A tensor product theorem related to perfect crystals

Masato Okado, a Anne Schilling, b and Mark Shimozono c

a Department of Informatics and Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan
b Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616-8633, USA
c Department of Mathematics, 460 McBryde Hall, Virginia Tech, Blacksburg, VA 24061-0123, USA

Received 27 April 2002
Communicated by Masaki Kashiwara

Abstract

Kang et al. provided a path realization of the crystal graph of a highest weight module over a quantum affine algebra, as certain semi-infinite tensor products of a single perfect crystal. In this paper, this result is generalized to give a realization of the tensor product of several highest weight modules. The underlying building blocks of the paths are finite tensor products of several perfect crystals. The motivation for this work is an interpretation of fermionic formulas, which arise from the combinatorics of Bethe Ansatz studies of solvable lattice models, as branching functions of affine Lie algebras. It is shown that the conditions for the tensor product theorem are satisfied for coherent families of crystals previously studied by Kang, Kashiwara and Misra, and the coherent family of crystals \( \{ B_{k,l}^{i} \}_{l \geq 1} \) of type \( A^{(1)}_n \).

© 2003 Elsevier Inc. All rights reserved.

MSC: Primary 17B67, 17B37, 81R10; Secondary 05E10, 82B23

1. Introduction

In the seminal paper [KMN1], the crystal graph \( B(\lambda) \) of the irreducible integrable module of highest weight \( \lambda \) over a quantum affine algebra \( U_a(g) \), is realized as a subset \( P(p^{(i)}, B) \) of the semi-infinite tensor product of copies of the crystal graph \( B \) of a single finite-dimensional \( U_a(g) \)-module, where \( U_a(g) \) is the subalgebra of \( U_a(g) \) corresponding to the derived subalgebra \( g^0 \) of \( g \). The suitable finite crystals \( B \) are called perfect and the
elements of $\mathcal{P}(p^{(\lambda)}, B)$ are called paths. The path realization of $B(\lambda)$ is particularly well-suited to the study of branching functions for the coset $g/g_I\{0\}$, where $g_I\{0\}$ is the simple Lie subalgebra of $g$ whose Dynkin diagram is obtained from that of $g$ by removing the vertex labeled 0 in [Kac].

The purpose of this paper is to give a similar realization of a finite tensor product of such highest weight modules. Let $B_i (i = 1, 2, \ldots, m)$ be a perfect crystal of level $l_i$ of a quantum affine algebra $U_q(g)$ such that $l_1 \geq l_2 \geq \cdots \geq l_m \geq l_{m+1} = 0$, and $\lambda_i$ a dominant integral weight of level $l_i - l_{i+1}$. We construct a set of paths $\mathcal{P}(p^{(\lambda_1, \ldots, \lambda_m)}, B_1 \otimes \cdots \otimes B_m)$ with the reference path $p^{(\lambda_1, \ldots, \lambda_m)}$ (Section 3.3). For technical reasons we also require assumptions (A1) and (A2) (Section 4.1). These assumptions are satisfied by many perfect crystals known so far, such as $B_i$ for any nonexceptional affine Lie algebra $g$ given in [KKM] (Example 4.4) and $B^{k,l}$ for $g = A_n$ given in [KMN2] (Example 4.5). With this setup we have the following isomorphism of crystals (Theorem 4.13)

$$\mathcal{P}(p^{(\lambda_1, \ldots, \lambda_m)}, B_1 \otimes \cdots \otimes B_m) \cong B(\lambda_1) \otimes \cdots \otimes B(\lambda_m).$$

(1.1)

The main result of [KMN1] is the case $m = 1$. A special case of $m = 2$ for $g = A_n^{(1)}$ is treated in [HKKOT].

Our motivation for (1.1) comes from a conjecture related to fermionic formulas given in [HKOTT]. Fermionic formulas are certain polynomials which originate from the combinatorics of the Bethe Ansatz in solvable lattice models. They are related to the representation theory of affine Lie algebras by a conjecture in [HKOTT] which states that they are equal to one-dimensional sums defined in terms of a conjectural family of crystals of finite-dimensional $U'_q(g)$-modules. If this series of conjectures is shown to be true, we see from (1.1) that fermionic formulas give truncated branching functions for the coset $g/g_I\{0\}$ provided that the relevant crystals are all perfect. For type $A_n^{(1)}$ the desired family of crystals exists and all crystals therein are perfect [KMN2], and the equality between the one-dimensional sum and the corresponding fermionic formula was established in [KSS]. Therefore, the conjectures mentioned above are all settled in this case.

To extend the result of [KMN1] to the case of several tensor factors, we employ some substantial results on affine crystals. The first is the path realization of the crystal graph of the lower triangular part $U_q(g)_{\text{\scriptsize{\,lower}}}^+ U_q(g)$ given in [KKM] using coherent families of perfect crystals. The second is the theory of crystals with core [KK]. Given a perfect crystal $B$ in a coherent family and $B(\lambda)$, the crystal $B \otimes B(\lambda)$ is a crystal with core, and there is a perfect crystal $B'$ and a weight $\lambda'$ such that $B \otimes B(\lambda) \cong B(\lambda') \otimes B'$. This result allows one to exchange a finite crystal past a highest weight crystal. Given the above results, finding the correct reference path and path space, more or less reduces to a computation involving the basic isomorphisms of [KMN1] and [KK] together with combinatorial $R$-matrices, which take the form $B \otimes B' \cong B' \otimes B$ where $B$ and $B'$ are perfect.

However, the resulting isomorphism (1.1) so obtained, is only known to preserve weights up to the null root $\delta$. As in [KMN1] one defines an energy function

$$E : \mathcal{P}(p^{(\lambda_1, \ldots, \lambda_m)}, B_1 \otimes \cdots \otimes B_m) \rightarrow \mathbb{Z}.$$
which determines the multiple of $\delta$ that makes (1.1) weight-preserving. However, unlike the situation in [KMN1], our paths are inhomogeneous in the sense that they have several different perfect crystals as tensor factors. For inhomogeneous paths the evaluation of the energy function requires the explicit computation of combinatorial $R$-matrices, which are the identity map in the homogeneous case. On minimal elements (defined in Section 2.3), it appears that the combinatorial $R$-matrices can be computed explicitly using automorphisms acting on perfect crystals. For type $A^{(1)}_n$ this is proved in [SS]. For general type we must assume that such a result holds for the perfect crystals being used. Several new and important observations regarding the value of the energy function (in particular, its value on minimal elements) are required to complete the proof of (1.1).

The plan of the paper is as follows. In Section 2 we prepare necessary notation and review crystals. The path space $\mathcal{P}(p^{(\lambda_1, \ldots, \lambda_m)}, B_1 \otimes \cdots \otimes B_m)$ is defined in Section 3. In Section 4 the required assumptions are explained and the main theorem proved. Section 5 is devoted to the demonstration that the series of crystals $\{B^{k,l}\}_{l \geq 1}$ of type $A^{(1)}_n$ given in [KMN2] forms a coherent family of perfect crystals for any fixed $k$. This fact is necessary to prove the main theorem for type $A^{(1)}_n$. We attach Appendix A for proofs of formulas of $\tilde{e}_0$, $\tilde{f}_0$ used in Section 5.

2. Crystals

2.1. Notation

Let $g$ be an affine Lie algebra and $I$ the index set of its Dynkin diagram. Note that 0 is included in $I$. Let $\alpha_i, h_i, \Lambda_i$ ($i \in I$) be the simple roots, simple coroots, and fundamental weights for $g$. Let $\delta = \sum_{i \in I} a_i \alpha_i$ denote the standard null root and $c = \sum_{i \in I} a^\vee_i h_i$ the canonical central element, where $a_i, a^\vee_i$ are the positive integers given in [Kac]. Let $P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta$ be the weight lattice and $P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i \oplus \mathbb{Z} \delta$ the dominant weights.

Let $U_q(g)$ be the quantum affine algebra associated to $g$. For the definition of $U_q(g)$ and its Hopf algebra structure, see, e.g., Section 2.1 of [KMN1]. For $J \subset I$ we denote by $U_q(g_J)$ the subalgebra of $U_q(g)$ generated by $e_i, f_i, t_i$ ($i \in J$). In particular, $U_q(g_{\{0\}})$ is identified with the quantized enveloping algebra for the simple Lie algebra whose Dynkin diagram is obtained by deleting the 0th vertex from that of $g$. We also consider the quantum affine algebra without derivation $U_q'(g)$. The weight lattice of $g'$ is called the classical weight lattice $P_{cl} = P/\mathbb{Z} \delta$. We canonically identify $P_{cl}$ with $\bigoplus_{i \in I} \mathbb{Z} \Lambda_i \subset P$. For the precise treatment, see Section 3.1 of [KMN1]. We further define the following subsets of $P_{cl}$: $P_{cl}^0 = \{ \lambda \in P_{cl} \mid \langle \lambda, c \rangle = 0 \}$, $P_{cl}^+ = \{ \lambda \in P_{cl} \mid \langle \lambda, h_i \rangle \geq 0 \ \text{for any} \ i \}$, $(P_{cl}^+)_l = \{ \lambda \in P_{cl}^+ \mid \langle \lambda, c \rangle = l \}$. For $\lambda, \mu \in P_{cl}$, we write $\lambda \geq \mu$ to mean $\lambda - \mu \in P_{cl}^+$.

2.2. Crystals and crystal bases

We summarize necessary facts in crystal theory. Our basic references are [K1], [KMN1] and [AK].
A crystal $B$ is a set $B = \bigsqcup_{\lambda \in P} B_{\lambda}$ (wt $b = \lambda$ if $b \in B_{\lambda}$) with the maps

$$
\tilde{e}_i : B_{\lambda} \rightarrow B_{\lambda + \alpha_i} \sqcup \{0\}, \quad \tilde{f}_i : B_{\lambda} \rightarrow B_{\lambda - \alpha_i} \sqcup \{0\},
$$

$$
\varepsilon_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}, \quad \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}
$$

for all $i \in I$ such that

$$
\begin{align*}
\text{for } b \in B_{\lambda}, & \quad \varphi_i(b) = \langle h_i, \lambda \rangle + \varepsilon_i(b), \quad (2.1) \\
\text{for } b \in B, & \quad \varepsilon_i(b) = \varphi_i(b) = -\infty \implies \tilde{e}_i b = \tilde{f}_i b = 0. \quad (2.4)
\end{align*}
$$

A crystal $B$ can be regarded as a colored oriented graph by defining

$$
b \xrightarrow{i} b' \iff \tilde{f}_i b = b'.
$$

If we want to emphasize $I$, $B$ is called an $I$-crystal.

Important examples of crystals are given by the crystal bases of integrable $U_q(g)$ (or $U'_q(g)$)-modules. They satisfy

$$
\text{for any } b \text{ and } i, \text{ there exists } n > 0 \text{ such that } \tilde{e}_i^n b = \tilde{f}_i^n b = 0.
$$

In such cases the maps $\varepsilon_i, \varphi_i$ are given by

$$
\begin{align*}
\varepsilon_i(b) &= \max\{n \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^n b \neq 0\}.
\end{align*}
$$

We also set

$$
\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b) \Lambda_i, \quad \varphi(b) = \sum_{i \in I} \varphi_i(b) \Lambda_i.
$$

If $B$ is $P_\Delta$-weighted, i.e., wt $b \in P_\Delta$ for any $b \in B$, (2.1) is equivalent to $\varphi(b) - \varepsilon(b) = \text{wt } b$.

For two crystals $B_1$ and $B_2$ a morphism of crystals from $B_1$ to $B_2$ is a map $\psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ such that

$$
\begin{align*}
\psi(0) &= 0, \quad (2.5) \\
\psi(\tilde{e}_i b) &= \tilde{e}_i \psi(b) \quad \text{for } b, \tilde{e}_i b \in B_1 \quad \text{and} \quad \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \quad \text{for } b, \tilde{f}_i b \in B_1. \quad (2.6)
\end{align*}
$$
for $b \in B_1$,
\[ \varepsilon_i(b) = \varepsilon_i(\psi(b)), \quad \varphi_i(b) = \varphi_i(\psi(b)) \text{ if } \psi(b) \in B_2. \] (2.7)

for $b \in B_1$,
\[ \text{wt } b = \text{wt } \psi(b) \text{ if } \psi(b) \in B_2. \] (2.8)

A morphism of crystals $\psi : B_1 \to B_2$ is called an embedding if $\psi$ is injective.

