Local Estimates for the Hausdorff Dimension of an Attractor

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1. INTRODUCTION

Most of the known dissipative nonlinear PDE's can be studied basically under two categories: the parabolic ones that include the 2D Navier-Stokes equations, the Kuramoto-Sivashinsky equations, etc., and the hyperbolic equations that include the sine-Gordon as well as other weakly damped equations [T]. An important characteristic of the equations that fall into the first category is the regularizing effect of the solution operator. Typically, in these equations, the solution becomes smoother in time as a function of only space variables.

On the other hand, the hyperbolic equations fail to exhibit a similar behaviour. However, in the model equation

$$\ddot{u} + \alpha \dot{u} - \Delta u + g(u) = f, \quad u(0) = u_0$$

(1.1)

the solution operator $S(t)$ can be decomposed into the sum of two operators $S_1(t)$ and $S_2(t)$. In this decomposition, $S_2(t)$ denotes the solution operator for the linear problem

$$\ddot{v} + \alpha \dot{v} - \Delta v = h, \quad v(0) = v_0$$

(1.2)

and $S_1(t) = S(t) - S_2(t)$. Under some restrictions on $g$ and $f$, it can be
shown that the solutions of (1.2) decay exponentially in time, whereas \( S_1(t) \) enjoys a regularizing effect [GT]. This special property of the solution operator allows us to extend the theory of local Lyapunov exponents developed in [EFT] to this class of hyperbolic dissipative PDE's. In this paper, we develop the proper modification and generalize the results of [EFT]. Apart from the infrequent remarks contrasting the present setup to the one given in [EFT] the paper is almost self-contained. In the last section we will make use of some estimates derived in [EFT].

We first define the solution operator as a nonlinear semigroup acting on a Hilbert space. The two main ingredients in this theory are the existence of a compact invariant set which attracts all solutions, that is, a global attractor, and the existence of a Frechet type derivative of the solution operator on this attractor. The asymptotic behaviour of the equation can then be discussed by analyzing the eigenvalues (or the \( s \)-numbers) of the derivative operator. The results connecting the Hausdorff dimension of the global attractor to the negativity of some Lyapunov exponents are results in this direction [CF, CFT].

For the model equation (1.1) the existence of the global attractor and the estimates of its Hausdorff dimension have already been discussed in [GT]. As is the case with all PDE's the Frechet type derivative of the solution operator can be recovered as the solution of the linearized problem

\[
\dot{w} + \alpha \dot{w} - \alpha w + g'(u)w = 0, \quad w(0) = u - u_0. \tag{1.3}
\]

In other words, if \( S'(t, u_0) \) denotes the solution operator for (1.3) then \( S'(t, u_0) \) and \( S(t) \) are related through

\[
|S(t)u - S(t) u_0 - S'(t, u_0)(u - u_0)| \leq c \cdot o(|u - u_0|) \tag{1.4}
\]

Due to the regularizing effect of the parabolic type equations, \( S'(t, u_0) \) is a compact operator in such a case. On the other hand, for the model equation (1.1) since \( S'(t, u_0) \) is the sum of \( S'_1(t, u_0) \) and \( S'_2(t, u_0) \), \( S'(t, u_0) \) can only be assumed to be the sum of a compact operator with a strict contraction. Hence the appearance of a continuous spectrum for \( S'(t, u_0) \) is unavoidable. In these circumstances, we replace the eigenvalues of \( S'(t, u_0) \) with the corresponding \( s \)-numbers, and the theory of local and global Lyapunov exponents are introduced via \( s \)-numbers.

This paper is organized as follows: we start with a nonlinear semigroup acting on the global attractor and then define a flow of positive operators on a continuous function space using this semigroup. The Lyapunov exponents for the semigroup then appear as the \( L \)-exponents of the flow. Hence the basic result of the third section concerning the connection between local and global \( L \)-exponents can also be interpreted as a similar statement for Lyapunov exponents. In the fourth section we also discuss
further relations between local Lyapunov exponents that lead us to the definition of the Douady–Oesterlé dimension given in the fifth section, and we prove in the sixth section the local estimate of the Hausdorff dimension via the local Douady–Oesterlé dimension. Namely, we show the existence of a critical path in the attractor for which the local DO-dimension dominates the Hausdorff dimension of the attractor. The last section is devoted to a nontrivial example of such a critical path, and especially for a Lorentz type equation we show that the critical path is one of the stationary solutions.

2. FROM NONLINEAR SEMIGROUPS TO FLOWS OF POSITIVE OPERATORS

Let $H$ be a separable Hilbert space and let $S(t)$ be a continuous nonlinear semigroup acting on $H$, that is,

$$t \to S(t)$$

is a continuous map from $\mathbb{R}^+$ into $\mathcal{C}(H, H)$, (2.1)

$$S(t + s) = S(t) S(s) \quad \text{for every} \quad s, t \in \mathbb{R}^+.$$ (2.2)

We further assume the existence of a compact invariant set $X$, on which $S(t)$ is bijective,

$$S(t)X = X.$$ (2.3)

For every $t > 0$, and $u_0 \in H$, there exists a bounded linear operator $L$, such that

$$|S(t)u - S(t)u_0 - L(u - u_0)|_H \leq C \cdot o(|u - u_0|_H)$$ (2.4)

for $u, u_0 \in X, \ u \in B_r(u_0), \ c = c(t)$. Let us denote such an operator by $S'(t, u_0)$.

The Lyapunov exponents associated with $S(t)$ on $X$ are defined through the $s$-numbers associated with $S'(t, u_0)$:

$$s_j^2(t, u_0) = \sup_{F \subseteq H} \inf_{\dim F = j} \left( \frac{1}{|f||f|_H = 1} \right)$$ (2.5)

for $j = 1, 2, \ldots$.

