1$
$0,0$	$x_2 x_1 f_{-6,-3}^{(2)}$	$x_2 x_1 f_{-6,-3}^{(1)}$	0	$f_{0,0}^{(2)} \to \Delta_2$
$1,1$	$f_{1,1}^{(1)}$	$x_2 f_{-1,-1}^{(1)}$	$x_2^2 x_1^2 f_{1,1}^{(1)}$	$x_1 f_{1,1}^{(2)} \to B_1$
$2,2$	$x_2 f_{-1,-1}^{(2)}$	$x_2 f_{1,1}^{(1)}$	0	$f_{2,2}^{(2)} \to \Delta_2$
$4,1$	$x_1 f_{1,1}^{(2)}$	$x_2 x_1^2 f_{-1,-1}^{(1)} + x_1 f_{1,1}^{(1)}$	0	$f_{4,1}^{(2)} \to \Delta_2$

$\underline{\beta}$	$f_{\underline{\beta}}^{(3)}$	$v_{\underline{\beta}}^{(2)}$	$L_{\underline{\beta}}^{(3)}$	
$0,0$	$f_{0,0}^{(2)}$	$f_{0,0}^{(2)}$	0	$f_{0,0}^{(3)} \to \Delta_3$
$2,2$	$f_{2,2}^{(2)}$	0	$x_2^3 x_1^2 f_{4,1}^{(2)}$	$x_1 f_{2,2}^{(3)} \to B_2$

4, 1	$f^{(2)}_{4,1}$	$x_1^2 f^{(2)}_{2,2} + f^{(2)}_{4,1}$	0	$f^{(3)}_{4,1} \to \Delta_3$
5, 2	$x_1 f^{(3)}_{2,2}$	$x_2 x_1 f^{(2)}_{4,1}$	0	$f^{(3)}_{5,2} \to \Delta_3$

$\underline{\beta}$	$f^{(4)}_{\underline{\beta}}$	$v^{(3)}_{\underline{\beta}}$	$L^{(4)}_{\underline{\beta}}$	
0, 0	$f^{(3)}_{0,0}$	$f^{(3)}_{0,0}$	0	$f^{(4)}_{0,0} \to \Delta_4$
4, 1	$f^{(3)}_{4,1}$	$x_1 f^{(3)}_{5,2} + f^{(3)}_{4,1}$	0	$f^{(4)}_{4,1} \to \Delta_4$
5, 2	$f^{(3)}_{5,2}$	$x_2 x_1 f^{(3)}_{4,1}$	0	$f^{(4)}_{5,2} \to \Delta_4$

Now extend that surface by $x_4^2 + x_4 x_1 + x_3 x_1 = 0$, with $W_4 = \begin{pmatrix} 8 & 10 & 6 & 6 \\ 2 & 4 & 6 & 0 \end{pmatrix}^T$. This does *not* satisfy the conditions of prop 2.1, and indeed, with $g := x_3^2/(x_2 x_1)$, the missing function above, $wt(x_4) = (8, 2) = wt(g)$. But $y_4 := x_4 + x_3^2/(x_2 x_1)$ satisfies $y_4^2 + g(x_1 + 1) + y_4 x_1 = 0$. This gives $W_4 = \begin{pmatrix} 7 & 10 & 6 & 6 \\ 1 & 4 & 6 & 0 \end{pmatrix}^T$, which does satisfy prop 2.1. Let $D := x_1$, and start with $\Delta_0^* := \{f^{(0)}_{-6,0} = 1,\ f^{(0)}_{4,4} = x_3,\ f^{(0)}_{1,1} = y_4,\ f^{(0)}_{2,2} = g,\ f^{(0)}_{9,3} = y_4 g f^{(0)}_{11,5} = y_4 x_3, \}$ with $f^{(0)}_{\underline{\beta}}/D$ having weight $\underline{\beta}$.

$\underline{\beta}$	$f^{(1)}_{\underline{\beta}}$	$v^{(0)}_{\underline{\beta}}$	$L^{(1)}_{\underline{\beta}}$	
−6, 0	$f^{(0)}_{-6,0}$	0	1	$x_1 f^{(1)}_{-6,0} \to B_0$
0, 0	$x_1 f^{(1)}_{-6,0}$	$x_1 f^{(0)}_{-6,0}$	0	$f^{(1)}_{0,0} \to \Delta_1$
1, 1	$f^{(0)}_{1,1}$	$f^{(0)}_{2,2} + f^{(0)}_{1,1}$	$f^{(0)}_{2,2}$	$x_1 f^{(1)}_{1,1} \to B_0$
2, 2	$f^{(0)}_{2,2} + f^{(1)}_{1,1}$	$f^{(0)}_{4,4}$	0	$f^{(1)}_{2,2} \to \Delta_1$
4, 4	$f^{(0)}_{4,4}$	$x_2 f^{(0)}_{2,2}$	0	$f^{(1)}_{4,4} \to \Delta_1$
7, 1	$x_1 f^{(1)}_{1,1}$	$x_1 f^{(0)}_{2,2}$	0	$f^{(1)}_{7,1} \to \Delta_1$
9, 3	$f^{(0)}_{9,3} + f^{(1)}_{1,1}$	$x_2 x_1^3 f^{(0)}_{9,3} + x_1 f^{(0)}_{11,5} + x_2 x_1^2 f^{(0)}_{-6,0} + f^{(0)}_{9,3} + f^{(0)}_{2,2}$	0	$f^{(1)}_{9,3} \to \Delta_1$
11, 5	$f^{(0)}_{11,5}$	$x_2 x_1^2 f^{(0)}_{4,4} + x_2 x_1 f^{(0)}_{9,3} + x_2 x_1 f^{(0)}_{4,4} + x_2(x_1 + 1) f^{(0)}_{2,2}$	0	$f^{(1)}_{11,5} \to \Delta_1$