For two crystals $B_1$ and $B_2$, the tensor product $B_1 \otimes B_2$ is defined.

\[ B_1 \otimes B_2 = \{ b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2 \}. \]

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by
\[ \tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \] (2.9)
\[ \tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2). \end{cases} \] (2.10)

Here $0 \otimes b$ and $b \otimes 0$ are understood to be $0$. $\varepsilon_i, \varphi_i$ and wt are given by
\[ \varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)), \] (2.11)
\[ \varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)), \] (2.12)
\[ \text{wt } (b_1 \otimes b_2) = \text{wt } b_1 + \text{wt } b_2. \] (2.13)

With this tensor product operation, $I$-crystals form a tensor category.

The following crystal $T_\lambda$ will be used later.

**Example 2.1.** For $\lambda \in P_{\text{cl}}$ consider the set $T_\lambda = \{ t_\lambda \}$ with one element. Set $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$ for $i \in I$ and $\text{wt } t_\lambda = \lambda$. Given a crystal $B$ one can take the tensor product
\[ T_\lambda \otimes B \otimes T_\mu = \{ t_\lambda \otimes b \otimes t_\mu \mid b \in B \}. \]

Then we have
\[ \text{wt}(t_\lambda \otimes b \otimes t_\mu) = \lambda + \mu + \text{wt } b, \]
\[ \tilde{e}_i(t_\lambda \otimes b \otimes t_\mu) = t_\lambda \otimes \tilde{e}_i b \otimes t_\mu, \quad \varepsilon_i(t_\lambda \otimes b \otimes t_\mu) = \varepsilon_i(b) - (h_i, \lambda), \]
\[ \tilde{f}_i(t_\lambda \otimes b \otimes t_\mu) = t_\lambda \otimes \tilde{f}_i b \otimes t_\mu, \quad \varphi_i(t_\lambda \otimes b \otimes t_\mu) = \varphi_i(b) + (h_i, \mu). \]

**Definition 2.2 ([AK]).** We say a $P$ (or $P_{\text{cl}}$)-weighted crystal is regular, if for any $i, j \in I$ ($i \neq j$), $B$ regarded as $[i, j]$-crystal is a disjoint union of crystals of integrable highest weight modules over $U_q(\mathfrak{g}(i, j))$. 
Let $V(\lambda)$ be the integrable highest weight $U_q(g)$-module with highest weight $\lambda \in P^+$ and highest weight vector $u_\lambda$. It is shown in [K1] that $V(\lambda)$ has a crystal base $(L(\lambda), B(\lambda))$. We regard $u_\lambda$ as an element of $B(\lambda)$ as well. $B(\lambda)$ is a regular $P$-weighted crystal. A finite-dimensional integrable $U'_q(g)$-module $V$ does not necessarily have a crystal base. If $V$ has a crystal base $(L, B)$, then $B$ is a regular $P^0_{\text{cl}}$-weighted crystal with finitely many elements.

Let $W$ be the affine Weyl group associated to $g$, and $s_i$ be the simple reflection corresponding to $\alpha_i$. $W$ acts on any regular crystal $B$ [K2]. The action is given by

$$S_{si}b = \begin{cases} \tilde{e}_{i}\langle h_i, wt b \rangle b & \text{if } \langle h_i, wt b \rangle \geq 0, \\ \tilde{f}_{i}\langle h_i, wt b \rangle b & \text{if } \langle h_i, wt b \rangle \leq 0. \end{cases}$$

An element $b$ of $B$ is called $i$-extremal if $\tilde{e}_ib = 0$ or $\tilde{f}_ib = 0$. $b$ is called extremal if $S_w b$ is $i$-extremal for any $w \in W$ and $i \in I$.

**Definition 2.3** ([AK] Definition 1.7). Let $B$ be a regular $P^0_{\text{cl}}$-weighted crystal with finitely many elements. We say $B$ is simple if it satisfies

1. There exists $\lambda \in P^0_{\text{cl}}$ such that the weights of $B$ are in the convex hull of $W\lambda$.
2. $\#B_{\lambda} = 1$.
3. The weight of any extremal element is in $W\lambda$.

Acting by the Weyl group of the canonical simple Lie subalgebra, one may choose the above weight $\lambda$ such that

$$\langle h_i, \lambda \rangle \geq 0 \quad \text{for } i \in I \setminus \{0\}. \quad (2.14)$$

For $B$ simple, let $u(B) \in B$ be the unique element of weight $\lambda$ such that $\lambda$ satisfies (2.14).

**Proposition 2.4** ([AK] Lemmas 1.9 and 1.10). Simple crystals have the following properties.

1. A simple crystal is connected.
2. The tensor product of simple crystals is also simple.

2.3. Category $C^{\text{fin}}$

We recall the category $C^{\text{fin}}$ defined in [HKKOT]. Let $B$ be a regular $P^0_{\text{cl}}$-weighted crystal with finitely many elements. For $B$ we introduce the level of $B$ by

$$\text{lev } B = \min \{ \langle c, \varepsilon(b) \rangle \mid b \in B \} \in \mathbb{Z}_{\geq 0}.$$ 

Note that $\langle c, \varepsilon(b) \rangle = \langle c, \varphi(b) \rangle$ for any $b \in B$. We also set $B_{\text{min}} = \{ b \in B \mid \langle c, \varepsilon(b) \rangle = \text{lev } B \}$ and call an element of $B_{\text{min}}$ minimal.
Definition 2.5 ([HKKOT] Definition 2.5). We denote by $C_{\text{fin}}(g)$ (or simply $C_{\text{fin}}$) the category of crystals $B$ satisfying the following conditions:

1. $B$ is a crystal base of a finite-dimensional $U'_q(g)$-module.
2. $B$ is simple.
3. For any $\lambda \in P^+_\text{cl}$ such that $\langle c, \lambda \rangle \geq \text{lev} B$, there exists $b \in B$ satisfying $\varepsilon(b) \leq \lambda$. It is also true for $\varphi$.

We call an element of $C_{\text{fin}}(g)$ a finite crystal.

Remark 2.6.

(i) Condition (1) implies that $B$ is a regular $P^0_{\text{cl}}$-weighted crystal with finitely many elements.

(ii) Set $l = \text{lev} B$. Condition (3) implies that the maps $\varepsilon$ and $\varphi$ from $B_{\text{min}}$ to $(P^+_{\text{cl}})^l$ are surjective (cf. (4.6.5) in [KMN1]).

(iii) Practically, one has to check condition (3) only for $\lambda \in P^+_\text{cl}$ such that there is no $i \in I$ satisfying $\lambda - \Lambda_i \geq 0$ and $\langle c, \lambda - \Lambda_i \rangle \geq \text{lev} B$. In particular, if $a_i^{(l)} = 1$ for any $i \in I$, that is, when $g = A^{(1)}_l$, $C^{(1)}_n$, the surjectivity of $\varepsilon$ and $\varphi$ assures (3).

(iv) The authors do not know a crystal satisfying (1) and (2), but not (3).

Let $B_1$ and $B_2$ be two finite crystals. Definition 2.5 (1) and the existence of the universal $R$-matrix assures that there is a natural isomorphism of crystals

$$B_1 \otimes B_2 \cong B_2 \otimes B_1.$$ (2.15)

The following lemma is immediate.

Lemma 2.7. Let $B_1$, $B_2$ be finite crystals.

1. $\text{lev}(B_1 \otimes B_2) = \max(\text{lev} B_1, \text{lev} B_2)$.
2. If $\text{lev} B_1 \geq \text{lev} B_2$, then $(B_1 \otimes B_2)_{\text{min}} = \{b_1 \otimes b_2 \mid b_1 \in (B_1)_{\text{min}}, \varphi(b_1) \geq \varepsilon(b_2)\}$.
3. If $\text{lev} B_1 \leq \text{lev} B_2$, then $(B_1 \otimes B_2)_{\text{min}} = \{b_1 \otimes b_2 \mid b_2 \in (B_2)_{\text{min}}, \varphi(b_1) \leq \varepsilon(b_2)\}$.

$C_{\text{fin}}(g)$ is a tensor category, as it is closed under the tensor product operation.

2.4. Perfect crystals

An object $B$ of $C_{\text{fin}}(g)$ is called perfect if the maps $\varepsilon$ and $\varphi$ are bijections from $B_{\text{min}}$ onto $(P^+_{\text{cl}})^l$, where $l = \text{lev} B$. If $B$ is perfect of level $l$, there is a bijection $\sigma : (P^+_{\text{cl}})^l \rightarrow (P^+_{\text{cl}})^l$ defined by $\sigma = \varepsilon \circ \varphi^{-1}$. It is called the associated automorphism of $B$. This nomenclature is explained below.

The next proposition is immediate from Lemma 2.7.
Proposition 2.8. Let $B$ be a perfect crystal of level $l$ with associated automorphism $\sigma$. Then $B^{\otimes m}$ is perfect of the same level with associated automorphism $\sigma^m$.

For a finite crystal $B$ write $B^{\leq \lambda} = \{ b \in B | \varepsilon(b) \leq \lambda \}$.

Theorem 2.9 ([KMN1]). Assume $\text{rank} g > 2$. Let $B$ be a perfect crystal of level $l$. For any $\lambda \in (P^+_\text{cl})_k$ with $k \geq l$, there exists an isomorphism

$$B(\lambda) \otimes B \cong \bigoplus_{b \in B^{\leq \lambda}} B(\lambda + \text{wt} b)$$

of $P_{\text{cl}}$-weighted crystals. In particular, if $k = l$, we have

$$B(\lambda) \otimes B \cong B(\sigma^{-1} \lambda),$$

where $\sigma$ is the associated automorphism of $B$.

We now recall the notion of a coherent family of perfect crystals [KKM]. Let $\{B^l\}_{l \geq 1}$ be a family of perfect crystals $B^l$ of level $l$. Take the index set $J = \{(l, b) | l \in \mathbb{Z}_{>0}, b \in B_{\text{min}}^l \}$. Let $T_\epsilon$ be the crystal in Example 2.1. A crystal $B^\infty$ with distinguished element $b^\infty$ is called a limit of $\{B^l\}_{l \geq 1}$ if it satisfies the following conditions:

$$\text{wt} b^\infty = 0, \varepsilon(b^\infty) = \varphi(b^\infty) = 0, \quad (2.16)$$

for any $(l, b) \in J$, there exists an embedding of crystals

$$f(l, b) : T_\epsilon(b) \otimes B^l \otimes T_{-\varphi(b)} \rightarrow B^\infty$$

sending $t_\epsilon(b) \otimes b \otimes t_{-\varphi(b)}$ to $b^\infty$, \quad (2.17)

$$B^\infty = \bigcup_{(l, b) \in J} \text{Im} f(l, b). \quad (2.18)$$

If a limit exists for the family $\{B^l\}$, we say $\{B^l\}$ is a coherent family of perfect crystals. Set $B^\infty_{\text{min}} = \{ b \in B^\infty | \langle c, \varepsilon(b) \rangle = 0 \}$. Then both $\varepsilon$ and $\varphi$ map $B^\infty_{\text{min}}$ bijectively to $P^0_{\text{cl}}$. It follows from [KKM, Lemma 4.6] that the bijection given by $\sigma = \varepsilon \circ \varphi^{-1}$, is a linear automorphism of $P^0_{\text{cl}}$.

Conjecture 2.10 [KK].

(1) For any coherent family of perfect crystals,

$$\sigma \text{ extends to a linear automorphism of } P_{\text{cl}}$$

such that $\sigma \varphi(b) = \varepsilon(b)$ for any $b \in B^l_{\text{min}}$. \quad (2.19)

(2) $\sigma$ is induced by a Dynkin diagram automorphism. That is, there is an automorphism $\tau : I \rightarrow I$ of the Dynkin diagram of $g$ such that $\sigma(A_i) = \Lambda_{\tau(i)}$ for all $i \in I$. 

Lemma 2.11. Let \( \{B^i\} \) be a coherent family of perfect crystals. Suppose there is an \( i_0 \in I \) such that \( \langle c, \Lambda_{i_0} \rangle = 1 \) and \( \text{wt} \varphi_{B^i}^{-1}(l \Lambda_{i_0}) = l \cdot \text{wt} \varphi_{B^{i_0}}^{-1}(\Lambda_{i_0}) \) for all \( l \in \mathbb{Z}_{>0} \). Then (2.19) holds.