Let us remark that in the case where $S'(t, u_0)$ is compact, $s$-numbers of $S'(t, u_0)$ coincide with the eigenvalues of $(S'(t, u_0) * S'(t, u_0))^{1/2}$, the positive part of $S'(t, u_0)$. It is also easy to see that $s_1(t, u_0)$ is nothing but the operator norm of $S'(t, u_0)$. We now show that one can recover the products of $s$-numbers as the operator norm of some operators constructed from
S'(t, u₀). To this end, we briefly review the basic properties of the outer products of Hilbert spaces and related concepts.

\[ H^{\wedge k} = \text{the linear span of } \{v_1 \wedge v_2 \wedge \cdots \wedge v_k : v_i \in H, i = 1, 2, \ldots, k\}, \quad (2.6) \]

where \( v_1 \wedge v_2 \wedge \cdots \wedge v_k \) denotes a skew-symmetric tensor in \( \otimes^k H \). Under the standard inner product

\[ \langle v_1 \wedge v_2 \wedge \cdots \wedge v_k, u_1 \wedge u_2 \wedge \cdots \wedge u_k \rangle = \det [(v_i, u_j)]_{i,j=1}^k \quad (2.7) \]

the vector space \( H^{\wedge k} \) becomes a pre-Hilbert space. Let \( H^k \) denote its completion. If \( A \) is a bounded linear operator on \( H \), then \( A^{\wedge k} \) defined on \( H^{\wedge k} \) by

\[ A^{\wedge k}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = Av_1 \wedge Av_2 \wedge \cdots \wedge Av_k \quad (2.8) \]

can be extended to a bounded linear operator on \( H^k \).

A standard result relates the products of \( s \)-numbers to the operator norm of \( S'(t, u₀)^{\wedge k} \) \([T]\). More precisely,

\[ w_k(t, u₀) = s_1(t, u₀) \cdot s_2(t, u₀) \cdots s_k(t, u₀) \\
= \sup_{|ξ|_H = 1} |S'(t, u₀) ξ_1 \wedge S'(t, u₀) ξ_2 \wedge \cdots \wedge S'(t, u₀) ξ_k| \\
= \|S'(t, u₀)^{\wedge k}\|_{\mathcal{L}(H^{\wedge k})}. \quad (2.9) \]

The relation (2.9) allows us to define the local and global Lyapunov exponents in two equivalent ways.

**DEFINITION 2.1.** We define recursively,

\[ \mu_1 + \mu_2 + \cdots + \mu_N = \lim_{t \to \infty} t^{-1} \log \sup_{u₀ \in X} w_N(t, u₀) \\
= \lim_{t \to \infty} t^{-1} \log \sup_{u₀ \in X} \|S'(t, u₀)^{\wedge N}\|_{\mathcal{L}(H^{\wedge N})}, \quad (2.10) \]

and

\[ (\mu_1 + \cdots + \mu_N)(u₀) = \lim_{t \to \infty} t^{-1} \log w_N(t, u₀) \\
= \lim_{t \to \infty} t^{-1} \log \|S'(t, u₀)^{\wedge N}\|_{\mathcal{L}(H^{\wedge N})}, \quad (2.11) \]

where \( \mu_i \)'s are called the **global Lyapunov exponents** and \( \mu_i(u₀) \)'s are called the **local Lyapunov exponents at** \( u₀ \).
Remark 2.2. The global Lyapunov exponents can also be defined as the spectral radius of some positive operators, hence the use of the normal limit in (2.10) is justified.

Remark 2.3. It is clear that the sum of the global Lyapunov exponents dominates the sum of local Lyapunov exponents. Our next aim is to show the existence of an element $u_0$ in $X$, for which

$$
(\mu_1 + \mu_2 + \cdots + \mu_N)(u_0) = \mu_1 + \mu_2 + \cdots + \mu_N.
$$

Let $S_k$ denote the unit sphere of $H^k$, and for a continuous bounded real-valued function $f$ on $X \times S_k$ we define

$$(T, f)(u_0, \xi) = |S'(t, u_0)^{\wedge k}\xi| f(S(t) u_0, \left(\frac{S'(t, u_0)^{\wedge k}\xi}{|S'(t, u_0)^{\wedge k}\xi|}\right)).$$

PROPOSITION 2.4. $T$, as defined in (2.13) is a positive operator mapping $C_b(X \times S_k)$ into itself. Moreover, $\{T_t\}_{t \geq 0}$ enjoys the following properties:

(a) $T_{t+s} f = T_t(T_s f)$ for every $t, s \in \mathbb{R}^+$

(b) the map $t \rightarrow \Gamma_t(u_0) \doteq \sup_{\xi \in S_k} (T_t 1)(u_0, \xi)$

is continuous from $\mathbb{R}^+$ into $C(X)$, here $1$ stands for the identically one function.

Proof. Note that $\Gamma_t(u_0) = \|S'(t, u_0)^{\wedge k}\|_{\mathcal{L}(H^{\wedge k})}$, hence the continuity of the map $t \rightarrow \Gamma_t(u_0)$ follows directly from the continuity of the map $t \rightarrow S'(t, u_0)$ as a map from $\mathbb{R}^+$ into $\mathcal{L}(H)$. On the other hand, the flow condition $T_{t+s} = T_t T_s$ is a direct consequence of

$$S'(t + s, u_0) = S'(t, S(s) u_0) S'(s, u_0)$$

and the elementary fact about the outer products of bounded linear operators, namely,

$$(A B)^{\wedge k} = A^{\wedge k} B^{\wedge k}.$$

Finally, it is straightforward to verify that $T_t$ maps $C_b(X \times S_k)$ into itself and that is a linear, positive operator.

Now we can rewrite the local and global Lyapunov exponents in terms of the flow $\{T_t\}_{t \geq 0}$. Since a positive on $C_b(X \times S_k)$ achieves its operator norm with the function $1$, we can deduce that

$$\|T_t\|_{\mathcal{L}(C_b(X \times S_k))} = \|T_t 1\|_{\mathcal{L}(C_b(X \times S_k))} = \sup_{u_0 \in X} \|S'(t, u_0)^{\wedge k}\|_{\mathcal{L}(H^{\wedge k})}.$$  

(2.14)
Consequently (2.10) and (2.11) are transformed into

\[ \mu_1 + \mu_2 + \cdots + \mu_N = \lim_{t \to \infty} t^{-1} \log \| T_t \|_{\mathcal{C}(\mathcal{E}_b)}, \tag{2.15} \]

\[ (\mu_1 + \mu_2 + \cdots + \mu_N)(u_0) = \lim_{t \to \infty} t^{-1} \log \Gamma_t(u_0). \tag{2.16} \]

3. Positive Operators on \( \mathcal{C}_b(X \times S) \)

In order to prove (2.12), we use (2.15) and (2.16) and view it as a problem related with positive operators on the appropriate space. To this end, we build up our machinery first for discrete flows, then transfer the results to the continuous flows of positive operators.

