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# COMPARISON OF EIGENVALUE, LOGARITHMIC LEAST SQUARES AND LEAST SQUARES METHODS IN ESTIMATING RATIOS

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**Abstract**—Three methods—the eigenvalue, logarithmic least squares, and least squares methods—used to derive estimates of ratio scales from a positive reciprocal matrix are analyzed. The criteria for comparison are the measurement of consistency, dual solutions, and rank preservation. It is shown that the eigenvalue procedure, which is metric-free, leads to a structural index for measuring inconsistency, has two separate dual interpretations and is the only method that guarantees rank preservation under inconsistency conditions.

## 1. INTRODUCTION

Given a matrix of data $A = (a_{ij})$, $a_{ij} > 0$, $a_{ji} = 1/a_{ij}$, where $a_{ij}$, $i, j = 1, \ldots, n$ are estimates of underlying ratios $(\alpha_i/\alpha_j)$, there are three methods commonly used to derive best estimates of $\alpha = (\alpha_1, \ldots, \alpha_n)$ using $A$. They are: the eigenvalue method (EM), the logarithmic least squares method (LLSM), and the least squares method (LSM). Our purpose in this paper is to compare them. Other methods such as row averaging and column normalization followed by row averaging will not be considered. All methods yield the same answer, $r_i$, $i = 1, \ldots, n$, when $a_{ij}$ is given in the form of a ratio, i.e., $a_{ij} = r_i/r_j$; otherwise the solutions are different but often close. One problem with making comparisons of such methods is for example that EM and LSM do not lead to a closed form solution. Numerical methods are required to estimate them. Let us first give a formal statement of the three methods. Our comparison of the methods relates to the question of the effect of the consistency of $A$ on rank preservation.

## 2. THE METHODS

*Definition*: A matrix $A$ is said to be reciprocal if $a_{ji} = a_{ij}^{-1}$ for all $i, j = 1, 2, \ldots, n$.
*Definition*: A reciprocal matrix is said to be consistent if $a_{ij}a_{jk} = a_{ik}$ for all $i, j, k = 1, 2, \ldots, n$.

The foregoing condition may be written to show that the coefficients in the $j$th row of $A$ are ratios of coefficients in the $i$th row. Thus $a_{jk} = a_{ik}/a_{ij}$.

When $A$ is consistent, all $a_{ij}$ can be derived from $(n - 1)$ given values which form a star (as in comparing one element with all others) a chain or, more generally, a spanning tree. Consistency here is a condition for relations among the data. It is not the usual requirement of convergence in probability of an estimate to its true value. A necessary and sufficient condition for a positive reciprocal matrix to be consistent is $a_{ij} = r_i/r_j$, $i, j = 1, 2, \ldots, n$. This is precisely the condition under which all these methods coincide.

Thus, we say that a set of judgments and their corresponding solutions are inconsistent if the reciprocal matrix of comparisons does not satisfy the definition of consistency. Obviously, there are different degrees of inconsistency. These can be measured through a comparison of the inconsistency of $A$ with the inconsistency of a positive reciprocal matrix with random entries. We shall address this question in greater detail below.

*The Eigenvalue Method (EM)*

An estimate of $\alpha_i$ is the vector derived by solving the eigenvalue problem $Aw = \lambda_{\max} w$ or

$$\sum_{j=1}^{n} a_{ij} w_j = \lambda_{\max} w_i, \qquad i = 1, 2, \ldots, n \tag{1}$$

where the priorities $w_i$, $i = 1, 2, \ldots, n$ are the components of the right-eigenvector corresponding to the principal eigenvalue (the largest eigenvalue) of $A$.

$\lambda_{\max}$ is obtained by solving the characteristic polynomial

$$\det(A - \lambda I) = 0 \tag{2}$$

where $I$ is the identity matrix.

The eigenvalues of $A$ can all be complex except for one. The exceptional eigenvalue whose existence is assured by the Perron-Frobenius theorem is denoted by $\lambda_{\max}$ and is real and positive. In addition, for a reciprocal matrix $A$, $\lambda_{\max} \geq n$. The corresponding eigenvector solution $w$ is also real and positive, and unique to within a multiplicative constant. Hence, the general rule for the solvability of our problem for an arbitrary positive reciprocal matrix $A$ is to determine the characteristic value $\lambda$ such that $\lambda = \lambda_{\max}$. It is only for this value that the existence of the desired ratio scale $w$ can be affirmed.

The EM solution is obtained iteratively as

$$\lim_{k \to \infty} \frac{A^k e}{\| A^k \|} \tag{3}$$

where $e = (1, 1, \ldots, 1)^T$ and $\| A^k \| = e^T A^k e$. Geometrically $A^k$ gives the sums of all products of $k$ coefficients. In graph theoretic language, it may be interpreted as the cumulative dominance of a vertex over each other vertex along chains of length $k$. As a result, for large values of $k$, every coefficient in $A$ would contribute to the calculation of $w_i$ for all $i$.