Proof. Let \( v_l = \varphi_{B^{i_0}}^{-1}(l \Lambda_{i_0}) \) for \( l \in \mathbb{Z}_{>0} \). The hypotheses imply that \( \sigma_{B^i}(l \Lambda_{i_0}) = l \sigma_{B^{i_0}}(\Lambda_{i_0}) \). It follows that there is a unique well-defined linear automorphism \( \sigma \) of \( P_{cl}^0 \) such that \( \sigma(\Lambda_{i_0}) = \Lambda_{i_0} \) and \( \sigma|_{P_{cl}^0} = \varepsilon \circ \varphi^{-1} \). Let \( \lambda \in (P_{cl}^+) \) and \( b \in B^l_{\text{min}} \) such that \( \varphi_{B^i}(b) = \lambda \). Then \( \lambda - l \Lambda_{i_0} \in P_{cl}^0 \). Write \( b' = f(l, v_l)(t \varepsilon(v_l) \otimes b \otimes t_{-l \Lambda_{i_0}}) \). By the hypotheses we have

\[
0 = \sigma\varphi(b') - \varepsilon(b') \\
= \sigma(\varphi_{B^i}(b) - l \Lambda_{i_0}) - (\varphi_{B^{i_0}}(b) - \sigma_{B^{i_0}}(l \Lambda_{i_0})) \\
= \sigma(\lambda) - l \sigma(\Lambda_{i_0}) - \sigma_{B^{i_0}}(\lambda) + l \sigma_{B^{i_0}}(\Lambda_{i_0}) \\
= \sigma(\lambda) - \sigma_{B^{i_0}}(\lambda).
\]

This is precisely (2.19). \( \square \)

Lemma 2.12. Suppose (2.19) holds. Then there is a permutation \( \tau : I \rightarrow I \) such that \( \sigma(\Lambda_i) = \Lambda_{\tau(i)} \) for all \( i \in I \).

Proof. By assumption \( \sigma = \sigma_{B^{i_0}} \) on \( (P_{cl}^+) \). Thus \( \sigma \) permutes \( (P_{cl}^+) \) for \( l > 0 \). By linearity it follows that \( \sigma \) permutes the dominant weights of level \( l \) that cannot be expressed as the sum of two nonzero dominant weights of smaller level. But these are precisely the fundamental weights of level \( l \). \( \square \)

Note that this does not prove that \( \tau \) must preserve the Dynkin diagram.

2.5. Crystals with core

In [KK] Kang and Kashiwara developed the theory of crystals with core. Let \( B \) be a regular crystal. For \( b \in B \) define

\[
\mathcal{E}(b) = \{ \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} b \mid l \geq 0, i_1, \ldots, i_l \in I \} \setminus \{0\}.
\]

Definition 2.13 ([KK] Definition 3.4). We say a regular crystal \( B \) has a core if \( \mathcal{E}(b) \) is a finite set for any \( b \in B \). In this case, we define the core \( C(B) \) of \( B \) by

\[
C(B) = \{ b \in B \mid \mathcal{E}(b') = \mathcal{E}(b) \text{ for every } b' \in \mathcal{E}(b) \},
\]

and \( B \) is called a crystal with core.

The following results are proven in [KK].
Theorem 2.14 ([KK] Theorem 5.4). Suppose rank $g > 2$. Let $\{B^i\}_{i \geq 1}$ be a coherent family of perfect crystals with the associated automorphism $\sigma$ satisfying (2.19). Take positive integers $k, l$ such that $k < l$ and a weight $\lambda \in (P_+^\vee)^k$. Then we have an isomorphism of $P_{\lambda}$-weighted crystals

$$B(\lambda) \otimes B^l \simeq B^{l-k} \otimes B(\sigma^{-1}\lambda).$$

(2.20)

Proposition 2.15 ([KK]). On the core of both sides of (2.20) we have

1. $C(B^{l-k} \otimes B(\sigma^{-1}\lambda)) = B^{l-k} \otimes u_{\sigma^{-1}1}.$
2. $C(B(\lambda) \otimes B^l) = u_2 \otimes (B^2)^{(\lambda)}$ with a suitable subset $(B^2)^{(\lambda)}$ of $B^l$. Moreover, $(B^l)^{(\lambda)} \cap B_{\text{min}}^l = \{b \in B^l \mid \varepsilon(b) = -\lambda \in (P^\vee_+)_{l-k}\}.$
3. For any distinct elements $b \in B(\lambda) \otimes B^l$ and $b' \in C(B(\lambda) \otimes B^l)$, there exists a sequence $i_1, \ldots, i_l \in I$ (l $\geq 1$) such that $b' \in \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} b$.

3. Paths

In this section we review the set of paths $\mathcal{P}(p, B)$ defined in [HKKOT] and prepare necessary facts.

3.1. Energy function

Let $B_1$ and $B_2$ be finite crystals. Suppose $b_1 \otimes b_2 \in B_1 \otimes B_2$ is mapped to $\tilde{b}_2 \otimes \tilde{b}_1 \in B_2 \otimes B_1$ under the isomorphism (2.15). A $\mathbb{Z}$-valued function $H$ on $B_1 \otimes B_2$ is called an energy function if for any $i$ and $b_1 \otimes b_2 \in B_1 \otimes B_2$ such that $\tilde{e}_i(b_1 \otimes b_2) \neq 0$, it satisfies

$$H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) + 1 & \text{if } i = 0, \varepsilon_0(b_1) \geq \varepsilon_0(b_2), \varepsilon_0(\tilde{b}_2) \geq \varepsilon_0(\tilde{b}_1), \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0, \varepsilon_0(b_1) < \varepsilon_0(b_2), \varepsilon_0(\tilde{b}_2) < \varepsilon_0(\tilde{b}_1), \\ H(b_1 \otimes b_2) & \text{otherwise.} \end{cases}$$

(3.1)

When we want to emphasize $B_1 \otimes B_2$, we write $H_{B_1, B_2}$ for $H$. The existence of such a function can be shown in a similar manner to Section 4 of [KMN1] based on the existence of the combinatorial $R$-matrix. The energy function is unique up to an additive constant, since $B_1 \otimes B_2$ is connected. By definition, $H_{B_1, B_2}(b_1 \otimes b_2) = H_{B_2, B_1}(\tilde{b}_2 \otimes \tilde{b}_1)$.

If the tensor product $B_1 \otimes B_2$ is homogeneous, i.e., $B_1 = B_2$, we have $b_2 = b_1$, $\tilde{b}_1 = b_2$. Thus (3.1) reduces to

$$H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) + 1 & \text{if } i = 0, \varepsilon_0(b_1) \geq \varepsilon_0(b_2), \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0, \varepsilon_0(b_1) < \varepsilon_0(b_2), \\ H(b_1 \otimes b_2) & \text{if } i \neq 0. \end{cases}$$

(3.2)

Let $M = [i_1, \ldots, i_l]$ be a multiset from $I$. A partition of $M$ is a decomposition $M = [j_1, \ldots, j_m] \cup [j'_1, \ldots, j'_m]$ such that $[i_1, \ldots, i_l] = [j_1, \ldots, j_m, j'_1, \ldots, j'_m]$. For $r \in I$ and a multiset $M = [i_1, \ldots, i_l]$ from $I$ let $\# r[i_1, \ldots, i_l]$ denote the number of letters $r$ in $M$.

The next lemma is an easy consequence of the definition of $H$. 

Lemma 3.1. Suppose $b_1 \otimes b_2$ is mapped to $\tilde{b}_2 \otimes \tilde{b}_1$ under the isomorphism $B_1 \otimes B_2 \sim B_2 \otimes B_1$ and $b'_1 \otimes b'_2 = \tilde{e}_{i_1} \cdot \cdot \cdot \tilde{e}_{i_1}(b_1 \otimes b_2)$. If we have

$$\tilde{e}_{i_1} \cdot \cdot \cdot \tilde{e}_{i_1}(b_1 \otimes b_2) = \tilde{e}_{j_1}^{*} \cdot \cdot \cdot \tilde{e}_{j_1}^{*}(b_1 \otimes b_2)$$

and

$$\tilde{e}_{i_1} \cdot \cdot \cdot \tilde{e}_{i_1}(\tilde{b}_2 \otimes \tilde{b}_1) = \tilde{e}_{j_1}^{*} \cdot \cdot \cdot \tilde{e}_{j_1}^{*}(\tilde{b}_2 \otimes \tilde{b}_1),$$

then

$$H(b'_1 \otimes b'_2) - H(b_1 \otimes b_2) = \#0[j_1', \ldots, j_m'] - \#0[j_1, \ldots, j_m].$$

The following proposition reduces the energy function of tensor products to that of their components. See Section 2.13 of [OSS].

Proposition 3.2. Consider the tensor product $B = B_1 \otimes B_2 \otimes \cdot \cdot \cdot \otimes B_m$.

(1) Set $B^* = B_1 \otimes \cdot \cdot \cdot \otimes B_{m-1}$. For $b_1 \otimes \cdot \cdot \cdot \otimes b_m \in B$, define $b^{(j)}_m (1 \leq j \leq m)$ by

$$B_j \otimes \cdot \cdot \cdot \otimes B_{m-1} \otimes B_m \sim B_m \otimes B_j \otimes \cdot \cdot \cdot \otimes B_{m-1},$$

$$b_j \otimes \cdot \cdot \cdot \otimes b_{m-1} \otimes b_m \mapsto b^{(j)}_m \otimes b_j \otimes \cdot \cdot \cdot \otimes b_{m-1}. \quad (3.3)$$

We understand $b^{(m)}_m = b_m$. Then we have

$$H_{B^* B_m}((b_1 \otimes \cdot \cdot \cdot \otimes b_{m-1}) \otimes b_m) = \sum_{1 \leq j \leq m-1} H_{B_j B_m}(b_j \otimes b^{(j+1)}_m).$$

(2) Set $B^! = B_2 \otimes \cdot \cdot \cdot \otimes B_m$. For $b_1 \otimes \cdot \cdot \cdot \otimes b_m \in B$, define $b^{(j)}_1 (1 \leq j \leq m)$ by

$$B_1 \otimes B_2 \otimes \cdot \cdot \cdot \otimes B_j \sim B_2 \otimes \cdot \cdot \cdot \otimes B_j \otimes B_1,$$

$$b_1 \otimes b_2 \otimes \cdot \cdot \cdot \otimes b_j \mapsto \hat{b}_2 \otimes \cdot \cdot \cdot \otimes \hat{b}_j \otimes b^{(j)}_1. \quad (3.4)$$

We understand $b^{(1)}_1 = b_1$. Then we have

$$H_{B_1 B^!}(b_1 \otimes (b_2 \otimes \cdot \cdot \cdot \otimes b_m)) = \sum_{2 \leq j \leq m} H_{B_1 B_j}(b^{(j-1)}_1 \otimes b_j).$$

3.2. Energy $D_B$

We wish to define an energy function $D_B : B \rightarrow \mathbb{Z}$ for tensor products of perfect crystals of the form $B^{r,s}$.

Let $B = B^{r,s}$ be perfect. Then there exists a unique element $b^* \in B$ such that $\varphi(b^*) = \text{lev}(B) \Lambda_0$. Define $D_B : B \rightarrow \mathbb{Z}$ by

$$D_B(b) = H_{BB}(b^* \otimes b) - H_{BB}(b^* \otimes u(B)),$$
where \( u(B) \) is defined in Section 2.2. From now on we assume that the energy function is normalized by

\[
H_{B_1, B_2} \left( u(B_1) \otimes u(B_2) \right) = 0. \tag{3.5}
\]

Now suppose by induction that \( B_i \) is a tensor product of perfect crystals of the form \( Br,s \) such that \( DB_i \) has been defined for \( 1 \leq i \leq m \). For \( B = B_1 \otimes \cdots \otimes B_m \) and \( b_i \in B_i \), define

\[
D_B(b_1 \otimes \cdots \otimes b_m) = \sum_{1 \leq i \leq m} D_{B_i}(b_i^{(1)}) + \sum_{1 \leq i < j \leq m} H_{B_i, B_j}(b_i \otimes b_j^{(i+1)}), \tag{3.6}
\]

where \( b_j^{(i)} \) is defined by (3.3).

**Proposition 3.3** ([OSS] Proposition 2.13). The above function \( D_B \) is well-defined, that is, it depends only on \( B \) and not on the grouping of tensor factors within \( B \).

**Example 3.4.** Let \( B_i \) have the form \( Br,s \) for \( 1 \leq i \leq 4 \) and let \( B = B_1 \otimes B_2 \otimes B_3 \otimes B_4 \). One may define \( D_B \) using three groups of tensor factors by the grouping \((B_1) \otimes (B_2 \otimes B_3) \otimes (B_4)\) or using two tensor factors by the grouping \((B_1 \otimes B_2) \otimes (B_3 \otimes B_4)\). Proposition 3.3 asserts that the resulting functions \( D_B \) are equal.

**Proposition 3.5** ([OSS] Proposition 2.15). Let \((i_1, i_2, \ldots, i_m)\) be a permutation of the set \{1, 2, \ldots, m\}, and set \( \tilde{B} = B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_m} \). Suppose that \( b \in B \) is mapped to \( \tilde{b} \in \tilde{B} \) under the isomorphism \( B \cong \tilde{B} \). Then we have \( D_B(b) = D_{\tilde{B}}(\tilde{b}) \).