**Definition 3.1.** Let \( X \) be a compact set and let \( S \) be any set. Let \( T \) be a positive, linear operator on \( \mathcal{C}_b(X \times S) \), the set of bounded continuous functions on \( X \times S \). \( T \) is said to satisfy **norm continuity** if for every \( n \in \mathbb{N} \), the function \( \Gamma_n \) defined by

\[ \Gamma_n(u_0) = \sup_{\xi \in S} (T^n 1)(u_0, \xi) \tag{3.1} \]

is a continuous function on \( X \).

We now recall a fundamental concept from Choquet–Foias that relates the pointwise behaviour of \( T^n \) with its uniform behaviour [ChF].

**Definition 3.2.** Let \( T \) and \( \Gamma_n \) be given as in the above definition; then \( T \) is said to satisfy **property** \( P_0 \) if for every \( u_0 \) in \( X \), there exists \( n = n(u_0) \) such that \( \Gamma_n(u_0) < 1 \).

Although property \( P_0 \) is a pointwise property it is enough to guarantee the fact that the spectral radius of \( T \) is less than one [ChF].

**Proposition 3.3.** If \( T \) has property \( P_0 \) then

\[ e^{\mu} = \lim_{n \to \infty} \| T^n \|^{1/n} = \inf_n \| T^n \|^{1/n} < 1. \tag{3.2} \]

**Proof.** Set \( g_r(u_0) = \inf \{ \Gamma_n(u_0) : 1 \leq n \leq r \} \); first we show the existence of an integer \( r \), such that \( g_r(u_0) < 1 \). We proceed by way of contradiction and assume the existence of \( u_r \) s.t. \( g_r(u_r) \geq 1 \) for every \( r \). By the compactness of \( X \), \( \{ u_r \} \subseteq \mathbb{N} \) has a convergent subnet \( \{ u_{\lambda} \}_{\lambda \in A} \). Let \( u_0 \) denote the limit of this subnet. It follows by the property \( P_0 \) that there exists \( m \in \mathbb{N} \) such that \( \Gamma_m(u_0) < 1 \). Since \( \Gamma_m \) is continuous there exists \( \lambda_0 \in A \) such that \( \Gamma_m(u_{\lambda_0}) < 1 \).
for $\lambda \geq \lambda_0$. Hence, if we pick $\lambda$ in $A$ satisfying both $\lambda \geq \lambda_0$ and $\lambda \geq m$, then clearly $g_2(u_\lambda) \leq \Gamma_m(u_\lambda) < 1$, which is the desired contradiction. Therefore, we have shown the existence of an integer $r$ such that

$$g_r(u) < 1 \quad \text{for every } u \in X. \quad (3.3)$$

Set $h_r(u_0, \xi) \equiv \inf\{(T^n)(u_0, \xi): 1 \leq n \leq r\}$; then clearly

$$h_r(u_0, \xi) \leq g_r(u_0) \leq \sup_{u_0 \in X} g_r(u_0) = \theta < 1. \quad (3.4)$$

We now prove some simple facts related with these functions $h_r$. From $h_r < 1$ and $Th_r < T'1$ for $i = 1, 2, \ldots, r$ it follows that $Th_r \leq h_r \leq \theta$; iterating this inequality we can deduce that $T'\cdots h_r \leq \theta$, for $i = 0, 1, 2, \ldots, r$, and consequently, $T'h_r \leq T'\theta = \theta T'1$. Taking the infimum over $i$, in the last inequality we deduce that

$$T'h_r < \theta h_r, \quad (3.5)$$

hence by taking the infimum, $T'1 \leq a'h_r$. Consequently,

$$T^{(n+1)r}1 = T^nT'1 \leq a'T^n h_r \leq a'\theta^n, \quad (3.6)$$

where in the last inequality we used the iterates of (3.5). Since $\theta < 1$, by choosing $n$ large enough $a'\theta^n$ can be made smaller than one. Combining this fact with the positivity of $T$, we conclude that:

There exists $m$ such that $\|T^m1\| < 1. \quad (3.7)$

Since from the definition of spectral radius it follows that

$$e^\mu = \lim_{n \to \infty} \|T^n\|^{1/n} = \inf_n \|T^n\|^{1/n}, \quad (3.8)$$

we can easily deduce from (3.7) that $e^n < 1.$

**Definition 3.4.** For a positive linear operator $T$ satisfying the norm continuity, the local and global $L$-exponents are defined respectively by

$$\mu(u_0) = \lim_{n \to \infty} \left[\Gamma_n(u_0)\right]^{1/n} \quad (3.9)$$

and

$$\mu = \lim_{n \to \infty} \|T^n\|^{1/n}. \quad (3.10)$$

The following proposition relates the local and global $L$-exponents and paves the way for (2.12).
HAUSDORFF DIMENSION OF AN ATTRACTOR

**Proposition 3.5.** There exists an element $u_0$ in $X$ such that

$$\Gamma_n(u_0) \geq e^{\gamma_n} \quad \text{for all } n. \quad (3.11)$$

Moreover, the same element satisfies

$$\lim_{n \to \infty} [\Gamma_n(u_0)]^{1/n} = \inf_n [\Gamma_n(u_0)]^{1/n} = e^{\gamma_0}. \quad (3.12)$$

**Proof.** Since $\Gamma_n(u) \geq 0$, the case where $\gamma = -\infty$ is trivial for (3.11). Henceforth, we will assume that $\gamma = e^{\omega}$ is positive. For $\varepsilon$ small enough, the linear operator $T_\varepsilon = (\gamma - \varepsilon)^{-1} T$ is still positive and satisfies the norm continuity. If for some $\varepsilon$ small enough, $T_\varepsilon$ satisfies property $P_0$, then by Proposition 3.3 there exists $n$ large enough so that $(1 - T_\varepsilon^n) \leq 1$. In other words, $\|T^n\| \leq (\gamma - \varepsilon)^n$ for all $n$ large enough. Hence, $\lim_{n \to \infty} \|T^n\|^{1/n} \leq \gamma = e^{\omega} - \varepsilon$, contradicting the fact that $e^{\omega}$ is the spectral radius of $T$. Therefore, for every $\varepsilon$ small enough $T_\varepsilon$ does not satisfy the property $P_0$, that is, there exists $u_\varepsilon$ in $X$ such that

$$\Gamma_{\varepsilon,n}(u_\varepsilon) = \sup_{\xi \in X} (T_\varepsilon^n 1)(u_\varepsilon, \xi) \geq 1 \quad \text{for all } n. \quad (3.13)$$