*The Logarithmic Least Squares Method (LLSM)*

An estimate of $\alpha_i$ is the vector derived by minimizing

$$\sum_{i,j=1}^{n} \left( \ln a_{ij} - \ln \frac{u_i}{u_j} \right)^2 . \tag{4}$$

The solution to this minimization problem is given by

$$u_i = \left( \prod_{j=1}^{n} a_{ij} u_j \right)^{1/n} . \qquad i = 1, 2, \ldots, n \tag{5}$$

Imposing the condition $\sum_{i=1}^{n} u_i = 1$, one obtains

$$u_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{1/n} \Bigg/ \sum_{i=1}^{n} \left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}, \qquad i = 1, 2, \ldots, n. \tag{6}$$

Unlike EM, in LLSM the coefficients in other rows make no direct contribution to the calculation of $u_i$. Thus inconsistent relations among row $i$ and other rows are not reflected in $u_i$. LLSM is a procedure for implementing the idea that the reciprocal of a function of a set of variables, any two of which are interchangeable, is equal to that function applied to the reciprocal of the variables, and thus the geometric mean of a row of $A$ is the reciprocal of the corresponding column of $A$. This results in a loss of discrimination of inconsistency as we shall see later.

*The Least Squares Method (LSM)*

An estimate of $\alpha_i$ is the vector $v$ derived by minimizing

$$\sum_{i,j=1}^{n} \left(a_{ij} - \frac{v_i}{v_j}\right)^2. \tag{7}$$

Note that there is neither a closed form solution for this problem, nor a widely used numerical method of solution.

The general LSM approach goes beyond estimating vectors to estimating matrices by matrices of lower rank [2, 4, 5]. From

$$\text{Trace}(AA^T) = \sum_{i,j=1}^{n} a_{ij}^2$$

we have

$$\text{Trace}(A - V)(A - V)^T = \sum_{i,j=1}^{n} \left[a_{ij} - \frac{v_i}{v_j}\right]^2$$

where $V = (v_i/v_j)$.

Now, for any positive matrix $X$, $XX^T$ is symmetric and all its eigenvalues are real. Also $XX^T$ is positive and has a unique real positive largest eigenvalue. We have

$$AA^T \equiv P \Lambda P^T, \qquad A^TA \equiv Q \Lambda Q^T$$

where $\Lambda$ is a diagonal matrix whose entries are the eigenvalues of $AA^T$ (or $A^TA$) in descending order of magnitude; the eigenvectors of $AA^T$ are the corresponding columns of $P$ and those of $A^TA$ are the corresponding rows of $Q^T$. Hence a least-squares approximation of $A$ by a matrix of rank $r$ is given by [2]

$$P_r \Lambda^{1/2} Q_r^T$$

where $P_r$ and $Q_r^T$ are the parts of $P$ and $Q^T$ associated with the first $r$ columns of $\Lambda$. Since the matrix $V$ constructed from the vector $v$ is consistent, it follows that $A$ is approximated by a matrix of unit rank and we have:

$$V = P_1 \Lambda^{1/2} Q_1^T$$

*Remark:* The most frequently used metric to measure closeness and accuracy in $n$-dimensional spaces is the Euclidean metric. This is the metric of LSM. However, the Euclidean metric does not address the question of inconsistency. J. Fichtner [3], in a forthcoming dissertation coached by the first author, recently introduced a metric satisfying the usual axioms and having to do with inconsistency. Its minimization yields the EM solution, i.e., the EM criterion is linked to minimizing inconsistency.

Let $R$ be the set of all reciprocal matrices and let $R_C \subset R$ be the subset of consistent reciprocal matrices. For $A \in R$ and $B \in R_C$, this metric is given by

$$\delta(A, B) = \left[ \sum_{i=1}^{n} [w_i^{(A)} - w_i^{(B)}]^2 \right]^{1/2} + \frac{1}{2(n-1)} \left| \lambda_{\max}^{(A)} - \lambda_{\max}^{(B)} \right| + \frac{1}{2(n-1)} \left| \lambda_{\max}^{(A)} + \lambda_{\max}^{(B)} - 2n \right| \cdot \delta_{AB}$$

where $Aw^{(A)} = \lambda_{\max}^{(A)} w^{(A)}$, $Bw^{(B)} = \lambda_{\max}^{(B)} w^{(B)}$ and

$$\delta_{AB} = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\delta(A, W) = \min_{B \in R_C} \delta(A, B),\ \delta(A, W) = \frac{\lambda_{\max}^{(A)} - n}{n - 1},$$

and

$$W w^{(A)} = n w^{(A)}.$$

## 3. RELATIONSHIPS AMONG $w$, $u$ AND $v$

Assume that $w$, $u$ and $v$ are normalized to unity.

THEOREM 2: For any $n \geq 2$, if $A$ is consistent, then, $w$, $u$ and $v$ coincide.

*Proof:* Let $A$ be a consistent matrix. Then $a_{ij} = w_i/w_j$, $i, j = 1, 2, \ldots, n$ [5]. EM yields

$$Ww = nw, \qquad W = \left( \frac{w_i}{w_j} \right)$$

or

$$(W - nI)w = 0.$$

Since $n$ is the largest eigenvalue of $W$ clearly $w = 1/\sum_{i=1}^{n} w_i (w_1, \ldots, w_n)^T$ is the desired solution.

By using (6) LLSM yields

$$u_i = \frac{w_i}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} w_j\right)^{1/n}} \Bigg/ \frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} w_j\right)^{1/n}} = \frac{w_i}{\sum_{i=1}^{n} w_i}, \quad i, 1, 2, \ldots, n.$$

For LSM, since $A$ is consistent we can write $a_{ij} = r_i/r_j$, $i, j = 1, 2, \ldots, n$, and take $v_i = r_i$, $i = 1, 2, \ldots, n$.