If the tensor product \( B_1 \otimes \cdots \otimes B_m \) is homogeneous, i.e., \( B_1 = \cdots = B_m \), we have \( b_j^{(i+1)} = b_{i+1} \) for any \( 1 \leq i < j \leq m \) and \( b_i^{(1)} = b_i \) for \( 1 \leq i \leq m \). Then (3.6) reduces to

\[
D_B(b) = mD_{B_1}(b_1) + \sum_{j=1}^{m-1} (m - j)H_{B_j B_1}(b_j \otimes b_{j+1}). \tag{3.7}
\]

For later use we prepare a lemma, which is an immediate consequence of (3.2).

**Lemma 3.6.** Suppose \( B = B_1^{\otimes m} \). For \( b = b_1 \otimes \cdots \otimes b_m \in B \) and \( i \in I \) such that \( \tilde{e}_i b \neq 0 \) we have

\[
D_B(\tilde{e}_i b) = D_R(b) - \delta_{i0},
\]

unless \( i = 0 \) and \( \tilde{e}_0 b = \tilde{e}_0 b_1 \otimes b_2 \otimes \cdots \otimes b_m \).
3.3. Set of paths $\mathcal{P}(p, B)$

An element $p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1$ of the semi-infinite tensor product of $B$ is called a **reference path** if it satisfies $b_j \in B_{\min}$ and $\varphi(b_{j+1}) = \varepsilon(b_j)$ for any $j \geq 1$. Fix a reference path $p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1$. The set of paths $\mathcal{P}(p, B)$ is defined by

$$\mathcal{P}(p, B) = \{ p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1 \mid b_j \in B, b_k = b_k \text{ for } k \gg 1 \}.$$ 

An element of $\mathcal{P}(p, B)$ is called a **path**. For convenience we denote $b_k$ by $p(k)$ and $\cdots \otimes b_{k+2} \otimes b_{k+1}$ by $p[k]$ for $p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1$. For a path $p \in \mathcal{P}(p, B)$, set

$$E(p) = \sum_{j=1}^{\infty} j \left( H(p(j+1) \otimes p(j)) - H(p(j+1) \otimes p(j)) \right),$$

$$\text{wt } p = \varphi(p(1)) + \sum_{j=1}^{\infty} (\text{wt } p(j) - \text{wt } p(j)) - \left( E(p)/a_0 \right) \delta,$$

where $a_0$ is the 0th Kac label. $E(p)$ and $\text{wt } p$ are called the **energy** and **weight** of $p$. We distinguish $\text{wt } p \in P$ from $\text{wt } p = \varphi(p(1)) + \sum_{j=1}^{\infty} (\text{wt } p(j) - \text{wt } p(j)) \in P_{\text{cl}}$. Compare $E(p)$ with (3.7). It is a normalized energy for the semi-infinite tensor product of $B$.

**Remark 3.7.**

(i) If $B$ is perfect, the set of reference paths is in bijection with $(P_{\text{cl}}^+)^l$, where $l = \text{lev } B$. For $\lambda \in (P_{\text{cl}}^+)^l$, take a unique $b_1 \in B_{\min}$ such that $\varphi(b_1) = \lambda$. The condition $\varphi(b_{j+1}) = \varepsilon(b_j)$ fixes $p = \cdots \otimes b_j \otimes \cdots \otimes b_1$ uniquely.

(ii) In [KMN1] $p$ is called a ground state path, since $E(p) \geq E(p')$ for any $p \in \mathcal{P}(p, B)$. But if $B$ is not perfect, it is no longer true in general.

The following theorem is important.

**Theorem 3.8 ([HKKOT] Theorem 3.7).** Assume $\text{rank } g > 2$. Then $\mathcal{P}(p, B)$ is isomorphic to a direct sum of crystals $B(\lambda)$ ($\lambda \in P^+$) of integrable highest weight $U_q(g)$-modules.

For a set of paths $\mathcal{P}(p, B)$ let $\mathcal{P}(p, B)_0$ denote the set of its **highest weight** elements, i.e., elements satisfying $\hat{e}_i p = 0$ for any $i$. The following proposition describes the set of highest weight elements in $\mathcal{P}(p, B)$.

**Proposition 3.9 ([HKKOT] Proposition 3.9 and Corollary 3.10).**

$$\mathcal{P}(p, B)_0 = \{ p \in \mathcal{P}(p, B) \mid p(j) \in B_{\min}, \varphi(p(j+1)) = \varepsilon(p(j)) \text{ for } \forall j \}.$$ 

Moreover, if $p \in \mathcal{P}(p, B)_0$, then $\text{wt } p[j] = \varphi(p(j+1))$. 
From Theorem 3.8 and Proposition 3.9 we obtain

**Theorem 3.10** ([KMN1] Proposition 4.6.4). Let $B$ be a perfect crystal of level $l$ with the associated automorphism $\sigma$. For $\lambda \in (P^+_\mathfrak{c})^l$ define a reference path $p^{(\lambda)}$ by $p^{(\lambda)}(j) = \epsilon_B^{-1}(\sigma^j \lambda)$. Then we have an isomorphism of $P$-weighted crystals

$$\mathcal{P}(p^{(\lambda)}, B) \simeq B(\lambda).$$

4. Tensor product theorem

In this section we assume rank $\mathfrak{g} > 2$ because of frequent use of Theorems 2.9, 2.14, and 3.8.

4.1. Assumptions

In this subsection we explain two properties (A1) and (A2) for a set of perfect crystals that are assumed to prove our main theorem. Consider a set of perfect crystals $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$. We assume $\text{lev } B_1 \geq \text{lev } B_2 \geq \cdots \geq \text{lev } B_m$.

(A1) Each $B_i$ in $\mathcal{B}$ belongs to a certain coherent family satisfying the condition (2.19).

It should be emphasized that the coherent family containing $B_i$ may depend on $i$. Let $\sigma_i$ be the associated automorphism of $B_i$ in $\mathcal{B}$ ($i = 1, \ldots, m$).

**Proposition 4.1.** Under the above assumptions, $\sigma_i \sigma_j = \sigma_j \sigma_i$ on $P^\mathfrak{c}$ for any $i, j$.

**Proof.** Take any $\lambda, \mu \in P^\mathfrak{c}$. Let $B'_i$ (respectively, $B'_j$) be the perfect crystal of level $\langle c, \lambda \rangle$ (respectively, $\langle c, \mu \rangle$) which belongs to the same coherent family as $B_i$ (respectively, $B_j$). Let $B''_i$ be the perfect crystal of level $\langle c, \lambda + \mu \rangle$ which belongs to the same coherent family as $B_i$. From Theorems 2.9, 2.14 and Eq. (2.15) one has the following isomorphisms of $P^\mathfrak{c}$-weighted crystals

$$B(\lambda) \otimes B(\mu) \simeq B(\sigma_i \lambda) \otimes B'(\mu) \simeq B(\sigma_i \lambda) \otimes B(\sigma_i \mu) \otimes B''_i \simeq B(\sigma_i \lambda) \otimes B(\sigma_i \mu) \otimes B''_j \otimes B'_j.$$ 

Going backward, we have $B(\lambda) \otimes B(\mu) \simeq B(\lambda) \otimes B(\sigma_j^{-1} \sigma_i^{-1} \sigma_j \sigma_i \mu)$, and therefore $\mu = \sigma_j^{-1} \sigma_i^{-1} \sigma_j \sigma_i \mu$. From (2.19) we see that the statement is valid. □

Let $\tau$ be a Dynkin diagram automorphism, that is, a permutation of the vertex set $I$ which preserves the Dynkin diagram of $\mathfrak{g}$. By setting $\tau(\sum_i m_i \Lambda_i) = \sum_i m_i \Lambda_{\tau(i)}$ one can
extend it to a linear automorphism of $P_{cl}$. Since $\tau$ is a Dynkin automorphism it follows that $\tau(a_i) = a_{\tau(i)}$ for all $i \in I$. It may happen that $\tau$ can be lifted to an automorphism, denoted by $\tau^*$, of a perfect crystal $B$ satisfying

$$\tau^*(\tilde{e}_i b) = \tilde{e}_{\tau(i)} \tau^*(b)$$

$$\tau^*(\tilde{f}_i b) = \tilde{f}_{\tau(i)} \tau^*(b), \quad (4.1)$$

for any $i$ and $b \in B$ such that $\tilde{e}_i b \neq 0$ and $\tilde{f}_i b \neq 0$, respectively. If such a $\tau^*$ exists, let us call $\tau$ a proper automorphism for $B$. Note that (4.1) implies $\varepsilon(\tau^*(b)) = \tau(\varepsilon(b))$, $\varphi(\tau^*(b)) = \tau(\varphi(b))$ and hence $\tau^*$ is unique for $B$. It should be emphasized that the associated automorphism $\sigma$ of a perfect crystal $B$ is an automorphism on $P_{cl}$ and $\sigma$ can be a proper automorphism for a different crystal $B'$ from $B$. In what follows we simply write $\sigma b$ to mean $\sigma^*(b)$.

**Remark 4.2.** The above $\tilde{b}_i$ is determined easily as $\tilde{b}_i = \varphi_{B_i}^{-1}(\varphi_{B_i}(b_i) + \text{wt} b_j)$ using (2.12) and Lemma 2.7 (2).

The following proposition is sometimes useful to check whether (A2) is satisfied.

**Proposition 4.3.** Let $B_1, B_2$ be perfect crystals of level $l_1, l_2$ ($l_1 \geq l_2$). Suppose $B_1$ belongs to a certain coherent family of perfect crystals with the associated automorphism $\sigma$ that is proper for $B_2$. We assume

$$\text{the map } \varepsilon \times \varphi : B_2 \longrightarrow (P_{cl}^+)^2 \text{ is injective.} \quad (4.2)$$

Then if $b_1 \otimes b_2 \in (B_1 \otimes B_2)_{\text{min}}$, $b_1 \otimes b_2$ is mapped to $\sigma b_2 \otimes \hat{b}_1$ under the isomorphism $B_1 \otimes B_2 \cong B_2 \otimes B_1$.

**Proof.** Let $b_1 \otimes b_2$ be mapped to $\hat{b}_2 \otimes \hat{b}_1$ under the isomorphism. We are to show $\tilde{b}_2 = \sigma b_2$.

Set $\lambda = \sigma \varepsilon(b_2)$, $\mu = \varphi(b_2)$ and $B_0$ to be the perfect crystal of level $l_1 - \langle c, \lambda \rangle$ in the same coherent family as $B_1$. Note that from Lemma 2.7 (2), we have $\varepsilon(b_1) = \sigma \varphi(b_1) \geq \varepsilon(b_2) = \lambda$ and hence $l_1 \geq \langle c, \lambda \rangle$. Using Theorems 2.9, 2.14 and Proposition 2.15 we have

$$B(\lambda) \otimes B_1 \otimes B_2 \simeq B_0 \otimes B(\mu) \otimes B_2 \simeq B_0 \otimes \left( \bigoplus_v B(v)^{\oplus m_v} \right)$$

$$\simeq \left( \bigoplus_v B(\sigma v)^{\oplus m_v} \right) \otimes B_1,$$

$$u_\lambda \otimes b_1 \otimes b_2 \mapsto b_0 \otimes u_{\alpha^{-1}_\lambda} \otimes b_2 \mapsto b_0 \otimes u_\mu \mapsto u_{\sigma \mu} \otimes b'_1,$$
with some \( b_0 \in B_0, b'_1 \in B_1 \). Now notice that when \( B(\sigma^{-1}\lambda) \otimes B_2 \simeq \bigoplus_v B(v) \otimes B_0 \), we have \( B(\lambda) \otimes B_2 \simeq \bigoplus_v B(\sigma(v)) \otimes B_0 \) because \( \sigma \) is proper for \( B_2 \). Since \( b_1 \otimes b_2 \) is mapped to \( \tilde{b}_1 \otimes \tilde{b}_0 \) under \( B_1 \otimes B_2 \simeq B_2 \otimes B_1 \), by \( B(\lambda) \otimes B_2 \otimes B_1 \simeq (\bigoplus_v B(\sigma(v)) \otimes B_0) \otimes B_1 \), \( u_\lambda \otimes \tilde{b}_2 \otimes \tilde{b}_1 \) should be mapped to \( u_\sigma \mu \otimes \tilde{b}_2 \). Therefore, we have

\[
\lambda = \varepsilon(\sigma b_2) = \varepsilon(\tilde{b}_2), \quad \sigma \mu = \varphi(\sigma b_2) = \varphi(\tilde{b}_2).
\]

From the assumption (4.2) we get \( \tilde{b}_2 = \sigma b_2 \).