Using $\Gamma_{\varepsilon,n}(u) = (\gamma - \varepsilon)^{-n} \Gamma_n(u)$, we can conclude from (3.13) that

$$[\Gamma_n(u_\varepsilon)]^{1/n} \geq \gamma - \varepsilon. \quad (3.14)$$

Again, we invoke the compactness of $X$ to ensure the existence of a convergent subnet of $\{u_\varepsilon\}$ converging to $u_0$ in $X$. Passing to the limit as $\varepsilon \to 0$, we see that

$$\Gamma_n(u_0) \geq \gamma = e^{\omega} \quad \text{for all } n. \quad (3.15)$$

Utilizing (3.15), it is now easy to prove (3.12).

$$e^{\gamma} \leq \inf_n [\Gamma_n(u_0)]^{1/n} \leq \lim_{n \to \infty} [\Gamma_n(u_0)]^{1/n} \leq \lim_{n \to \infty} [\Gamma_n(u_0)]^{1/n} \leq \|T^n\|^{1/n} = e^{\gamma_0}. \quad (3.16)$$

Hence, (3.12) now follows easily.

**Corollary 3.6.** There exists an element $u_0$ in $X$ where the local $L$-exponent $\mu(u_0)$ coincides with the global $L$-exponent $\mu$.

**Proof.** Follows directly from the definitions of $\mu$ and $\mu(u_0)$.
Remark 3.7. It is clear from the proof of Proposition 3.5 that the element \( u_0 \) need not be unique, since the net \( \{ u_\varepsilon \} \) may have more than one convergent subnet.

We close this section with a result about the continuity of the \( L \)-exponents with respect to a continuous parameter.

**Proposition 3.8.** If \( T_x \) and \( T \) are positive linear operators satisfying the norm continuity, and if \( T_x \) converges to \( T \) in the strong operator topology on bounded linear operators on \( \mathcal{G}_b(X \times S) \), then

\[
\lim_{x} \mu_x \leq \mu,
\]

(3.16)

where \( \mu_x \) and \( \mu \) are the \( L \)-exponents of \( T_x \) and \( T \), respectively.

**Proof.** By Proposition 3.5, there exists \( u_x \) and \( u_0 \) in \( X \) such that

\[
n^{-1} \log \Gamma_{x,n}(u_x) \geq \mu_x \quad \text{for all } n,
\]

(3.17)

and

\[
n^{-1} \log \Gamma_{n}(u_0) \geq \mu \quad \text{for all } n.
\]

(3.18)

Let us assume that (3.16) does not hold, that is there exists a subnet \( \{ \mu_\beta \} \) of \( \{ \mu_x \} \) such that \( \lim_\beta \mu_\beta > \mu \). Then by the compactness of \( X \), \( \{ u_\beta \} \) has a subnet \( \{ u_\gamma \} \) converging say to \( u \) in \( X \). Since we also have \( T_x^n \Gamma \rightarrow T^n \Gamma \) in \( \mathcal{G}_b(X \times S) \), it follows that

\[
\Gamma_{x,n}(u_x)^{1/n} \rightarrow \Gamma_n(u_0)^{1/n} \quad \text{for every } n.
\]

(3.19)

\[
\lim_{\gamma} e^{\mu_\gamma} \leq \lim_{\gamma} \Gamma_{\gamma,n}(u_\gamma)^{1/n} \leq \Gamma_n(u_0)^{1/n} \leq \| T^n \|^n.
\]

(3.20)

Therefore,

\[
e^{\mu} < \lim_{\beta} e^{\mu_\beta} = \lim_{\gamma} e^{\mu_\gamma} \leq \inf_{n} \| T^n \|^{1/n} = e^\mu;
\]

(3.21)

this contradiction finishes the proof. \( \Box \)

Remark 3.9. In \([EFT]\) a simple example is furnished, showing that one can have strict inequality in (3.16).

4. SEMIFLOWS OF POSITIVE OPERATORS

In this section we develop the notion of \( L \)-exponent for semiflows of positive operators. It is in this framework that we formulate and prove the result (2.12). First we transfer the results from the discrete case, given in
the previous section, to semiflows that satisfy the conditions listed in Proposition 2.4.

**Definition 4.1.** \( \{T_t\}_{t > 0} \) is called a *semiflow of positive operators* on \( \mathcal{G}_b(X \times S) \) if

(a) \( T_t \) is a positive, bounded linear operator on \( \mathcal{G}_b(X \times S) \) for every \( t > 0 \),

(b) \( T_{t+s} f = T_t(T_s f) \) for every \( f \in \mathcal{G}_b(X \times S) \) and \( s, t > 0 \).

**Definition 4.2.** \( \{T_t\}_{t > 0} \) is said to satisfy *norm continuity* if the map \( t \rightarrow \Gamma_t \) is a continuous function from \( \mathbb{R}^+ \) into \( \mathcal{G}(X) \); here \( \Gamma_t \) is defined by

\[
\Gamma_t(u) = \sup_{\xi \in S} (T_t,1)(u, \xi).
\]

Let us remark that under these definitions, Proposition 2.4 now can be reinterpreted as saying: the semiflow of operators defined by (2.13) is a norm continuous semiflow of positive operators on \( \mathcal{G}_b(X \times S) \). In order to prove Proposition 3.5 for semiflows we introduce the \( L \)-exponents of semiflows and also define a property very similar to property \( P_0 \) given in 3.2.

**Definition 4.3.** For a semiflow of positive operators \( \{T_t\}_{t \geq 0} \), the *local* and *global \( L \)-exponents* are defined respectively by

\[
\mu(u) = \lim_{t \to \infty} \frac{1}{t} \log \Gamma_t(u),
\]

and

\[
\mu = \lim_{t \to \infty} \frac{1}{t} \log \|T_t\|.
\]

**Definition 4.4.** A semiflow of positive operators \( \{T_t\}_{t > 0} \) is said to have *property \( (P_0) \)* if for every \( u \) in \( X \) there exists \( t = t(u) \) positive such that \( \Gamma_t(u) < 1 \).

Clearly, if \( T_1 \) has property \( P_0 \) then \( \{T_t\}_{t > 0} \) has property \( (P_0) \). A partial converse is also true.

**Proposition 4.5.** If \( \{T_t\}_{t > 0} \) is a semiflow of positive operators with property \( (P_0) \) then there exists \( t_0 > 0 \) such that for every \( s \in (0, t_0] \), \( T_s \) has property \( P_0 \).