Consistency is a sufficient condition for $w = u = v$. An example of an inconsistent matrix $A$ for which $w$, $u$ and $v$ are nearly the same is

$$A = \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & 2 \\ 2 & \frac{1}{2} & 1 \end{bmatrix}.$$

The solution is approximately given by $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$. The LSM solution is not always unique.

THEOREM 3: $w$ and $u$ coincide for an arbitrary positive-reciprocal matrix $A$ for $n = 2, 3$.

*Proof:* A $2 \times 2$ matrix is consistent and hence Theorem 2 applies. Let $A = (w_i/w_j\, \epsilon_{ij})$ be a reciprocal matrix. $A = W \circ E$ where "o" denotes the elementwise product of $W$ and $E$ with $W = (w_i/w_j)$, and $E = (\epsilon_{ij})$, $\epsilon_{ji} = \epsilon_{ij}^{-1}$, $i, j = 1, 2, \ldots, n$. Since $W$ is consistent by construction, its principal right eigenvector coincides with that of $A$. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ is the principal right eigenvector of $E$ and $\lambda_{max}(A) = \lambda_{max}(E)$.

The matrix $E$ can be uniquely characterized in terms of a single parameter $\epsilon$. Thus we have

$$E = \begin{bmatrix} 1 & \epsilon & \frac{1}{\epsilon} \\ \frac{1}{\epsilon} & 1 & \epsilon \\ \epsilon & \frac{1}{\epsilon} & 1 \end{bmatrix}.$$

This can be shown by solving the system of equations

$$1 + \epsilon_{12} + \epsilon_{13} = \lambda_{max}$$

$$\frac{1}{\epsilon_{12}} + 1 + \epsilon_{23} = \lambda_{max}$$

$$\frac{1}{\epsilon_{13}} + \frac{1}{\epsilon_{23}} + 1 = \lambda_{max}$$

in three unknowns.

Using (6) we have:

$$u_i = \frac{\left(\prod_{j=1}^{n} \frac{w_i}{w_j}\epsilon_{ij}\right)^{1/n}}{\sum_{k=1}^{n}\left(\prod_{j=1}^{n} \frac{w_k}{w_j}\epsilon_{ij}\right)^{1/n}} = \frac{w_i\left(\prod_{j=1}^{n}\epsilon_{ij}\right)^{1/n}}{\sum_{k=1}^{n} w_k\left(\prod_{j=1}^{n}\epsilon_{kj}\right)^{1/n}}, \quad i = 1, 2, 3$$

Since $\Pi_{j=1}^{n} \epsilon_{ij} = 1$, on normalization (i.e., $\sum_{k=1}^{n} w_k = 1$), we have $w_i = u_i$, $i = 1, 2, 3$.

## 4. MEASUREMENT OF CONSISTENCY

EM has provided a useful structural criterion for the measurement of violations of the consistency criterion $a_{ij}a_{jk} = a_{ik}$. Presumably because of inconsistency the usefulness of the data in the matrix $A$ may be questionable and new data may be needed.

The inconsistency of the data is measured as follows. A consistent reciprocal matrix $A$ has unit rank and hence all but one of its eignevalues are zero [6, 7]. From $n = \text{trace}(A) = \sum_{i=1}^{n} \lambda_i$, we have $\lambda_{\max} - n = -\sum_{i=2}^{n} \lambda_i$ and $(\lambda_{\max} - n)/(n - 1) = -\sum_{i=2}^{n} \lambda_i/(n - 1)$ where $\lambda_1 \equiv \lambda_{\max}$. By definition the average inconsistency is given by

$$\mu(n) \equiv \frac{\lambda_{\max} - n}{n - 1} = -\frac{1}{n-1}\sum_{i=2}^{n} \lambda_i. \tag{8}$$

By abuse of language $\mu(n)$ is called the consistency index of $A$. By writing $a_{ij} = w_i/w_j\, \epsilon_{ij}$, $\epsilon_{ij} > 0$ and $\epsilon_{ij} = 1 + \delta_{ij}$, $\delta_{ij} > -1$, it is easy to show that $\mu(n)$ is a measure of the variance of $\delta_{ij}$ [7].

As an illustration of how to perform a test of inconsistency a particular scale of absolute numbers employed in making pairwise comparisons to quantify qualitative judgments in the analytic hierarchy process [7] was used. Here 500 reciprocal matrices of sizes $n = 3, 4, \ldots, 13$ were randomly generated using the scale values $\frac{1}{9}, \frac{1}{8}, \frac{1}{2}, 1, 2, \ldots, 8, 9$. Each time their largest eigenvalue was computed. The average was then taken. The results of this simulation are given in Table 1. Table 2 gives estimates of the mean consistency and the standard deviation. In [7], $\lambda_{\max}$ was tested for normality. Table 3 gives 95% confidence interval bounds on consistency using the information of Table 2. For example, if the consistency index of a $3 \times 3$ reciprocal matrix is $\mu(3) = .1087$, then the ratio of this index and its corresponding value .6090 for $n = 3$ (.1087/.6090 = .1785) obtained for randomly generated matrices (Table 3) is a measure of the closeness of the pairwise