**Example 4.4.** In [KKM] a coherent family of perfect crystals \( B = \{B^i\}_{i \geq 1} \) is given for any nonexceptional affine Lie algebra \( g \). For these \( B \) (A1) is easily verified using Lemma 2.11. Let \( \sigma \) be the associated automorphism. One can see this \( \sigma \) is proper for any \( B^i \), since the automorphism on \( B^i \) is determined uniquely by requiring \( \sigma b = b' \) if \( \varepsilon(b) = \varepsilon(b') \) for \( b, b' \in B^i \) and (4.1). Using the explicit formula for \( \varepsilon(b) \) and \( \varphi(b) \) in [KKM], one can also check that the condition (4.2) holds. Thus by Proposition 4.3, (A2) is also valid for this \( B \).

**Example 4.5.** Consider the set of perfect crystals \( B = \{B^{k,l}\}_{k \leq \nu, l \geq 1} \) of type \( A^{(1)}_\nu \) given in [KMN2]. We will show in the next section that \( \{B^{k,l}\}_{l \geq 1} \) for any fixed \( k \) forms a coherent family of perfect crystals with the associated automorphism \( \sigma_k \) satisfying (2.19) such that \( \sigma_k(A_\nu) = A_{\nu-k \mod n+1} \). Thus (A1) is valid for \( B \). It is also known that there exists an automorphism \( pr \) on any \( B^{k,l} \) called the promotion operator [S]. \( pr \) satisfies

\[
pr \circ \tilde{e}_i - 1 = \tilde{e}_i \circ pr \quad \text{and} \quad pr \circ \tilde{f}_i - 1 = \tilde{f}_i \circ pr
\]

for any \( i \) where indices are taken modulo \( n + 1 \). Therefore, defining an automorphism \( \sigma_{k'} \) on \( B^{k,l} \) by \( \sigma_{k'} = pr^{k'} \), \( \sigma_{k'} \) turns out to be a proper automorphism for \( B^{k,l} \). Furthermore, the second requirement in (A2) is already proven in Theorem 7.3 of [SS]. Hence (A2) is also valid for \( B \). We note that the condition (4.2) is no longer true for \( B^{k,l} \) in general.

In what follows in this section, we assume that any set of perfect crystals we consider satisfies (A1) and (A2).

### 4.2. Lemmas

We prepare several lemmas. First suppose \( B_1 \) and \( B_2 \) are perfect crystals of level \( l_1 \) and \( l_2 \) such that \( l_1 \geq l_2 \), and let \( \sigma \) be the associated automorphism of \( B_1 \).

**Lemma 4.6.** Let \( b_1, b'_1 \in B_1, \ b_2 \in B_2 \) and suppose \( b_2 \otimes b_1 \in (B_2 \otimes B_1)_{\min} \). Then there exists a sequence \( i_1, \ldots, i_l \in I \) such that

\[
b_2 \otimes b_1 = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} (b_2 \otimes b'_1).
\]

**Proof.** Let \( \lambda = \varepsilon(b_2) \) and consider the connected component \( \tilde{B} \) of \( B(\lambda) \otimes B_2 \otimes B_1 \) containing \( u_\lambda \otimes b_2 \otimes b_1 \). We have \( u_\lambda \otimes b_2 \otimes b'_1 \in \tilde{B} \) and also \( u_\lambda \otimes b_2 \otimes b_1 \in C(\tilde{B}) \) by Proposition 2.15 (2). Apply Proposition 2.15 (3). \( \square \)
Set \( r = \sigma^{-1}(0) \) and define a function \( \gamma \) on \( B_2 \) by

\[
\gamma(b_2) = \left( \Lambda_\gamma^n - \Lambda_0^n \right) \dim wt b_2, \quad \Lambda_\gamma^n = \left( a_i / a_i^n \right) \Lambda_i,
\]

where \( a_i \) (respectively, \( a_i^n \)) is the \( i \)th Kac (respectively, dual Kac) label and \( (\cdot) \) is the invariant bilinear form on \( P \) as in [Kac]. If \( r = 0 \), \( \gamma(b) = 0 \). Otherwise, it has the following recursive property

\[
\gamma(\tilde{e}_i b_2) = \begin{cases} 
\gamma(b_2) - 1 & \text{if } i = 0, \\
\gamma(b_2) + 1 & \text{if } i = r, \\
\gamma(b_2) & \text{otherwise.}
\end{cases}
\]

**Lemma 4.7.** If \( b_1 \otimes b_2 \in (B_1 \otimes B_2)_{\min} \), we have \( H_{B_1}, B_2(b_1 \otimes b_2) = \gamma(b_2) \) up to global additive constant.

**Proof.** Take \( b_1 \otimes b_2, b'_1 \otimes b'_2 \in (B_1 \otimes B_2)_{\min} \) and suppose \( b_1 \otimes b_2 \mapsto \tilde{b}_2 \otimes \tilde{b}_1, b'_1 \otimes b'_2 \mapsto \tilde{b}'_2 \otimes \tilde{b}'_1 \) under the isomorphism \( B_1 \otimes B_2 \cong B_2 \otimes B_1 \). From (A2) we have \( \tilde{b}_2 = \sigma b_2, \tilde{b}'_2 = \sigma b'_2 \). Take a sequence \( i_1, \ldots, i_l \in I \) such that \( \tilde{b}'_2 = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} \tilde{b}_2 \) and \( l \) is minimal. Then one can get a sequence \( k_1, \ldots, k_n \in I \) such that \( \tilde{e}_{k_n} \cdots \tilde{e}_{k_1}(b_2 \otimes b_1) = \tilde{e}_{i_1} \tilde{b}_2 \otimes b_1 '' \) for some \( b_1 '' \in B_1 \). Using the previous lemma one can assume \( b_1 '' \equiv b_1 ' \). Now suppose \( \tilde{e}_{k_n} \cdots \tilde{e}_{k_1}(b_1 \otimes b_2) = \tilde{e}_{i_1} \tilde{b}_1 ' \otimes \tilde{e}_{j_1} \tilde{b}_2 ' \otimes \tilde{e}_{i_2} \tilde{b}_2 ' \) due to the minimality of \( l \),

\[
\sigma \left( \tilde{e}_{i_1} \tilde{b}_1 ' \tilde{e}_{i_2} \tilde{b}_2 ' \right) = \tilde{e}_{i_1} \tilde{e}_{i_2} \sigma b_2
\]

implies

\[
\sum_{a=1}^{l'} \alpha_{i_a} - \sum_{a=1}^{l} \alpha_{\sigma^{-1}(i_a)} \in \mathbb{Z}_{\geq 0}.\delta.
\]

Similarly, going from \( b'_1 \otimes b'_2 \) to \( b_1 \otimes b_2 \) by applying \( \tilde{e}_i \)'s, one obtains

\[
\tilde{e}_{k_n} \cdots \tilde{e}_{k_1} \tilde{b}_2 ' \otimes \tilde{b}_1 ' = \tilde{e}_{j_n} \cdots \tilde{e}_{j_1} \tilde{b}_2 ' \otimes \tilde{b}_1 ',
\]

\[
\tilde{e}_{k_n} \cdots \tilde{e}_{k_1} (b_1 ' \otimes b_2 ') = \tilde{e}_{j_n} \cdots \tilde{e}_{j_1} b_1 ' \otimes \tilde{e}_{j_m} \tilde{b}_2 ' \otimes \tilde{e}_{j_1} \tilde{b}_2 '.
\]

\[
\sum_{a=1}^{m'} \alpha_{i_a} - \sum_{a=1}^{m} \alpha_{\sigma^{-1}(j_a)} \in \mathbb{Z}_{\geq 0}.\delta.
\]

Calculating the difference of the energy function during this process from Lemma 3.1, one gets

\[
0 = \mp\left[ 0[i_1, \ldots, i_{r'}, j_1, j_1 \ldots j_m] - \mp\left[ 0[i_1, \ldots, i_{r'}, j_1, j_1 \ldots j_m] \right. \right. \right. \right. \right. \]
Therefore, we have 

\[ l' = l, m' = m \]

and 

\[ \{i'_1, \ldots, i'_l\} = \{\sigma^{-1}(i_1), \ldots, \sigma^{-1}(i_l)\} \]

as multiset.

Thus noting \( b'_2 = \tilde{e}_{\sigma^{-1}(i)} \cdots \tilde{e}_{\sigma^{-1}(i_l)} b_2 \), one gets

\[
H(b'_1 \otimes b'_2) - H(b_1 \otimes b_2) = \#_0 [i_1, \ldots, i_l] - \#_0 [\sigma^{-1}(i_1), \ldots, \sigma^{-1}(i_l)]
\]

as desired. \( \square \)

Let \( B = \{B_1, B_2, \ldots, B_m\} \) be a set of perfect crystals satisfying (A1), (A2), and \( \text{lev} B_1 \geq \text{lev} B_2 \geq \cdots \geq \text{lev} B_m \).

**Lemma 4.8.** Suppose \( b_1 \otimes \cdots \otimes b_m \in (B_1 \otimes \cdots \otimes B_m)_{\min} \). Then \( b_1^{(j-1)} \otimes b_j \in (B_1 \otimes B_j)_{\min} \) for any \( 2 \leq j \leq m \). Here \( b_1^{(j)} \) is defined in (3.4).

**Proof.** Let \( b_1 \otimes b_2 \otimes \cdots \otimes b_{j-1} \) be mapped to \( \tilde{b}_2 \otimes \cdots \otimes \tilde{b}_{j-1} \otimes b_1^{(j-1)} \) by \( B_1 \otimes B_2 \otimes \cdots \otimes B_{j-1} \otimes B_1 \). From the fact that \( \tilde{b}_2 \otimes \cdots \otimes \tilde{b}_{j-1} \otimes b_1^{(j-1)} \otimes b_j \cdots \otimes b_m \) is minimal and using Lemma 2.7, (2.11), (2.12), we have

\[
\varphi(b_1^{(j-1)}) = \varphi(\tilde{b}_2 \otimes \cdots \otimes \tilde{b}_{j-1} \otimes b_1^{(j-1)}) \geq \varepsilon(b_j \otimes \cdots \otimes b_m) \geq \varepsilon(b_j)
\]

and know \( b_1^{(j-1)} \) is minimal. Hence \( b_1^{(j-1)} \otimes b_j \in (B_1 \otimes B_j)_{\min} \) again by Lemma 2.7. \( \square \)

Let \( \sigma \) be the associated automorphism of \( B_1 \) and set \( B^\dagger = B_2 \otimes \cdots \otimes B_m \). For an element \( b^\dagger = b_2 \otimes \cdots \otimes b_m \in B^\dagger \) define \( \sigma b^\dagger \) by

\[
\sigma b^\dagger = \sigma b_2 \otimes \cdots \otimes \sigma b_m.
\]

Note that this definition allows (4.1) to hold on the tensor product crystal. Let \( \gamma_{B_j} \) be the function \( \gamma \) (4.3) on \( B_j \). The following lemma is now immediate from Lemma 4.8, (A2), Remark 4.2, Proposition 3.2 (2) and Lemma 4.7.

**Lemma 4.9.** Let \( b_1 \otimes b^\dagger \in (B_1 \otimes B^\dagger)_{\min} \). Then

1. \( b_1 \otimes b^\dagger \) is mapped to \( \sigma b^\dagger \otimes \tilde{b}_1 \) under the isomorphism \( B_1 \otimes B^\dagger \otimes B_1 \). Here \( \tilde{b}_1 = \varphi_{B_1}(b_1) + \text{wt } b^\dagger \).
2. \( H_{B_1, B^\dagger}(b_1 \otimes b^\dagger) = \gamma_{B^\dagger}(b^\dagger) = \sum_{j=2}^m \gamma_{B_j}(b_j) \) up to global additive constant.
Next we consider the tensor product $B_1 \otimes L_1 \otimes \cdots \otimes B_m \otimes L_m$. Set $B_1^* = B_1^L \otimes L_1$, $B_2^* = B_2^L \otimes L_2$, $\cdots$, $B_m^* = B_m^L \otimes L_m$. Here is the last lemma.

**Lemma 4.10.** Let $L$ be a multiple of the order of $\sigma$. Suppose $b_1 \in (B_1^*)_{\text{min}}$. Then the map $L \rightarrow \mathbb{Z}$ given by $b_1 \mapsto H_{B_1^*L}(b_1 \otimes b_1)$ is constant on the set of elements $b_1$ such that $b_1 \otimes b_1$ is minimal.