**Proof.** First we write the property \( (P_0) \) more carefully and deduce some simple results.

\[
\forall u \in X, \exists t_u > 0 \text{ such that } \Gamma_{t_u}(u) < \theta_u < 1.
\]
Since $\Gamma_i$ is a continuous function, there exists a neighborhood $N_i$ of $u$, on which $(4.4)$ remains valid. Using the compactness of $X$, we can extract a finite subcover $\{N_i\}_{i=1}^k$ from the open cover $\{N_u\}_{u \in X}$. By relabeling, if it is necessary, we can ensure that $0 < t_1 < t_2 < \cdots < t_k$. Hence

$$\Gamma_i(u) < \theta_i < 1 \quad \text{for} \quad i = 1, 2, \ldots, k. \quad (4.5)$$

Given $\varepsilon$ positive and $\{t_1, t_2, \ldots, t_k\}$ a set of positive real numbers, one can find another positive number $t_0$ and $n, \in$ for $i = 1, 2, \ldots, k$ such that

$$|n_i t_0 - t_i| < \varepsilon \quad \text{for} \quad i = 1, 2, \ldots, k. \quad (4.6)$$

On the other hand, by the continuity of the map $t \to \Gamma_i$, there exists $\varepsilon > 0$ small enough so that for $|s - t| < \varepsilon$

$$\Gamma_i < \theta_i < 1 \implies \Gamma_i \in \mathbb{R}.$$ (4.7)

Now we choose $t_0$ and $n, \in$ to satisfy (4.6), hence by the virtue of (4.7 we conclude that

$$\Gamma_{n_i t_0}(u) < 1 \quad \text{for} \quad u \in N_i, \quad i = 1, 2, \ldots, k. \quad (4.8)$$

In other words, $\Gamma_{n_i t_0}(u) < 1$ for $u \in N_i$; this implies directly that $T_{n_i t_0}$ satisfy the property $P_0$. The same argument carries through for any positive real number $s$ less than $t_0$ by choosing $n, \in$ accordingly. \)

Remark 4.6. If $\{T_i\}$ is a semiflow satisfying the property $(P_0)$ then by Proposition 4.5 and Proposition 3.3 there exists $t_0 > 0$ such that $(n t_0)^{-1} \log \|T_0^n\| < 0$ for $n$ large enough. Consequently, $\lim_{t \to \infty} t^{-1} \log \|T_i\| < 0$.

Remark 4.7. As is the case with the $L$-exponent of a positive linear operator, one can show that

$$\lim_{t \to \infty} t^{-1} \log \|T_t\| = \inf_{t > 0} t^{-1} \log \|T_t\|. \quad (4.9)$$

This fact follows from the subadditivity of the net $\{\log \|T_t\|\}_{t > 0}$ combined with the fact that $\log \|T_t\|$ remains bounded on a compact interval of $t$. Hence we see that

$$\mu = \lim_{t \to \infty} t^{-1} \log \|T_t\| = \inf_{t > 0} t^{-1} \log \|T_t\|. \quad (4.10)$$
The analog of Proposition 3.5 can now be proven.

**Proposition 4.8.** There exists an element \( u_0 \) in \( X \) such that

\[
\log \Gamma_t(u_0) \geq \mu t \quad \text{for} \quad t > 0.
\]

Moreover, the same element \( u_0 \) satisfies

\[
\mu(u_0) = \mu.
\]

**Proof.** As in Proposition 3.5, we can assume that \( \mu > -\infty \), and set \( a = e^\mu \). Then \( \{S_t^\varepsilon\}_{t > 0} \) is also a semiflow of positive operators satisfying the norm continuity, where \( S_t^\varepsilon = (a - \varepsilon)^{-1} T_t \). If \( \{S_t^\varepsilon\}_{t > 0} \) satisfies the property \( (P_0) \) for small \( \varepsilon \), then by Proposition 4.5 there exists a positive \( t_0 \) such that \( S_t^{\varepsilon_0} \) has property \( P_0 \). Hence by Remark 4.6

\[
\lim_{t \to \infty} t^{-1} \log \|S_t^\varepsilon\| < 0,
\]

that is,

\[
\lim_{t \to \infty} t^{-1} \log \|T_t\| < \log(a - \varepsilon),
\]

contradicting the fact that the left hand side of the inequality is equal to \( \mu = \log a \). Therefore, for all positive \( \varepsilon \) small enough, there exists \( u_\varepsilon \) in \( X \) such that

\[
1 \leq \Gamma_{t,\varepsilon}(u_\varepsilon) = (a - \varepsilon)^{-1} \Gamma_t(u_\varepsilon) \quad \text{for} \quad t > 0.
\]

Again invoking the compactness of \( X \), we extract a convergent subnet of \( \{u_\varepsilon\} \), converging say to \( u_0 \) in \( X \). Passing to the limit as \( \varepsilon \to 0 \), we conclude that

\[
\Gamma_t(u_0) \geq e^\mu t \quad \text{for} \quad t > 0.
\]

Consequently,

\[
\mu \leq \inf_{t > 0} t^{-1} \log \Gamma_t(u_0) \leq \lim_{t \to \infty} t^{-1} \log \Gamma_t(u_0)
\]

\[
\leq \lim_{t \to \infty} t^{-1} \log \Gamma_t(u_0) = \lim_{t \to \infty} t^{-1} \log \|T_t\| = \mu,
\]

which implies that

\[
\mu = \lim_{t \to \infty} t^{-1} \log \Gamma_t(u_0) - \inf_{t > 0} t^{-1} \log \Gamma_t(u_0). \quad \blacksquare
\]
COROLLARY 4.9. Let \( w_N(t, u_0) \) as defined as in (2.9); then there exists \( u_0 = u_0(N) \) in \( X \) such that
\[
 t^{-1} \log w_N(t, u_0) \leq (\mu_1 + \mu_2 + \cdots + \mu_N)(u_0) \quad \text{for } t > 0.
\] (4.19)

Moreover, \( u_0 \) satisfies
\[
 (\mu_1 + \mu_2 + \cdots + \mu_N)(u_0) = \mu_1 + \mu_2 + \cdots + \mu_N.
\] (4.20)

Proof. First we remark that if \( \{ T_t \}_{t \geq 0} \) is the semiflow defined by (2.13) then
\[
 \Gamma_t(u_0) = w_k(t, u_0)
\] (4.21)
as noted before, thence Proposition 4.8 applied to this setting gives the desired result. \[ \blacksquare \]

Corollary 4.9 was the result we were mentioning from the very beginning; now we introduce a more complex class of semiflows that we will be needing when estimating the Hausdorff dimension. First, we list some basic properties of local and global Lyapunov exponents.