Table 1

| $n$ | Interval of variation | $\bar{\lambda}_{\max}(n)$ Sample mean | $S_{\lambda_{\max}}(n)$ Sample standard deviation |
|---|---|---|---|
| 3 | ( 3.0000, 9.7691) | 4.0762 | 1.3695 |
| 4 | ( 4.1013, 11.3006) | 6.6496 | 1.8380 |
| 5 | ( 5.3855, 15.3237) | 9.4178 | 2.1032 |
| 6 | ( 7.1376, 17.2361) | 12.3129 | 2.1007 |
| 7 | (10.0610, 20.3545) | 15.0001 | 2.0305 |
| 8 | (12.4807, 23.5383) | 17.9518 | 1.9045 |
| 9 | (14.9457, 25.3345) | 20.5652 | 1.8240 |

Table 2

| Matrix size ($n$) | Mean consistency $\bar{\mu}(n)$ | Standard deviation of the mean consistency |
|---|---|---|
| 3 | 0.5381 | 0.0433 |
| 4 | 0.8832 | 0.0475 |
| 5 | 1.1045 | 0.0470 |
| 6 | 1.2525 | 0.0420 |
| 7 | 1.3334 | 0.0371 |
| 8 | 1.4217 | 0.0322 |
| 9 | 1.4457 | 0.0288 |

comparison judgments to random judgments. A reciprocal matrix is then said to be near consistency if this ratio is 10% or less [7, 9].

A proposal has been put forth in [1] to measure inconsistency by means of the correlation coefficient

$$R = \left[\frac{n \sum_{i=1}^{n} x_i^2}{\sum_{i<j} b_{ij}^2}\right]^{1/2}$$

where $x_i = 1/n \sum b_{ij}$, and $b_{ij} = \ln a_{ij}$. But this is not a good measure as can be seen from the following example.

$$A = \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & 2 & 2 \\ 2 & \frac{1}{2} & 1 \end{bmatrix}$$

$\mu(n)$/[Random $\mu(n)$] = $[(3.5 - 3)/2]/.6090 \cong .43$ which is very poor, and the corresponding value of $R$ is $R = 0$ which is also very poor. If we replace $a_{13} = \frac{1}{2}$ and $a_{31} = 2$ by $a_{13} = \frac{1}{9}$ and $a_{31} = 9$, then $\mu(n)$/[Random $\mu(n)$] = $[(4.60 - 3)/2] \cong 1.38$ which is much worse, yet $R = .5104$ which is much better.

## 5. DUAL SOLUTIONS

Let $A$ be a positive reciprocal matrix and consider $A^T$. The corresponding EM, LLSM and LSM estimates of $\alpha$ derived by using $A^T$ and denoted by $w^*$, $u^*$, and $v^*$, respectively, are called the dual solutions of $w$, $u$, and $v$. The problem here is to determine the rela-

Table 3

| Matrix size ($n$) | Critical point ($\alpha$ = 5%) |
|---|---|
| 3 | .6090 |
| 4 | .9610 |
| 5 | 1.1820 |
| 6 | 1.3220 |
| 7 | 1.3940 |
| 8 | 1.4750 |
| 9 | 1.4930 |

tionship between a solution and its dual. $w$ and $w^*$ are principal right and left eigenvectors, respectively, of $A$.

THEOREM 4: For an arbitrary positive reciprocal consistent matrix $A$, the normalized componentwise reciprocal of $w^*$ is equal to $w$.

*Proof:* Since $A$ is consistent any normalized row of $A$ yields $w^*$. On the other hand, $w$ is any normalized column of $A$. By construction, the normalized components of any column of $A$ are the reciprocals of the normalized components of any row of $A$, and the theorem follows.

THEOREM 5: For $n = 3$, the normalized solutions $w$ and $w^*$ are reciprocals.

*Proof:* For $n = 3$, $w = u$ and we have from (6)

$$w_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{1/n} \Bigg/ \sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}, \quad i = 1, 2, \ldots, n,$$

and

$$w_i^* = \left(\prod_{j=1}^{n} a_{ji}\right)^{1/n} \Bigg/ \sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ji}\right)^{1/n}, \quad i = 1, \ldots, n.$$

Normalization of the reciprocals of $w_i^*$, $i = 1, \ldots, n$, yields $w$.

$w$ and $w^*$ are the right- and left-eigenvector of $A$, respectively. In general, $w$ and $w^*$ need not be reciprocals. When $A$ is consistent, the conversion from left- to right-eigenvector presents no problem, for then the entire set of reciprocal comparisons and their relations can be inferred from the comparisons themselves. When $A$ is inconsistent this is no longer true except for $n = 3$ (see Theorem 5) where the structure of the matrix again dictates the mathematical relations between the comparisons and their reciprocals. Now there is a significant distinction to be made between left and right-eigenvectors. The right-eigenvector arises from answering the following kind of question: How much more does one element dominate another with respect to a given criterion? The left-eigenvector arises from answering the question: How much more is the smaller element dominated by the larger one with respect to the criterion? Our ability to answer these two questions is not the same. The second one is much more difficult. To answer the first question, the smaller element is used as a unit for making the comparison. The answer to the second question requires that the first element serve as a unit to be divided into parts, each roughly equal to the second element which now is used as the unit. It is much easier for one to take multiples of a smaller unit than to divide a larger one into fractions. In fact there is doubt as to our ability to do the latter without inverting the comparison to the first kind. If people were forced to make the second type of comparisons they would be less certain of the correctness of their judgment and larger errors would result.