**Proof.** Write $b_1 = b_1^1 \otimes \cdots \otimes b_1^{l_1}, b_1^i \in B_1^i$. By Proposition 3.2 (1) and Lemma 4.9 we have

$$H_{B_1^*L}(b_1 \otimes b_1) = \sum_{i=1}^{L} H_{B_1^i}(b_1^{L+1-i} \otimes \sigma^{i-1}b_1) = c + \sum_{i=1}^{L} y_{B_1^i}(\sigma^{i-1}b_1)$$

where $c$ is a constant. However, for any weight $\lambda$, one has

$$\left( A^{\gamma} \left| \sum_{i=1}^{L} \sigma^{i-1}\lambda \right. \right) = \left( A^{\gamma} \left| \sum_{i=1}^{L} \sigma^{i}\lambda \right. \right) = \left( A^{\gamma} \left| \sum_{i=1}^{L} \sigma^{i-1}\lambda \right. \right).$$

Hence $\sum_{i=1}^{L} y_{B_1^i}(\sigma^{i-1}b_1) = 0$ by (4.3). $\Box$

### 4.3. Main theorem

Let $B = \{B_1, B_2, \ldots, B_m\}$ be a set of perfect crystals satisfying (A1) and (A2). Set $l_i = \text{lev} B_i$ and assume $l_1 \geq l_2 \geq \cdots \geq l_m \geq l_{m+1} := 0$. Let $\sigma_i$ be the associated automorphism of $B_i$ and $B_i^i$ be a perfect crystal of level $l_i - l_{i+1}$ which belongs to the same coherent family as $B_i$. Take $\lambda_i \in (P_+^{cl})_{l_i - l_{i+1}}$ for all $i = 1, 2, \ldots, m$. Then we have the isomorphisms of $P_{cl}$-weighted crystals

$$B(\lambda_1) \otimes B(\lambda_2) \otimes \cdots \otimes B(\lambda_m)$$

$$\simeq (B(\sigma_1 \lambda_1) \otimes B_1^i) \otimes (B(\sigma_2 \lambda_2) \otimes B_2^i) \otimes \cdots \otimes (B(\sigma_m \lambda_m) \otimes B_m^i)$$

$$\simeq B(\sigma_1 \lambda_1) \otimes B(\sigma_2 \sigma_1 \lambda_2) \otimes \cdots \otimes B(\sigma_1 \sigma_2 \cdots \sigma_m \lambda_m) \otimes (B_1 \otimes B_2 \otimes \cdots \otimes B_m) \quad (4.5)$$

by applying Theorems 2.9 and 2.14 successively.

**Proposition 4.11.** Under the isomorphism (4.5) $u_{\lambda_1} \otimes u_{\lambda_2} \otimes \cdots \otimes u_{\lambda_m}$ is sent to $u_{\sigma_1 \lambda_1} \otimes u_{\sigma_1 \sigma_2 \lambda_2} \otimes \cdots \otimes u_{\sigma_1 \sigma_2 \cdots \sigma_m \lambda_m} \otimes (b_1^{(1)} \otimes b_2^{(1)} \otimes \cdots \otimes b_m^{(m)}) \quad (b_i^{(i)} \in B_i)$ where $b_i^{(i)}$ is given by

$$b_i^{(i)} = \varepsilon_{B_i}^{-1}(\sigma_{i} \lambda_1 + \sigma_{i} \sigma_{i+1} \lambda_{i+1} + \cdots + \sigma_{i} \sigma_{i+1} \cdots \sigma_{m} \lambda_{m})$$

for $i = 1, 2, \ldots, m$. 
Proof. We consider the $m = 2$ case first. Under (4.5) for $m = 2$ we have
\[ u_{\sigma_1, \lambda_1} \otimes u_{\sigma_2, \lambda_2} \mapsto (u_{\sigma_1, \lambda_1} \otimes b^{(1)}) \otimes (u_{\sigma_2, \lambda_2} \otimes b^{(2)}) \mapsto u_{\sigma_1, \lambda_1} \otimes u_{\sigma_1, \sigma_2 \lambda_2} \otimes (b^{(1)} \otimes b^{(2)}). \]

To determine $b^{(i)}$, calculate $\epsilon(u_{\sigma_i, \lambda_i} \otimes b^{(i)})$, which must be 0, using (2.11). We know $b^{(2)} = b^{(2)}$. To obtain $b^{(1)}$, use the fact that $\epsilon$ is unchanged under $B'_1 \otimes B(\sigma_1) \otimes B(\sigma_2) \otimes B(\sigma_3) \otimes B(\sigma_4) \otimes B(\sigma_4) = b^{(2)}$. The cases when $m > 2$ are similar. \qed

Iterating the isomorphism (4.5) we are led to consider a set of paths based on the finite crystal $B = B_1 \otimes B_2 \otimes \cdots \otimes B_m$. In view of Proposition 4.11 we define a distinguished path $p^{(\lambda_1, \ldots, \lambda_m)}$ associated with a set of weights $\lambda_1, \ldots, \lambda_m$ by
\[ p^{(\lambda_1, \ldots, \lambda_m)}(j) = b^{(j)}_1 \otimes \cdots \otimes b^{(m)}_j \quad (j \geq 1, b^{(i)}_j \in B_i \text{ for } 1 \leq i \leq m). \]

From Proposition 4.11 the highest weight element of $B(\lambda_1) \otimes \cdots \otimes B(\lambda_m)$ is sent to $u_{\sigma_1, \lambda_1} \otimes \cdots \otimes u_{\sigma_m, \lambda_m} \otimes p^{(\lambda_1, \ldots, \lambda_m)}(L) \otimes \cdots \otimes p^{(\lambda_1, \ldots, \lambda_m)}(1)$ under the $L$ fold iteration of (4.5), which implies that $p^{(\lambda_1, \ldots, \lambda_m)}(L) \otimes \cdots \otimes p^{(\lambda_1, \ldots, \lambda_m)}(1)$ is minimal in $B^{\otimes L}$. This fact shows

**Proposition 4.12.** $p^{(\lambda_1, \ldots, \lambda_m)}$ is a reference path in $B_1 \otimes \cdots \otimes B_m$. \vspace{10pt}

We now have a set of paths $P(p^{(\lambda_1, \ldots, \lambda_m)}, B_1 \otimes \cdots \otimes B_m)$. Our main theorem is the following.

**Theorem 4.13.** Let $B_i$ be a perfect crystal of level $l_i$ with the associated automorphism $\sigma_i$ for $i = 1, \ldots, m$. Suppose $l_1 \geq l_2 \geq \cdots \geq l_m$. For $\lambda_i \in (P_{\text{cl}}^+)_{l_i-l_{i+1}}$ ($i = 1, \ldots, m; l_m+1 = 0$) let $p^{(\lambda_1, \ldots, \lambda_m)}$ be a reference path in $B_1 \otimes \cdots \otimes B_m$ defined in (4.6). Then we have an isomorphism of $P$-weighted crystals
\[ P(p^{(\lambda_1, \ldots, \lambda_m)}, B_1 \otimes \cdots \otimes B_m) \simeq B(\lambda_1) \otimes \cdots \otimes B(\lambda_m). \]

**Proof.** We have the isomorphism of $P_{\text{cl}}$-weighted crystals by iterating (4.5).
\[ B(\lambda_1) \otimes \cdots \otimes B(\lambda_m) \simeq B(\sigma_1^{L_1} \lambda_1) \otimes \cdots \otimes B(\sigma_m^{L_m} \lambda_m) \otimes (B_1 \otimes \cdots \otimes B_m)^{\otimes L}. \]

For an element $v_1 \otimes \cdots \otimes v_m \in B(\lambda_1) \otimes \cdots \otimes B(\lambda_m)$ take sufficiently large $L$ such that under the above isomorphism
\[ v_1 \otimes \cdots \otimes v_m \mapsto u \otimes p(L) \otimes \cdots \otimes p(1), \]
where \( u = u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_m} \) and \( p(j) \in B_1 \otimes \cdots \otimes B_m \). We also assume that \( L (> 0) \) is a multiple of the order of \( \sigma_i \) for all \( i \) and \( p(L) = p(L) \) where \( p(L) \) is the \( L \)th component of the reference path \( p = p^{(\lambda_1, \ldots, \lambda_m)} \). To prove the theorem it suffices to show

\[
- \sum_{i=1}^m \langle d, \text{wt} v_i \rangle = \sum_{j=1}^{L-1} j \left( H(p(j+1) \otimes p(j)) - H(p(j+1) \otimes p(j)) \right),
\]

(4.7)

where \( d \) is the scaling element of \( g \) as in [Kac] and \( H \) is the energy function on \( B \otimes B \) (\( B = B_1 \otimes \cdots \otimes B_m \)). See (3.8), (3.9) and recall \( \langle d, \delta \rangle = a_0 \).

Consider the isomorphism of crystals

\[
B^\otimes_L \simeq B_1^\otimes_L \otimes \cdots \otimes B_m^\otimes_L,
\]

where \( B = B_1 \otimes \cdots \otimes B_m \). Suppose \( p(L) \otimes \cdots \otimes p(1) \mapsto b_1 \otimes \cdots \otimes b_m \), \( p(L) \otimes \cdots \otimes p(1) \mapsto b_1 \otimes \cdots \otimes b_m \) \((b_1, b_i \in B_i^\otimes_L)\) under the isomorphism. By (3.7) and Proposition 3.5 we have

\[
\text{r.h.s. of (4.7)} = D(b_1 \otimes \cdots \otimes b_m) - D(b_1 \otimes \cdots \otimes b_m).
\]

We wish to show (4.7) by induction on \( m \). We first observe that it suffices to check (4.7) when \( v_1 \otimes \cdots \otimes v_m \) is a highest weight element. If not, we may certainly apply \( \tilde{e}_i \) for some \( i \) with the change \( -\delta_{i0} \) on both sides of (4.7) from (3.7) and Lemma 3.6. Thus we may assume that \( v_1 \otimes \cdots \otimes v_m \) is a highest weight element. This implies \( v_1 = u_{\lambda_1} \) and that \( b_1 \otimes \cdots \otimes b_m \) is a minimal element of \( B_1^\otimes_L \otimes \cdots \otimes B_m^\otimes_L \), and therefore that \( b_1 = b_1 \). Write \( B_1^* = B_1 \), \( B_2^* = B_2 \otimes \cdots \otimes B_m^\otimes_L \), \( b_1^* = b_2 \otimes \cdots \otimes b_m \) and \( b_1 \otimes \cdots \otimes b_m \). We have

\[
D(b_1 \otimes b_1) - D(b_1 \otimes b_1) = D(b_1 \otimes b_1) - D(b_1 \otimes b_1)
\]

\[
= H_{B_1^* B_2^*}(b_1 \otimes b_1) - H_{B_1^* B_2^*}(b_1 \otimes b_1) + D(b_1^*) - D(b_1^*)
\]

\[
= D(b_1^*) - D(b_1^*)
\]

by (3.6) for two tensor factors, (A2), the fact that the order of \( \sigma_1 \) divides \( L \), and Lemma 4.10.

Noting \( \langle d, \text{wt} u_{\lambda_1} \rangle = 0 \) we have reduced (4.7) to the case that the first tensor factor is missing, and the theorem is proved by induction. \( \Box \)

5. Type A perfect crystals

In this section we review the perfect crystal \( B^{k,l}_n \) of type \( A_n^{(1)} \), given in [KMN2], and show for fixed \( k \) \((1 \leq k \leq n)\) that \( \{B^{k,l}_n\}_{l \geq 1} \) forms a coherent family of perfect crystals.
5.1. Crystal $B^{k,l}$

Let $B^{k,l}$ (1 $\leq k \leq n$, $l \geq 1$) be the perfect crystal of level $l$ of type $A_n^{(1)}$ given in [KMN2]. As a $U_q(A_n)$-crystal it coincides with the crystal base $B(l\Lambda_k)$ of the highest weight $U_q(A_n)$-module with highest weight $l\Lambda_k$, where $\Lambda_k$ is the $k$th fundamental weight of $A_n$. Therefore, $B^{k,l}$ can be identified with the set of semistandard tableaux of $k \times l$ rectangle shape over the alphabet $\{1, 2, \ldots, n+1\}$ as described in [KN]. Thus with each $b \in B^{k,l}$ one associates a table $(m_{jj'})$ for $1 \leq j \leq k$, $1 \leq j' \leq l$, such that $m_{jj'} \in \{1, 2, \ldots, n+1\}$, $m_{jj'} \leq m_{j,j'+1}$ and $m_{jj'} < m_{j+1,j}$. For our purpose we associate another table $x = x(b)$ with $b$, namely

$$x = (x_{ji})_{1 \leq j \leq k, i \leq j \leq j+k'}, \quad x_{ji} = \sharp \{j' | m_{jj'} = i\}. \quad (5.1)$$

Notice that $x_{ji}$ defined by (5.1) satisfies

$$x_{ji} = 0 \text{ unless } j \leq i \leq j + k'$$

due to the semistandardness of the tableau. In view of this we set $x_{ji} = 0$ when $1 \leq i < j$ or $j + k' < i \leq n + 1$. We also set $x_{0i} = x_{k+1,i} = 0$ (1 $\leq i \leq n + 1$) for convenience.