DEFINITION 4.10. Let \( \bar{s}_N(t) = \sup_{u_0 \in X} s_N(t, u_0) \); then the \( N \)th upper Lyapunov exponent \( \bar{\mu}_N \) is defined as
\[
 \bar{\mu}_N = \lim_{t \to \infty} t^{-1} \log \bar{s}_N(t).
\]

PROPOSITION 4.11. Let \( \mu_k(u) \), \( \mu_k \), and \( \bar{\mu}_k \) denote the \( k \)th local, global, and upper Lyapunov exponents, respectively. Then
\begin{enumerate}
\item \( \mu_N \leq \cdots \leq \mu_2 \leq \mu_1 = \bar{\mu}_1. \)
\item \( \mu_N \leq \bar{\mu}_N \leq N^{-1}(\mu_1 + \mu_2 + \cdots + \mu_N), \)
\item \( \mu_N \leq \sup_{u_0 \in X} \mu_N(u_0) \leq \bar{\mu}_N. \)
\end{enumerate}

Proof. (a) and (b) follow directly from the fact that s-numbers are decreasing. To prove (c), we first write
\[
 w_N(t, u) = w_{N-1}(t, u) \cdot s_N(t, u),
\]
and deduce that
\[
 (\mu_1 + \cdots + \mu_N)(u) = \lim_{t \to \infty} t^{-1} \log w_N(t, u)
\]
\[
 \leq \lim_{t \to \infty} t^{-1} \log w_{N-1}(t, u) + \lim_{t \to \infty} t^{-1} \log \bar{s}_N(t)
\]
\[
 \leq (\mu_1 + \cdots + \mu_{N-1})(u) + \bar{\mu}_N.
\]
Hence, \( \mu_N(u) \leq \bar{\mu}_N \) for any \( u \) in \( X \). On the other hand, if \( u_0 \) is an element satisfying (4.20) then the fact \((\mu_1 + \cdots + \mu_{N+1})(u_0) \leq \mu_1 + \cdots + \mu_{N+1} \) implies that \( \mu_N \leq \hat{\mu}_N(u_0) \).

It is possible to define semiflows to deduce more general relations between local and global Lyapunov [E]. Here we will be satisfied with the simplest among them. Let \( k \in \mathbb{N}, \alpha \in (0, 1) \) be given, and set \( k' = k + 1 \). For \( f \in \mathcal{C}_b(X \times S_k \times S_{k'}) \) we define

\[
(T_t f)(u, \xi, \xi') = \|S'(t, u)\wedge k\xi\|^\xi \|S'(t, u)\wedge k'\xi'\|^{1-\alpha} \cdot f \left( \frac{S'(t, u)\wedge k\xi'}{|S'(t, u)\wedge k\xi'|}, \frac{S'(t, u)\wedge k'\xi'}{|S'(t, u)\wedge k'\xi'|} \right). \tag{4.22}
\]

**Proposition 4.12.** \( \{T_t\}_{t \geq 0} \) as defined in (4.22) is a semiflow of positive operators on \( \mathcal{C}_b(X \times (S_k \times S_{k'})) \) satisfying the norm continuity condition.

**Proof.** The proof runs along the same lines as in the proof of Proposition 2.4. We only point out that

\[
\Gamma_t(u) = \|S'(t, u)\wedge k\|^\xi \|S'(t, u)\wedge k'\|^{1-\alpha} \tag{4.23}
\]

from which the norm continuity follows easily. (See [E] for more details.)

Before applying the Proposition 4.8 once again, let us note that by (4.23) and (2.9)

\[
\Gamma_t(u) = w_k(t, u) s_{k+1}^{1-\alpha}(t, u) = w_k(t, u) s_{k+1}^{1-\alpha}(t, u). \tag{4.24}
\]

**Corollary 4.13.** There exists \( u_0 \) in \( X \) such that

\[
\lim_{t \to \infty} t^{-1} \log w_k(t, u_0) s_{k+1}^{1-\alpha}(t, u_0) = \lim_{t \to \infty} t^{-1} \log \sup_{u \in X} w_k(t, u) s_{k+1}^{1-\alpha}(t, u). \tag{4.25}
\]

5. **Douady–Oesterlé Dimension of an Invariant Set**

From now on, we assume that \( N \) is the first integer such that \((\mu_1 + \mu_2 + \cdots + \mu_{N+1})(u_0) < 0 \) for all \( u_0 \) in \( X \), that is, \( \mu_1 + \mu_2 + \cdots + \mu_{N+1} < 0 \).
DEFINITION 5.1. The Duoady–Oesterlé dimension of $X$, denoted by $\mathbf{DO}$-dimension, is defined as

$$d_0(X) = \inf\{D > 0 : (\sup_{u \in X} w_N(t, u) \cdot s_{N+1}^D(t, u)) \text{ converges to zero exponentially as } t \to \infty \}. \quad (5.1)$$

Similarly the local $\mathbf{DO}$-dimension at $u_0$ is defined by

$$d_0(u_0) = \inf\{D > 0 : w_N(t, u_0) s_{N+1}^D(t, u_0) \text{ converges to zero exponentially as } t \to \infty \}. \quad (5.2)$$

Notation 5.2. We set

$$\beta(t, u_0, D) = w_N(t, u_0) \cdot s_{N+1}^D(t, u_0), \quad (5.3)$$

and

$$\beta(t, D) = \sup_{u \in X} \beta(t, u, D). \quad (5.4)$$

Using Corollary 4.13, we relate the local and global $\mathbf{DO}$-dimensions.

**PROPOSITION 5.3.** (a) $N \leq \sup_{u_0 \in X} d_0(u_0) \leq d_0(X) \leq N + 1$,

(b) there exists $u_0$ in $X$ such that

$$d_0(u_0) = d_0(X).$$

*Proof.* (a) $d_0(X) \leq N + 1$ follows directly from the assumption $\mu_1 + \mu_2 + \ldots + \mu_{N+1} < 0$. Also from the same assumption, one can deduce that $\mu_1 + \ldots + \mu_N \geq 0$. Hence for the critical $u_0 = u_0(N)$,

$$t^{-1} \log w_N(t, u_0) \geq \mu_1 + \ldots + \mu_N \geq 0 \quad \text{for} \quad t > 0,$$

that is, $w_N(t, u_0) \geq 1$ for $t > 0$. Clearly, $s_{N+1}(t, u_0) < 1$; hence, for $D < N$, $\beta(t, u_0, D)$ converge to infinity as $t \to \infty$. Therefore $N \leq d_0(u_0)$. The second inequality follows directly from (5.3) and (5.4).