The scale one seeks from dominance comparisons is obtained through the right-eigenvector. Here, the elements listed on the left of the matrix are compared with those listed at the top. Were one to compute the right-eigenvector of the transpose of the matrix, the new right-eigenvector would not represent dominance comparisons and, because of the difficulties mentioned above, it would not be a meaningful scale. Thus, even though left and right-eigenvectors are related through the structure of the matrix, only the right-eigenvector gives a meaningful scale derived from $A$.

THEOREM 6: The LLSM solutions $u$ and $u^*$ are reciprocals.

*Proof:* Identical to that of Theorem 5 for all $n$.

THEOREM 7: The LSM solutions $v$ and $v^*$ are reciprocals.

*Proof:* $v$ is obtained by minimizing

$$F \equiv \sum_{i,j=1}^{n} \left(a_{ij} - \frac{v_i}{v_j}\right)^2 = \sum_{i<j} (a_{ij} - v_i/v_j)^2 + (a_{ji} - v_j/v_i)^2.$$

Let $B = A^T$. $v^*$ is obtained from minimizing

$$G \equiv \sum_{i,j=1}^{n} \left(b_{ij} - \frac{v_i^*}{v_j^*}\right)^2 = \sum_{i<j} (b_{ij} - v_i^*/v_j^*)^2 + (b_{ji} - v_j^*/v_i^*)^2.$$

Since $b_{ij} = a_{ji} = a_{ij}^{-1}$, we have

$$G = \sum_{i<j} (a_{ji} - v_i^*/v_j^*)^2 + (a_{ij} - v_j^*/v_i^*)^2.$$

Substituting $v_i^* = 1/v_i$, $i = 1, 2, \ldots, n$, in $G$ we have

$$G = \sum_{i<j} (a_{ji} - v_j/v_i)^2 + (a_{ij} - v_i/v_j)^2 = F$$

and the result follows.

## 6. RANK PRESERVATION

By rank order we mean the order relationship between $x_i$ and $x_j$, where $x = w$, $u$, or $v$. How should this relationship be interpreted in terms of what we know about $A$?

We indicated earlier that the values and hence also the order of $w_i$, $i = 1, 2, \ldots, n$ is the result of complex calculations having to do with chains of arbitrary length to keep track of consistency relations. This turns out to be important for capturing inconsistencies to preserve rank order. However, all three methods behave similarly with respect to rank under certain conditions. For example, we noted earlier that with consistency these methods yield the same solution and hence ranking is the same.

Now let use assume that $A$ is inconsistent and hence its columns may yield different rankings. Generally on observing that $a_{ij} \geq 1$ one might expect $x_i \geq x_j$ to hold. But this cannot always be the case. Another interesting observation is that if row $i$ dominates row $j$, then the methods should preserve rank. This turns out to be true. Still other intuitive guesses have to do with taking the arithmetic mean or the geometric mean (LLSM) of the rows. Rank preservation can be easily shown to break down under these assumptions.

*Definition:* A method of solution is said to preserve rank weakly if $a_{ij} \geq 1$ implies $x_i \geq x_j$ where $x = w$, $u$ or $v$

THEOREM 8 (Intuitive Expectation): If $A$ is consistent then EM, LLSM and LSM preserve rank weakly.

*Proof:* If $A$ is consistent then $a_{ij} = x_i/x_j$ from which the proof follows.

Note that with consistency $x_i \geq x_j$ implies $a_{ij} \geq 1$.

*Definition:* A method of solution is said to preserve rank strongly if $a_{ih} \geq a_{jh}$ for all $h$ implies $x_i \geq x_j$.

The next theorem shows the sufficiency of row dominance without requiring consistency.

THEOREM 9 (Row Dominance): For EM and LLSM, given $i$ and $j$, $a_{ik} \geq a_{jk}$ for all $k$, implies $x_i \geq x_j$.

*Proof:* For EM we have

$$\lambda_{\max} w_i = \sum_{k=1}^{n} a_{ik} w_k \geq \sum_{k=1}^{n} a_{jk} w_k = \lambda_{\max} w_j$$

which yields $w_i \geq w_j$.

For LLSM we have

$$u_i = \left( \prod_{k=1}^{n} a_{ik} \right)^{1/n} \geq \left( \prod_{k=1}^{n} a_{jk} \right)^{1/n} = u_j, \quad \text{and} \quad u_i \geq u_j.$$

For LSM, this theorem has been proven in [8].

*Corollary 1:* $a_{ik} \geq a_{jk}$ for all $k$, implies $a_{ij} \geq 1$.

*Proof:* Put $k = j$ obtaining $a_{ij} \geq a_{jj}$.

*Corollary 2:* Let $a_{ij} \geq 1$. If $a_{ik}/a_{ij} \geq a_{jk}$, $k = 1, 2, \ldots, n$, then $x_i \geq x_j$.

*Corollary 3:* If $a_{ij} \leq 1$ and $a_{jk} \geq a_{ik}/a_{ij}$ for all $k$, then $x_j \geq x_i$.

The following is a generalization of Theorem 9 to products.