It is shown in [KMN2] that $B^{k,l}$ is perfect of level $l$. We give below its associated automorphism $\sigma = \sigma_k$, which depends only on $k$, and the minimal elements. Set $\sigma_k(\Lambda_i) = A_i \mod n+1$. $\sigma_k$ on $P_\Lambda$ is given by extending it $\mathbb{Z}$-linearly. Let $\lambda = \sum_{i=0}^{n} \lambda_i \Lambda_i$ be in $(P_\Lambda^+)_{\leq l}$, that is, $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{Z}_{\geq 0}$, $\sum_{i=0}^{n} \lambda_i = l$. The table $x(b)$ of the minimal element $b$ such that $\varepsilon(b) = \lambda$ is given by

$$x_{jj} = \lambda_0 + \sum_{a=0}^{k-1} \lambda_{a+k'},$$

$$x_{ji} = \lambda_{i-j} \quad (j < i < j + k'),$$

$$x_{j,j+k'} = \sum_{a=0}^{j-1} \lambda_{a+k'} \quad (5.2)$$

for $1 \leq j \leq k$.

5.2. Actions of $\tilde{e}_a$, $\tilde{f}_a$ ($a \neq 0$)

We give the actions of $\tilde{e}_a$, $\tilde{f}_a$ ($a \neq 0$) in terms of the coordinate $x(b) = (x_{ji})$. Set $\beta = \max(0, a - k')$, $\gamma = \min(k, a)$. For fixed $x = x(b)$ define

$$\Gamma(c) = \sum_{j=a+\beta}^{c-1} (x_{ja} - x_{j+1,a+1}) \quad (5.3)$$

for $\beta + 1 \leq c \leq \gamma$. Let $\Gamma_{\min}$ be the minimum of $\Gamma(c)$. Set
\[ c_0 = \min\{c \mid \beta + 1 \leq c \leq \gamma, \Gamma(c) = \Gamma_{\min}\}, \]
\[ c_1 = \max\{c \mid \beta + 1 \leq c \leq \gamma, \Gamma(c) = \Gamma_{\min}\}. \]

Then the values \( \varepsilon_a(b), \varphi_a(b) \) are given by
\[ \varepsilon_a(b) = c_0 - \sum_{j=\beta}^{c_0-1} (x_{j+1\cdot a+1} - x_{ja}), \tag{5.4} \]
\[ \varphi_a(b) = \gamma \sum_{j=c_1}^\gamma (x_{ja} - x_{j+1\cdot a+1}). \tag{5.5} \]

If \( \varepsilon_a(b) > 0 \), writing \( x' = x(\tilde{e}_a b) \) we have
\[ x'_{ji} = x_{ji} - \delta_{j\cdot \beta\cdot a+1} + \delta_{j_{c_0} \cdot \beta\cdot a}. \tag{5.6} \]

Also if \( \varphi_a(b) > 0 \), writing \( x' = x(\tilde{f}_a b) \) we have
\[ x'_{ji} = x_{ji} - \delta_{j\cdot c_1\cdot a} + \delta_{j_{c_1} \cdot a+1}. \tag{5.7} \]

These formulas are obtained by interpreting the rule on the tableau given in [KN] in terms of the coordinate \( x(b) \).

5.3. Actions of \( \tilde{e}_0, \tilde{f}_0 \)

The following method (which we shall call the KKMMNN method) for computing \( \tilde{e}_0 \) was stated in [KMN2, Proposition 6.3.11] but the proof was not included. Proofs of this method and (5.13) and (5.15) are given in Appendix A.

The set \( C \) of all sequences \( 1 = c_0 < c_1 < \cdots < c_{k-1} < c_k = n + 1 \) is a poset under the relation \( c \subseteq c' \) defined by \( c_i \leq c'_i \) for all \( 0 \leq i \leq k \). \( C \) contains a unique \( \subseteq \)-minimum element \( c_{\min} = (1, 2, \ldots, k, n + 1) \in C \) and a unique \( \subseteq \)-maximum element \( c_{\max} = (1, k' + 1, \ldots, n + 1) \) where \( k' = n + 1 - k \).

Fix \( x = x(b) \). Let \( \Delta_x = \Delta : C \to \mathbb{Z}_{\geq 0} \) be defined by
\[ \Delta(x) = \sum_{j=1}^{k} \sum_{c_{j-1} < \cdot < c_j} x_{ji}. \tag{5.8} \]

Consider the nonempty subposet \( C_{\min}^x = C_{\min}^x \) of \( C \) consisting of elements \( c \) such that \( \Delta(c) \) is minimal. The poset \( C_{\min}^x \) has a unique \( \subseteq \)-minimum element. For suppose not. Let \( c, c' \in C_{\min}^x \) be \( \subseteq \)-minima which are incomparable. Define \( c \land c' \in C \) and \( c \lor c' \in C \) be defined by
\[ (c \land c')_i = \min(c_i, c'_i), \]
\[ (c \lor c')_i = \max(c_i, c'_i). \]
for all $i$. It can be verified that

$$
\Delta(c \land c') - \Delta(c) = \Delta(c') - \Delta(c \lor c').
$$

(5.9)

Since $c$ and $c'$ are incomparable, $c \land c'$ is properly smaller than $c$. Thus the left-hand side of (5.9) is positive. But the positivity of the right-hand side contradicts $c' \in C_{\text{min}}$.

So for a given $x = x(b)$, there is a unique $c \in C$ such that

$$
\Delta(c) \leq \Delta(m) \quad \text{if} \quad m \supseteq c,
$$

(5.10)

$$
\Delta(c) < \Delta(m) \quad \text{if} \quad m \nsubseteq c
$$

(5.11)

for all $m \in C$. The formula of [KMN2] states that

1. $\tilde{e}_0 b = 0$ if and only if $c = c_{\text{min}}$ and $x_{ik} = 0$.
2. If $\tilde{e}_0 b \neq 0$, then writing $x' = x(\tilde{e}_0 b)$, one has

$$
x'_{ji} = x_{ji} - \delta_{i,cj} + \delta_{i,cj-1} \quad \text{for all} \quad 1 \leq j \leq k \quad \text{and} \quad 1 \leq i \leq n + 1.
$$

(5.12)

The value $\varepsilon_0(b)$ is given by

$$
\varepsilon_0(b) = l - x_{1,n+1} - \Delta(c).
$$

(5.13)

The formula for $\tilde{f}_0$ is similar. It can be shown as above, that for a given $x = x(b)$, there is a unique $c_{\text{max}}$-maximum element $c$ in $C_{\text{min}}$, that is, a unique $c \in C$ such that

$$
\Delta(c) \leq \Delta(m) \quad \text{if} \quad m \subseteq c,
$$

$$
\Delta(c) < \Delta(m) \quad \text{if} \quad m \nsubseteq c
$$

for all $m \in C$. Then $\tilde{f}_0 b$ is defined as follows.

1. $\tilde{f}_0 b = 0$ if and only if $c = c_{\text{max}}$ and $x_{1,k' + 1} = 0$.
2. If $\tilde{f}_0 b \neq 0$, then writing $x' = x(\tilde{f}_0 b)$, one has

$$
x'_{ji} = x_{ji} - \delta_{i,cj} + \delta_{i,cj-1} \quad \text{for all} \quad 1 \leq j \leq k \quad \text{and} \quad 1 \leq i \leq n + 1.
$$

(5.14)

One also has

$$
\varphi_0(b) = l - x_{11} - \Delta(c).
$$

(5.15)
5.4. Coherent family

We define the crystal $B^{k,\infty}$ by

$$B^{k,\infty} = \left\{ (v_{ji})_{1 \leq j \leq k, j \leq i \leq j+k'} \mid v_{ji} \in \mathbb{Z}, \sum_{i=j}^{j+k'} v_{ji} = 0 \text{ for any } j \right\}$$

with $b_{\infty} = (v_{ji} = 0 \text{ for any } j, i)$. We again set $v_{0i} = v_{k+1,j} = 0$ ($1 \leq i \leq n + 1$) for convenience. For $b = (v_{ji}) \in B^{k,\infty}$, $\varepsilon_a(b)$, $\varphi_a(b)$, $\tilde{e}_a b$, $\tilde{f}_a b$ for $a = 1, 2, \ldots, n$ are defined by the formulas (5.4)–(5.7) with $x_{ji}$ replaced by $v_{ji}$, but without the condition $\varepsilon_a(b) > 0$ before (5.6) and $\varphi_a(b) > 0$ before (5.7). For $\tilde{e}_0$ and $\tilde{f}_0$ use (5.12) and (5.14), but we warn that the formulas for $\varepsilon_0(b)$ and $\varphi_0(b)$ are modified as

$$\varepsilon_0(b) = -v_{k,n+1} - \Delta(c), \quad \varphi_0(b) = -v_{11} - \Delta(c). \quad (5.16)$$

Note that $\tilde{e}_a b$, $\tilde{f}_a b \neq 0$ for any $a = 0, 1, \ldots, n$ and $b \in B^{k,\infty}$. Clearly we have $\varepsilon_a(b_{\infty}) = \varphi_a(b_{\infty}) = 0$ for any $a = 0, 1, \ldots, n$.

We are to show that $B^{k,\infty}$ is a limit of $\{B^{k,l}\}_{l \geq 1}$ in the sense of Section 2.4. For $b_0 \in B^{k,l}_{\min}$ let $x(b_0) = (\xi_{ji})$ and $\varepsilon(b_0) = \lambda = \sum_{a=0}^{n} \lambda_a \Lambda_a$. See (5.2) for the explicit value of $\xi_{ji}$. Then $\varphi(b_0) = \sigma^{-1}_k \lambda = \sum_{a=0}^{k-1} \lambda_a + \sum_{a=k}^{n} \lambda_{a-k} \Lambda_a$. For $b \in B^{k,l}$ such that $x(b) = (x_{ji})$ we define the map

$$f_{(l,b_0)}: T_\lambda \otimes B^{k,l} \otimes T_{-\sigma^{-1}_k \lambda} \rightarrow B^{k,\infty}$$

by

$$f_{(l,b_0)}(t \otimes b \otimes t_{-\sigma^{-1}_k \lambda}) = b' = (v_{ji}),$$

where

$$v_{ji} = x_{ji} - \xi_{ji} \quad \text{for } 1 \leq j \leq k, \ j \leq i \leq j+k'.$$

Note that $f_{(l,b_0)}(t \otimes b_0 \otimes t_{-\sigma^{-1}_k \lambda}) = b_{\infty}$.

Let us show that $f_{(l,b_0)}$ is a morphism of crystals. For this purpose we prepare some properties of the functions $\Gamma$ (5.3) and $\Delta$ (5.8). Writing their $x$-dependence as $\Gamma_x$ and $\Delta_x$, one can check

$$\Gamma_x(c) = \Gamma_{x-\xi}(c),$$

$$\Delta_x(c) = \Delta_{x-\xi}(c) = \sum_{a=1}^{k'-1} \lambda_a.$$
Here $x - \xi$ means $(x_{ji})$ is replaced by $(x_{ji} - \xi_{ji})$. From these formulas it is straightforward to see that $f(l, b_0)$ satisfies (2.6). By (5.4), (5.5), (5.13), (5.15), (5.16) and Example 2.1, we also see that

$$
\varepsilon_a \left( T_\lambda \otimes B_{k,l} \otimes T_{-\sigma_{-1}^k} \right) = \varepsilon_a (b) - \langle h_a, \lambda \rangle = \varepsilon_a (b'),
$$

$$
\varphi_a \left( T_\lambda \otimes B_{k,l} \otimes T_{-\sigma_{-1}^k} \right) = \varphi_a (b) + \langle h_a, -\sigma_{-1}^k \lambda \rangle = \varphi_a (b')
$$

for $a = 0, 1, \ldots, n$.

Hence for any $b_0 \in B^{k,l}_{\min}$, $f(l, b_0) : T_\lambda \otimes B^{k,l} \otimes T_{-\sigma_{-1}^k} \to B^{k,\infty}$ is a morphism of crystals. By definition, it is clear that $f(l, b_0)$ is an embedding and that

$$
B^{k,\infty} = \bigcup_{(l, b_0)} \text{Im} f(l, b_0),
$$

Therefore, $B^{k,\infty}$ is the limit of the coherent family of perfect crystals $\{B^{k,l}\}_{l \geq 1}$.