$\underline{\beta}$	$f^{(2)}_{\underline{\beta}}$	$v^{(1)}_{\underline{\beta}}$	$L^{(2)}_{\underline{\beta}}$	
0, 0	$f^{(2)}_{0,0}$	$f^{(1)}_{0,0}$	0	$f^{(2)}_{0,0} \to \Delta_2$
2, 2	$f^{(1)}_{2,2}$	$f^{(1)}_{4,4} + f^{(1)}_{2,2}$	0	$f^{(2)}_{2,2} \to \Delta_2$

4, 4	$f^{(1)}_{4,4}$	$x_2 f^{(1)}_{2,2}$	$x_2 f^{(1)}_{7,1}$	$x_1 f^{(2)}_{4,4} \to B_1$
7, 1	$f^{(1)}_{7,1}$	$(x_1^2 + x_1) f^{(1)}_{2,2} + f^{(1)}_{7,1}$	0	$f^{(2)}_{7,1} \to \Delta_2$
9, 3	$f^{(1)}_{9,3}$	$x_2 x_1 f^{(1)}_{0,0} + x_1 f^{(1)}_{11,5} + x_2 x_1 f^{(1)}_{0,0} + f^{(1)}_{9,3}$	0	$f^{(2)}_{9,3} \to \Delta_2$
10, 4	$x_1 f^{(2)}_{4,4}$	$x_2 x_1 f^{(1)}_{7,1}$	0	$f^{(2)}_{10,4} \to \Delta_2$
11, 5	$f^{(1)}_{11,5} + f^{(2)}_{4,4}$	$x_2 x_1^2 f^{(1)}_{4,4} + x_2 x_1 f^{(1)}_{9,3} + x_2 x_1 f^{(1)}_{4,4} + x_2(x_1 + 1) f^{(1)}_{2,2}$	0	$f^{(2)}_{11,5} \to \Delta_2$

$\underline{\beta}$	$f^{(3)}_{\underline{\beta}}$	$v^{(2)}_{\underline{\beta}}$	$L^{(3)}_{\underline{\beta}}$	
0, 0	$f^{(2)}_{0,0}$	$f^{(2)}_{0,0}$	0	$f^{(3)}_{0,0} \to \Delta_3$
2, 2	$f^{(2)}_{2,2}$	0	$f^{(2)}_{10,4}$	$x_1 f^{(3)}_{2,2} \to B_1$
7, 1	$f^{(2)}_{7,1}$	$(x_1^2 + x_1) f^{(2)}_{2,2} + f^{(2)}_{7,1}$	0	$f^{(3)}_{7,1} \to \Delta_3$
8, 2	$x_1 f^{(3)}_{2,2}$	$x_1 f^{(2)}_{10,4} + x_1^2 f^{(2)}_{2,2}$	0	$f^{(3)}_{8,2} \to \Delta_3$
9, 3	$f^{(2)}_{9,3}$	$x_2 x_1^2 f^{(2)}_{0,0} + x_1 f^{(2)}_{11,5} + x_2 x_1 f^{(2)}_{0,0} + f^{(2)}_{10,4} + f^{(2)}_{9,3}$	0	$f^{(3)}_{9,3} \to \Delta_3$
10, 4	$f^{(2)}_{10,4}$	$x_2 x_1^2 f^{(2)}_{2,2} + x_2 x_1 f^{(2)}_{7,1}$	0	$f^{(3)}_{10,4} \to \Delta_3$
11, 5	$f^{(2)}_{11,5}$	$x_2 x_1 f^{(2)}_{10,4} + x_2 x_1 f^{(2)}_{9,3} + x_2 f^{(2)}_{10,4} + x_2 x_1 f^{(2)}_{2,2}$	0	$f^{(3)}_{11,5} \to \Delta_3$

$\underline{\beta}$	$f^{(4)}_{\underline{\beta}}$	$v^{(3)}_{\underline{\beta}}$	$L^{(4)}_{\underline{\beta}}$	
0, 0	$f^{(3)}_{0,0}$	$f^{(3)}_{0,0}$	0	$f^{(4)}_{0,0} \to \Delta_4$
7, 1	$f^{(3)}_{7,1}$	$(x_1 + 1) f^{(3)}_{8,2} + f^{(3)}_{7,1}$	0	$f^{(4)}_{7,1} \to \Delta_4$
8, 2	$f^{(3)}_{8,2}$	$x_1 f^{(3)}_{10,4} + x_1 f^{(3)}_{8,2}$	0	$f^{(4)}_{8,2} \to \Delta_4$
9, 3	$f^{(3)}_{9,3}$	$x_2 x_1^2 f^{(3)}_{0,0} + x_1 f^{(3)}_{11,5} + x_2 x_1 f^{(3)}_{0,0} + f^{(3)}_{10,4} + f^{(3)}_{9,3}$	0	$f^{(4)}_{9,3} \to \Delta_4$
10, 4	$f^{(3)}_{10,4}$	$x_2 x_1 f^{(3)}_{8,2} + x_2 x_1 f^{(3)}_{7,1}$	0	$f^{(4)}_{10,4} \to \Delta_4$
11, 5	$f^{(3)}_{11,5}$	$x_2 x_1 f^{(3)}_{10,4} + x_2 x_1 f^{(3)}_{9,3} + x_2 f^{(3)}_{10,4} + x_2 f^{(3)}_{8,2}$	0	$f^{(4)}_{11,5} \to \Delta_4$

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