THEOREM 10: $u_i \geq u_j$ if and only if $\prod_{k=1}^{n} a_{ik} \geq \prod_{k=1}^{n} a_{jk}$.

THEOREM 11: Let $n = 3$. For an arbitrary positive reciprocal matrix $A$, $w_i \geq w_j$ if and only if $\prod_{k=1}^{n} a_{ik} \geq \prod_{k=1}^{n} a_{jk}$.

*Proof:* Follows from (6) and Theorem 3.

*Definition:* A positive reciprocal matrix $A$ is ordinally transitive if for each $i = 1, 2, \ldots, n$, $a_{ij} \geq a_{ik}$ for some $j$ and $k$, implies $a_{jk} \leq 1$.

Hence, a positive reciprocal matrix $A$ is ordinally intransitive if $a_{ij} \leq a_{ik}$ implies $a_{jk} \geq 1$ for some $i, j$ and $k$.

THEOREM 12: In an ordinally transitive reciprocal matrix $A$, given $i$ and $i'$, either $a_{ih} \geq a_{i'h}$ for all $h$, or $a_{ih} \leq a_{i'h}$ for all $h$.

*Proof:* Consider rows $i$ and $i'$ and let $h$ be the subset of column indices for which $a_{ih} \geq a_{i'h}$ and let $h'$ be the remaining subset of indices for which $a_{ih'} \leq a_{i'h'}$.

Because $A$ is ordinally transitive, we have $a_{hi'} \geq a_{hi}$ implies that $a_{i'i} \leq 1$ and $a_{h'i} \geq a_{h'i'}$ implies that $a_{ii'} \leq 1$ or $a_{i'i} \geq 1$ which is absurd.

The following relates ordinal transitivity to rank preservation.

*Corollary:* If $A$ is ordinally transitive then EM, LLSM and LSM preserve rank strongly.

Now assume that for some $i$ and $j$, neither $a_{ih} \geq a_{jh}$ nor $a_{ih} \leq a_{jh}$ for all $h$. Thus $A$ is inconsistent. It follows that $a_{ij} \geq 1(= a_{jj})$ need not imply $x_i \geq x_j$. However, it turns out that $a_{ij}^{(k)} \geq a_{jj}^{(k)}$ does, where $a_{ij}^{(k)}$ is the $(i, j)$ entry of $A^k$.

We now develop a necessary and sufficient condition for rank preservation in terms of the row dominance of the powers of $A$. For emphasis, recall that an element $a_{ij}^{(k)}$ of $A^k$ gives the cumulative dominance of the $i$th element over the $j$th element along all chains of length $k$. That is precisely how one measures the consistency relation between that row and each column. In fact when $A$ is consistent we have from $A^k = n^{k-1}A$ that the entries of $A^k$ and those of $A$ differ by a constant thus maintaining consistency.

In general, consider $A^k = (a_{ij}^{(k)})$ where

$$a_{ij}^{(k)} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}.$$

THEOREM 13: For a positive reciprocal matrix $A$

$$\lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}} = \lim_{k\to\infty} \frac{a_{is}^{(k)}}{\sum_{i=1}^{n} a_{is}^{(k)}}, \quad h, s = 1, 2, \ldots, n$$

*Proof:* Let $B = NAN^{-1}$ be the Jordan canonical form of $A$ given by

$$B = \begin{bmatrix} \lambda_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{bmatrix}$$

where $\lambda_1 \equiv \lambda_{\max}$, and $B_p$, $p = 2, 3, \ldots, r$ is the $m_p \times m_p$ Jordan block defined by

$$B_p = \begin{bmatrix} \lambda_p & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_p & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda_p \end{bmatrix}$$

where $\lambda_p$, $p = 2, \ldots, r$ are distinct eigenvalues with multiplicities $m_2, \ldots, m_r$ respectively, and $\sum_{p=2}^{r} m_p = n - 1$. We have $A = N^{-1}BN$ and $A^k = N^{-1}B^kN$ where

$B^k$ is given by

$$B^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & B_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_r^k \end{bmatrix}$$

Let us denote $N^{-1} \equiv D = (d_{ij})$ and $N = (n_{ij})$. We have

$$A^k = DB^kN = \begin{bmatrix} n_{11}d_{11}\lambda_1^k + \cdots, & n_{12}d_{11}\lambda_1^k + \cdots, & \cdots, & n_{1n}d_{11}\lambda_1^k + \cdots \\ n_{11}d_{21}\lambda_1^k + \cdots, & n_{12}d_{21}\lambda_1^k + \cdots, & \cdots, & n_{1n}d_{21}\lambda_1^k + \cdots \\ \vdots & \vdots & \cdot & \vdots \\ n_{11}d_{n1}\lambda_1^k + \cdots, & n_{12}d_{n1}\lambda_1^k + \cdots, & \cdots, & n_{1n}d_{n1}\lambda_1^k + \cdots \end{bmatrix}.$$

Let $e = (1, 1, \ldots, 1)^T = a_1w_1 + \cdots + a_rw_r$, where $w_p$ is the principal right eigenvector corresponding to $\lambda_p$. We have

$$e^TA^k = a_1\lambda_1^k w_1^T + \cdots + a_r\lambda_1^k w_r^T = \left(n_{11}\sum_{i=1}^n d_{i1}\lambda_1^k + \cdots, \cdots, n_{1n}\sum_{i=1}^n d_{i1}\lambda_1^k + \cdots\right).$$