(b) Let $\alpha = N + 1 - D$; then for the semiflow defined in (4.22), and using (4.24)

$$\Gamma_t(u_0) = \beta(t, u_0, D) \quad \text{and} \quad \|T_t\| = \beta(t, D). \quad (5.5)$$

Then using the Proposition 4.8, we deduce the existence of $u_0$ in $X$ such that

$$\lim_{t \to \infty} t^{-1} \log \beta(t, u_0, D) = \lim_{t \to \infty} t^{-1} \log \beta(t, D). \quad (5.6)$$
If $D > d_0(u_0)$ then the left hand side of (5.6) is negative; hence $\beta(t, D)$ converges to zero exponentially, that is, $D > d_0(X)$. Since $D$ was arbitrary it follows that $d_0(X) \leq d_0(u_0)$. The other inequality is obvious from (a). 

The DO-dimensions are related to local and global Lyapunov dimensions already defined in [EFT]. We first recall the basic definitions.

**Definition 5.4.** Let $N$ be the first integer such that $\mu_1 + \mu_2 + \cdots + \mu_{N+1} < 0$; then the local and global Lyapunov dimensions are defined respectively by

\[
d_L(u_0) = N + \frac{(\mu_1 + \mu_2 + \cdots + \mu_N)(u_0)}{|(\mu_{N+1}(u_0)|},
\]

and

\[
d_L(X) = N + \frac{\mu_1 + \mu_2 + \cdots + \mu_N}{|\mu_{N+1}|}.
\]

**Proposition 5.5.** For every $u_0$ in $X$, $d_0(u_0) \leq d_L(u_0)$.

**Proof.** Let $D > d_L(u_0)$; then $(D - N)|\mu_{N+1}(u_0)| > (\mu_1 + \cdots + \mu_N)(u_0)$. Using the fact that $\mu_{N+1}(u_0) \leq \tilde{\mu}_{N+1} \leq N^{-1}(\mu_1 + \cdots + \mu_{N+1}) < 0$ we can rewrite the inequality as

\[
(D - N)(\mu_1 + \cdots + \mu_N)(u_0) + (D - N)(\mu_1 + \cdots + \mu_{N+1})(u_0) < 0. \tag{5.9}
\]

On the other hand,

\[
\lim_{t \to \infty} t^{-1} \log w_N(t, u_0) s^{D - N}_{N - 1}(t, u_0) \\
\leq \lim_{t \to \infty} t^{-1} \log w_N^{N+1-D}(t, u_0) + \lim_{t \to \infty} t^{-1} \log w_N^{D - N}(t, u_0) < 0; \tag{5.10}
\]

therefore, from the definition of $d_0(u_0)$, we conclude that $D > d_0(u_0)$, hence $d_L(u_0) \geq d_0(u_0)$. 

**Corollary 5.6.** $d_0(X) = \max_{u_0 \in X} d_0(u_0) \leq \sup_{u_0 \in X} d_L(u_0) \leq d_L(X)$.

**Proof.** The first equality follows from Proposition 5.3, whereas the second inequality is a direct consequence of Proposition 5.5. Finally, the last inequality, though it is a little involved, follows from definitions [E].
6. A LOCAL ESTIMATE FOR THE HAUSDORFF DIMENSION OF AN ATTRACTOR

If $S'(t, u_0)$ is a compact linear operator, then the existence of a critical element $u_0$ in $X$ that satisfies

$$d_H(X) \leq d_L(u_0) \quad (6.1)$$

was shown in [EFT], utilizing the idea of iterated coverings [DO]. Our next aim is to show the existence of such a critical element when $S'(t, u_0)$ is the sum of a compact operator with a strict contraction. We will mainly follow the proof given in [EFT], modifying it at the beginning and at the end.

**DEFINITION 6.1.** Let $X$ be a compact subset of a metric space; then the Hausdorff dimension of $X$ is defined to be

$$d_H(X) = \inf \{ d > 0 : \mu_d(X) = 0 \}.$$  

where

$$\mu_d(X) \equiv \sup_{\delta > 0} \mu_{d, \delta}(X) = \lim_{\delta \to 0^+} \mu_{d, \delta}(X),$$

and

$$\mu_{d, \delta}(X) \equiv \inf \left\{ \sum_{i=1}^{k} r_i^d : r_i \leq \delta, X \subseteq \bigcup_{i=1}^{k} B(u_i, r_i) \right\}.$$  

The following proposition gives a local estimate of the Hausdorff dimension.

**PROPOSITION 6.2.** If $N$ is the integer such that $\mu_1 + \cdots + \mu_{N+1} < 0$ then there exists $u_0$ in $X$ such that

(a) $d_H(X) \leq d_0(X) \leq d_0(u_0)$, and  
(b) $d_H(X) \leq d_L(u_0)$.

**Proof:** Once (a) is shown, (b) will follow directly from Proposition 5.5. Recall that we are working with an operator $S'(t, u_0)$ acting on a Hilbert space. We fix $t$ and $u_0$ for the moment, and set $S' = S'(t, u_0)$; then $H$ can be decomposed into two orthogonal subspaces $E$ and $F$ such that $E$ consists of the eigenvectors of the positive part of $S'$ corresponding to the eigenvalues $< s_{N+1}$ and $F$ is its orthogonal complement $[T, E]$. Then the following results are standard:
HAUSDORFF DIMENSION OF AN ATTRACTOR

(6.2)

\[ S'|_{F} \text{ has norm less than } s_{N+1}, \]

\( E \) is spanned by \( e_1, e_2, \ldots, e_N \), where \( S'e_i \perp S'e_j \) for \( i \neq j \). Moreover, both \( E \) and \( F \) are invariant under \( S' \).