Given two columns of $A^k$, $h$ and $s$ we have

$$\frac{a_{ih}^{(k)}}{\sum_{i=1}^n a_{ih}^{(k)}} = \frac{n_{1h}d_{i1}\lambda_1^k + \cdots}{n_{1h}\sum_{i=1}^n d_{i1}\lambda_1^k + \cdots} \quad \text{and} \quad \frac{a_{is}^{(k)}}{\sum_{i=1}^n a_{is}^{(k)}} = \frac{n_{1s}d_{i1}\lambda_1^k + \cdots}{n_{1s}\sum_{i=1}^n d_{i1}\lambda_1^k + \cdots}.$$

Since both numerators and denominators are polynomials in $\lambda_p^k$, $p = 1, 2, \ldots, r$, and $\lambda_1 = \lambda_{\max} > |\lambda_p|$, $p \neq 1$, we have for the $i$ entries of two arbitrary columns $h$ and $s$

$$\lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^n a_{ih}^{(k)}} = \lim_{k\to\infty} \frac{a_{is}^{(k)}}{\sum_{i=1}^n a_{is}^{(k)}} = \frac{d_{i1}}{\sum_{i=1}^n d_{i1}}.$$

*Definition:* A positive matrix $A$ is said to be $k$-dominant if there is a $k_0$ such that for $k \geq k_0$ either $a_{ih}^{(k)} \geq a_{i'h}^{(k)}$ or $a_{ih}^{(k)} \leq a_{i'h}^{(k)}$ for all $h$ and for any pair $i$ and $i'$.

*Corollary:* A positive matrix is $k$-dominant.

*Proof:* We have from Theorem 13 that the normalized columns of $A^k$ are the same in the limit. Since the elements in each row are identical, the result follows by choosing $k_0$ to be the maximum of its values for each pair of rows.

We now show that for an inconsistent matrix $A$ rank is determined in terms of the powers of $A$. To do this we demonstrate that there is a method of estimating $\alpha$ which coincides with the normalized limiting columns of $A$. This method is precisely EM.

THEOREM 14:

$$\lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^n a_{ih}^{(k)}} = w_i, \quad i = 1, 2, \ldots, n.$$

*Proof:*

From $\lim_{k\to\infty} A^k e/\| A^k \| = w$, we have $w_i = \lim_{k\to\infty} 1/\| A^k \| \sum_{h=1}^{n} a_{ih}^{(k)}$. Multiplying and dividing $a_{ih}^{(k)}$ by $\sum_{i=1}^{n} a_{ih}^{(k)}$ we have on distributing the limit with respect to the finite sum

$$w_i = \sum_{h=1}^{n} \lim_{k\to\infty} \left[ \frac{a_{ih}^{(k)}}{\| A^k \|} \cdot \frac{\sum_{i=1}^{n} a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}} \right] = \sum_{h=1}^{n} \left[ \lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}} \right] \left[ \lim_{k\to\infty} \frac{\sum_{i=1}^{n} a_{ih}^{(k)}}{\| A^k \|} \right].$$

By Theorem 13,

$$\lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}},$$

is the same constant for all $h$, hence we have

$$w_i = \lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}} \sum_{h=1}^{n} \left[ \lim_{k\to\infty} \frac{\sum_{i=1}^{n} a_{ih}^{(k)}}{\| A^k \|} \right].$$

Since $\| A^k \| = \sum_{i=1}^{n} \sum_{h=1}^{n} a_{ih}^{(k)}$, the proof is complete.

The foregoing also clarifies why one can take a matrix and obtain a rank order, then augment the matrix by a row and its reciprocal column and discover that the new ranking involves a reversal of the old ranking. This must happen because of the inconsistency relations among the old and the new rows of the matrix.

For $n = 3$, we have shown earlier that the inconsistencies of $A$ can be characterized in terms of a single parameter and hence the normalized row products of $A$ coincide with the normalized principal eigenvector. The following is an example for $n = 3$ in which the LSM solution yields a different ranking than EM and LLSM where of course $w$ and $u$ are identical.

$$\begin{bmatrix} 1 & 2 & 7 \\ \frac{1}{2} & 1 & 9 \\ \frac{1}{7} & \frac{1}{9} & 1 \end{bmatrix}$$

| EM | LLSM | LSM |
|---|---|---|
| .559 | .559 | .412 |
| .383 | .383 | .529 |
| .058 | .058 | .059 |

Here $\mu(n) = .05$, $\mu(n)/[\text{Random } \mu(n)] = .086$ which is good. $w_1 > w_2 > w_3$, $u_1 > u_2 > u_3$ but $v_2 > v_1 > v_3$.

For $n \geq 4$ unlike EM, whether $A$ is consistent or not LLSM makes no use of coefficients in other rows in the calculation of $u$. Consequently except for $n = 3$ where $u$ coincides with $w$, LLSM would tend to produce a ranking that is insensitive to inconsistencies in the matrix.

## 7. EXAMPLES AND COUNTEREXAMPLES

The following is an example with good consistency where the ranking by EM differs from the other two.

$$A = \begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{3} & \frac{1}{8} & 5 \\ 6 & 1 & 2 & 1 & 8 \\ 3 & \frac{1}{2} & 1 & \frac{1}{2} & 5 \\ 8 & 1 & 2 & 1 & 5 \\ \frac{1}{5} & \frac{1}{8} & \frac{1}{5} & \frac{1}{5} & 1 \end{bmatrix}$$

| EM | LLSM | LSM | |
|---|---|---|---|
| .081 | .073 | .055 | |
| .346 | .358 | .363 | |
| .180 | .187 | .201 | |
| .355 | .346 | .332 | |
| .038 | .036 | .049 | |
| 45.168 | 46.080 | 24.654 | = SSE(*) |

Here $\mu(n) = .089$ and $\mu(n)/[\text{Random } \mu(n)] = .079$. The rankings are

$$w_4 > w_2 > w_3 > w_1 > w_5$$

$$u_2 > u_4 > u_3 > u_1 > u_5$$

$$v_2 > v_4 > v_3 > v_1 > v_5$$

Here $A^k$ with $k = 4$ gives the same ranking as $w$. We have with rounding off

$$A^k = A^4 = \begin{bmatrix} 182.560 & 34.205 & 65.020 & 37.846 & 391.042 \\ 769.9 & 146.810 & 277.933 & 164.45 & 1652.25 \\ 401.35 & 76.380 & 144.667 & 85.435 & 861.375 \\ 789.983 & 150.875 & 285.45 & 169.435 & 1682.083 \\ 84.083 & 16.175 & 30.588 & 18.135 & 185.602 \end{bmatrix}$$

Here is an example with moderately poor consistency for which the three methods yield different rankings

$$A = \begin{bmatrix} 1 & 4 & 3 & 1 & 3 & 4 \\ \frac{1}{4} & 1 & 7 & 3 & \frac{1}{5} & 1 \\ \frac{1}{3} & \frac{1}{7} & 1 & \frac{1}{5} & \frac{1}{5} & \frac{1}{6} \\ 1 & \frac{1}{3} & 5 & 1 & 1 & \frac{1}{3} \\ \frac{1}{3} & 5 & 5 & 1 & 1 & 3 \\ \frac{1}{4} & 1 & 6 & 3 & \frac{1}{3} & 1 \end{bmatrix}$$

| EM | LLSM | LSM | |
|---|---|---|---|
| .321 | .316 | .185 | |
| .139 | .139 | .216 | |
| .035 | .035 | .037 | |
| .129 | .125 | .150 | |
| .237 | .236 | .215 | |
| .139 | .148 | .197 | |
| 89.847 | 85.279 | 60.049 | = SSE |

and $\mu(n) = .284$, and $\mu(n)/[\text{Random } \mu(n)] = .229$.

An example with very poor consistency in which all three methods coincide in value and therefore also in ranking is:

(*) SSE = Sum of Squares of Errors.

$$A = \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & 2 \\ 2 & \frac{1}{2} & 1 \end{bmatrix}$$

| EM | LLSM | LSM | |
|---|---|---|---|
| .333 | .333 | .333 | |
| .333 | .333 | .333 | |
| .333 | .333 | .333 | |
| 3.75 | 3.75 | 3.75 | = SSE |

$\mu(n) = .25$, $\mu(n)/[\text{Random } \mu(n)] = .431$.

Finally an example with very poor consistency in which the three methods yield three different rankings is

$$A = \begin{bmatrix} 1 & 4 & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & 1 & \frac{1}{3} & 4 \\ 2 & 3 & 1 & \frac{1}{2} \\ 5 & \frac{1}{4} & 2 & 1 \end{bmatrix}$$

| EM | LLSM | LSM | |
|---|---|---|---|
| .214 | .193 | .118 | |
| .245 | .184 | .175 | |
| .242 | .319 | .298 | |
| .299 | .304 | .408 | |
| 40.987 | 37.787 | 30.538 | = SSE |

$\mu(n = .806$, $\mu(n)/[\text{Random } \mu(n)] = .896$

$$w_4 > w_2 > w_3 > w_1$$

$$u_3 > u_4 > u_1 > u_2$$

$$v_4 > v_3 > v_1 > v_2$$

Here we have

$$A^{11} = \begin{bmatrix} 195132463.2 & 205257530 & 91373282.57 & 164496038.1 \\ 222651232.1 & 231069167.1 & 104356525 & 191688460.3 \\ 221646689.6 & 230654337.7 & 103670215.5 & 186199353.9 \\ 275564324.4 & 280849364.2 & 128581383.3 & 229448679.1 \end{bmatrix}$$

in which the fourth row dominates the second row which dominates the third row which dominates the first row coinciding with the ranking induced by EM.

## 8. CONCLUSION

The purpose of deriving a ratio scale estimate of $\alpha$ is to obtain a unidimensional scale which "best" fits the data represented by $A$. An important criterion which must be considered by all the methods is the criterion of consistency. When consistency obtains EM, LLSM and LSM produce identical solutions. Consideration of consistency gives rise to two properties, the existence of reciprocal dual solutions and the preservation of rank. When there is inconsistency in the data for whatever reason, that inconsistency must be dealt with as a fact, either accepted or reduced by improving the quality of the information. EM is a useful method for addressing the problem of inconsistency both with respect to dual solutions and to preservation of rank. In fact it is the only method that should be used when the data are not entirely consistent in order to make the best choice of alternative.

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