(6.3)

Hence, given \( h \) in \( H \),

\[ S'h = \sum_{i=1}^{N} (h, e_i) S'e_i + S'f \quad \text{with } f \in F. \]  

(6.4)

Letting \( h = u - u_0 \), (2.4) can be transformed into

\[ \begin{align*}
|S(t)u - S(t)u_0 - \sum_{i=1}^{N} (u - u_0, e_i) S'e_i| \\
\leq |S'f| + c(t) o(|u - u_0|) \\
\leq s_{N+1}|u - u_0| + c(t) o(|u - u_0|),
\end{align*} \]  

(6.5)

where the second inequality followed from (6.2) and \(|f| \leq |u - u_0|\). We set

\[ \mathcal{E}_r = \left\{ S(t)u_0 + \sum_{i=1}^{N} (h, e_i) S'e_i : h \in H, |h| \leq r \right\}. \]  

(6.6)

A standard covering lemma induces the following estimate [EFT]:

\[ m \]  
\[ \mu r/2 \]  
that is necessary to cover \( \mathcal{E}_r \)

\[ \leq c_N(s_1, s_2, \ldots, s_N) \mu^{-N} = \beta, \]  

(6.7)

where \( \mu = s_{N+1} < s_N \), and \( c_N \) is a constant depending only on \( N \). Note that by (6.5)

\[ \text{dist}_H(S(t)u, \mathcal{E}_r) \leq \left( s_{N+1} + c(t) \frac{o(r)}{r} \right) r \]  

(6.8)

for \(|u - u_0| < r\). Therefore, the minimum number of balls of radius \( \leq (\mu/2 + s_{N+1} + c(t)o(r)/r)r \) that is necessary to cover \( S(t)(B_r \cap X) \) is less than \( \beta \), with \( \beta \) as defined in (6.7).

After these preliminary remarks we return to the proof. In order to show that \( D > d_0(X) \) implies \( \mu_D(X) = 0 \), we proceed by way of contradiction. From the definition of \( \mu_D \), if \( \mu_D(X) > 0 \) then there exists \( \delta_0 > 0 \) such that \( \mu_{D, \delta}(X) > 0 \) for \( \delta < \delta_0 \). Therefore, there exists a cover of \( X \) with balls of radius less than \( \delta \) such that

\[ \sum_{k} r_k^D < 2 \mu_{D, \delta}(X), \]

(6.9)
where $r_k \leq \delta$ is the radius of the ball $B_k$ with center $u_k$. Since $\{B_k\}$ cover $X$, and $S(t)X = X$, we also conclude that

$$X \subseteq \bigcup_k S(t)(X \cap B_k). \tag{6.10}$$

We now cover the right hand side of (6.10) will balls of radius $\varkappa_k \delta$, for some $\varkappa_k < 1$, for each $B_k$. Consequently,

$$\mu_{d,x\delta}(X) \leq \sum_k \mu_{d,x\delta}(S(t)(X \cap B_k)) \leq \sum_k \beta_k(\varkappa_k r_k)^D, \tag{6.11}$$

where $\varkappa \geq \varkappa_k$ for every $k$, $\beta_k = c_N w_N(t, u_k) s_{-N+1}(t, u_k)$ and

$$\varkappa_k = \frac{1}{2} s_{-N+1}(t, u_k) + s_{N+1}(t, u_k) + c(t) \frac{o(r_k)}{r_k}. \tag{6.12}$$

Now we set $\varkappa = 5s_{-N+1}(t)$ and choose $t$ large enough so that $\varkappa < 1$, which is guaranteed by the fact $\bar{\mu}_{-N+1} < 0$. Moreover, since $D > d_0(X)$ by the definition of $d_0(X)$, $\beta(t, D)$ goes to zero exponentially; hence, by choosing $t$ even larger we can ensure that $25^D \beta(t, D) < 1$. Next we choose $\delta > 0$ small enough so that $\max\{\delta, c(t) (\delta)/\delta\} < s_{N+1}(t, u_0)$ for every $u_0$. With these choices (6.11) reduces to

$$\mu_{D,x\delta}(X) \leq \sum_k r_k^D (5s_{-N+1}(t, u_k)^D \beta_k) \leq 5^D \beta(t, D) \sum_k r_k^D \leq \frac{1}{2} \sum_k r_k^D < \mu_{D,x\delta}(X) \quad \text{by (6.9).} \tag{6.13}$$

But this contradicts the fact that $\mu_{D,x\delta}(X)$ is a decreasing function of $\delta$. ■

7. AN EXAMPLE LORENZ TYPE ODE’s

We consider a system of ODE’s very similar to the one considered by Lorenz, namely,

$$\begin{align*}
\dot{x} &= -\sigma x + \sigma y \\
\dot{y} &= -y - xz \\
\dot{z} &= -bz + xy - br,
\end{align*} \tag{7.1}$$

where we fix the value of $b$ as 2, and $\sigma > 2$, $r > 1$ are allowed to take any values. Note that $u_0 = (0, 0, -r)$ is a stationary solution of (7.1); we claim that for this $u_0$, $d_0(u_0) = d_0(X)$. This system is already considered in [EFT] and a local estimate of Hausdorff dimension is found, specially when $b = \frac{\sigma}{2}$.
\( \sigma = 10, \ r = 28 \). Here our aim is to exhibit another aspect of the computations done in \([EFT]\). We recall the two basic estimates we will use:

\[
\begin{align*}
\omega_3(t, u) &= \exp\{-(\sigma + 3)t\}, \\
\omega_2(t, u) &\leq \exp\{\alpha_1 t\},
\end{align*}
\]

where

\[
\alpha_1 = \frac{1}{2} \{-(\sigma + 5) + \sqrt{(\sigma - 1)^2 + 4\sigma r}\}.
\]

Also in the same analysis it is shown that for \( u_0 = (0, 0, -r) \),

\[
\omega_2(t, u_0) = \exp\{\alpha_1 t\}.
\]

Hence, it easily follows that

\[
\beta(t, u_0, D) = \beta(t, D) = \exp\{\alpha_1 (3 - D) - (\sigma + 3)(D - 2)\} t,
\]

where we took \( N = 2 \) and used the fact that \( \omega_2(t, u) \leq \omega_2(t, u_0) \) for every \( u \) in \( X \). Clearly, this implies that \( d_0(X) = d_0(u_0) \). Moreover, by Propositions 5.5 and 6.2,

\[
d_H(X) \leq d_0(u_0) \leq d_L(u_0).
\]

Let us note that (7.6) already implies

\[
d_H(X) \leq \frac{3\alpha_1 + 2(\sigma + 3)}{\alpha_1 + \sigma + 3} d_0(u_0).
\]

**REFERENCES**