Acknowledgments

The authors thank Masaki Kashiwara for stimulating discussions. M.O. thanks Goro Hatayama, Atsuo Kuniba and Taichiro Takagi for collaboration in the early stage of this work. M.O. is partially supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology. A.S. is partially supported by a Faculty Research Grant. M.S. is partially supported by the grant NSF DMS-0100918.

Appendix A. Formulas for $\tilde{e}_0$ and $\tilde{f}_0$

In this appendix we prove that two ways of computing $\tilde{e}_0 b$ for the perfect crystal $B^{k,l}$ of type $A^{(1)}_n$, one given by [KMN2] and the other by [S], are equivalent. The formulas for $\varepsilon_0, \varphi_0$ and $f_0$ are also proven.

A.1. Promotion method

In this subsection the method of [S] to compute $\tilde{e}_0$ is reviewed and suitably reformulated.

Let $b \in B^{k,l}$. The following algorithm to compute $\tilde{e}_0 b$ is easily seen to be equivalent to that in [S]. We shall use the notation $x$ interchangeably with $b$ when $x = x(b)$. Let $x_j$ denote the $j$th row. $x$ can also be identified with its row word $x_k x_{k-1} \cdots x_1$.

1. Remove the letters $n + 1$ from $x$, forming the subtableau $x|_{[n]}$. Since the shape of $x$ is the $k \times \ell$ rectangle, all of these letters $n + 1$ lie in the last row (the $k$th). Let $m = x_{k,n+1}$ be the number of such letters $n + 1$. 

(2) Slide \( x|_{[n]} \) to antinormal shape, obtaining the skew tableau \( z \) of shape \((\ell k)/(m)\). Let \( z_j \) be the \( j \)th row of \( z \).

(3) \( \tilde{e}_0 x \) is defined if and only if \( z_1 \) contains a letter 1.

(4) Suppose this holds. Let \( z' \) be obtained from \( z \) by removing the leftmost letter 1 in \( z_1 \).

(5) Slide the skew tableau \( z' \) to normal shape. It has shape \((\ell k - 1, \ell - m - 1)\).

(6) Append \( m + 1 \) letters \( n + 1 \) to the last row; the result is \( \tilde{e}_0 x \).

Note that in passing from \( b \) to \( \tilde{e}_0 b \), a letter 1 has been removed and a letter \( n + 1 \) added, with the other letters moving around.

It is useful to perform the sliding algorithms in a particular way. Define the row words (weakly increasing words) \( y_j \) for \( 1 \leq j \leq k \) as follows. Let \( y_k = (x_k)|_{[n]} \), that is, let \( y_k \) be obtained by removing the letters \( n + 1 \) from the last row \( x_k \) of \( x \). Then \( x|_{[n]} = y_k x_{k-1} \cdots x_2 x_1 \). Inductively (for \( j \) decreasing from \( k - 1 \) down to 1) let \( y_j \) be the unique row word of length \( \ell - m \) and \( z_{j+1} \) the unique row word of length \( \ell \), such that

\[
y_{j+1} x_j \equiv z_{j+1} y_j.
\]  

(A.1)

Also define \( z_1 = y_1 \). Then the tableau \( z \) defined above, is given by \( z = z_k \cdots z_2 z_1 \). The entries of \( y_j \) are given explicitly by the following relations.

\[
y_{ki} = x_{ki}
\]

\[
y_{kn+1} = 0
\]

\[
y_{ji} = \min \left( x_{ji}, \sum_{\alpha > i} (y_{j+1} \alpha - y_{j\alpha}) \right)
\]  

for \( 1 \leq j \leq k - 1 \).

(A.2)

The statements on \( y_k \) are immediate. To justify the equation for \( y_j \) with \( 1 \leq j \leq k - 1 \), note that (A.1) can be computed by a two-row jeu de taquin. Consider the two-row tableau \( y_{j+1} x_j \). We imagine that \( m \) letters are moving out of \( x_j \) to the other row, to create \( y_j \). The quantity \( y_{ji} \) (the number of letters \( i \) that remain in \( y_j \)) is the minimum of \( x_{ji} \) (the number of letters \( i \) that start out in \( x_j \)) and the number of letters in \( y_{j+1} \) that are available to block these letters \( i \) from moving to the other row. The latter quantity is the total number of letters in \( y_{j+1} \) that could block letters \( i \) (which is \( \sum_{\alpha > i} y_{j+1} \alpha \), the number of letters in \( y_{j+1} \) of value strictly greater than \( i \)), minus the number of letters in \( y_{j+1} \) that are already being used to block letters in \( x_j \) of value greater than \( i \). This last quantity is equal to the number of letters in \( x_j \) that are greater than \( i \) and not moving to the other row, which is \( \sum_{\alpha > i} y_{j, \alpha} \).

This given, it is immediate that

\[
z_{j+1, i} = y_{j+1, i} + x_{ji} - y_{ji}
\]  

for \( 1 \leq j \leq k - 1 \),

\[
z_{1, i} = y_{1, i}.
\]  

(A.3)

Continuing the computation of \( \tilde{e}_0 b \), one examines the first row \( y_1 = z_1 \). By the promotion method, one has

\[
\tilde{e}_0 (b) = y_{11}.
\]  

(A.4)
In particular,
\[ \tilde{e}_0 b \neq 0 \iff y_{11} > 0. \]  
(A.5)

Suppose \( \tilde{e}_0 b \neq 0 \). As prescribed,
\[ z'_{ji} = z_{ji} - \delta_{j,1} \delta_{i,1}. \]

Let \( x'_{ji} = x(\tilde{e}_0 b) \). Observe that one may use the same process to pass from \( \tilde{e}_0 b \) to \( z' \), as one does to pass from \( b \) to \( z \). Therefore, one may define \( y'_j \) as in (A.1) and \( y'_{ji} \) as in (A.2).

Then \( z' \) is related to \( x' \) via \( y' \) using (A.3).

We now prove by induction that there exists a unique sequence of values \( 1 = c_0 < c_1 < \cdots < c_k = n + 1 \) such that
\[ y'_j = y_j - \delta_{j,c_{j-1}}. \]  
(A.6)

It is true by definition for \( j = 1 \). By induction suppose \( c_0 < \cdots < c_{r-1} \) have been defined for some \( r \geq 1 \) and (A.6) holds for \( 1 \leq j \leq r \). Write \( y_r = uc_{r-1}v \) for some words \( u \) and \( v \) such that \( u \) does not contain \( c_{r-1} \). Then \( y'_r = uv \). Consider (A.1) for \( j = r \). It can be interpreted that one row inserts the letters of \( y_r \) from left to right, into the single-row tableau \( z_{r+1} \), to obtain the two-row tableau \( y_{r+1}x_r \). Analogous statements hold for the primed counterparts. By definition \( z'_{r+1} = z_{r+1} \). Clearly when inserting \( y'_r = uv \) into \( z'_{r+1} \) the subword \( u \) displaces the same letters as when \( y_r = uc_{r-1}v \) is inserted into \( z_{r+1} = z'_{r+1} \).

However, the insertion of \( c_{r-1} \) is skipped, and one proceeds to insert \( v \). It follows from the definition of row insertion, that the multiset of letters bumped by the insertion of the subword \( v \) of \( y'_r \), is a submultiset of the letters bumped by the insertion of the subword \( v \) of \( y_r \). The difference of these multisets is a single letter \( c_{r-1} \), which is greater than \( c_{r-1} \) since it must have been displaced by a letter in the subword \( c_{r-1}v \) of \( y_r \). Thus (A.6) holds for \( j = r \), finishing the induction.

Note that (A.6) implies that
\[ x'_{ji} = x_{ji} - \delta_{i,c_{j-1}} + \delta_{i,c_{j}}. \]  
(A.7)

Let \( \chi(P) \) be 0 or 1 according as the statement \( P \) is false or true. Note that \( y' \) is defined in terms of \( x' \) as in (A.2). Substituting (A.7) and (A.6) one has
\[ y_{ji} - \delta_{i,c_{j-1}} = \min \left( x_{ji} - \delta_{i,c_{j-1}} + \delta_{i,c_{j}}, \sum_{a > i} (y_{j+1,a} - \delta_{a,c_{j}}) - (y_{j,a} - \delta_{a,c_{j-1}}) \right) \]
\[ = \min \left( x_{ji} - \delta_{i,c_{j-1}} + \delta_{i,c_{j}}, \chi(c_{j-1} > i) - \chi(c_j > i) + \sum_{a > i} (y_{j+1,a} - y_{j,a}) \right). \]
Adding $\delta_{i,c_j-1}$ we have

$$y_{ji} = \min \left(x_{ji} + \delta_{i,c_j}, \chi (c_j - 1 \geq i) - \chi (c_j > i) + \sum_{\alpha > i} (y_{j+1,a} - y_{j,a}) \right). \quad (A.8)$$

In the case $c_{j-1} < i < c_j$, (A.8) becomes

$$y_{ji} = \min \left(x_{ji}, -1 + \sum_{\alpha > i} (y_{j+1,a} - y_{j,a}) \right).$$

Comparing with (A.2), it follows that

$$y_{ji} = x_{ji} < \sum_{\alpha > i} (y_{j+1,a} - y_{j,a}) \quad \text{for} \quad c_{j-1} < i < c_j. \quad (A.9)$$

In the case $i = c_j$, (A.8) becomes

$$y_{j,c_j} = \min \left(x_{j,c_j} + 1, \sum_{\alpha > c_j} (y_{j+1,a} - y_{j,a}) \right).$$

Comparing with (A.2), we have

$$y_{j,c_j} = \sum_{\alpha > c_j} (y_{j+1,a} - y_{j,a}). \quad (A.10)$$

In light of (A.9) and (A.10), it follows that the indices $c_j$ may be defined by the rule that

$$c_0 = 1 \quad (A.11)$$

for $j > 1$, $c_j$ is the minimum index $i > c_{j-1}$ such that

$$y_{j,i} = \sum_{\alpha > i} (y_{j+1,a} - y_{j,a}). \quad (A.12)$$

A.2. KKMMNN conditions imply promotion equations

Let $b \in B^{k,l}$, $x = x(b)$, and $c \in C$ such that (5.10) and (5.11) hold. Define $y$ in terms of $x$ by (A.2). It is shown in this subsection that $y$ must satisfy (A.9) and (A.10).

We will show by descending induction on $1 \leq j < k$ that

$$x_{j,c_j} \geq \sum_{\alpha > c_j} (y_{j+1,a} - y_{j,a}), \quad (A.13)$$

$$x_{j,m} < \sum_{\alpha > m} (y_{j+1,a} - y_{j,a}) \quad \text{for} \quad c_{j-1} < m < c_j. \quad (A.14)$$
By definition (A.2) this implies immediately that

\[ y_{j,c_j} = \sum_{\alpha > c_j} (y_{j+1,\alpha} - y_{j,\alpha}), \]  

(A.15)

\[ y_{j,m} = x_{j,m} \quad \text{for} \quad c_{j-1} < m < c_{j}, \]  

(A.16)

which also holds for \( j = k \) by definition.

Let us prove (A.13) and (A.14) for \( j \) assuming that (A.13)–(A.16) hold for all labels greater than \( j \). Define \( s \) and \( r_p \) for \( j \leq p \leq s \) as follows. Let \( r_p > r_{p-1} \) (\( r_j > c_j \) for \( p = j \)) be minimal such that

\[ y_{p,r_p} = \sum_{\alpha > r_p} (y_{p+1,\alpha} - y_{p,\alpha}), \]  

(A.17)

\[ r_p \geq c_{p+1} \] and \( r_s < c_{s+1} \). The minimality of \( r_p \) implies

\[ y_{p,m} = x_{p,m} \quad \text{for} \quad r_{p-1} < m < r_p \quad (c_j < m < r_j \text{ for } p = j). \]  

(A.18)

First we show by induction on \( p = j, j+1, \ldots, s \) that (A.13) is equivalent to

\[ \sum_{\beta = j+1}^{p} \sum_{c_{\beta-1} < \alpha < c_{\beta}} x_{\beta,\alpha} + \sum_{c_p < \alpha \leq r_p} y_{p+1,\alpha} \leq \sum_{c_j \leq \alpha < r_j} x_{j,\alpha} + \sum_{p}^{p} \sum_{\beta = j+1}^{p} \sum_{r_{\beta-1} < \alpha < r_{\beta}} x_{\beta,\alpha}. \]  

(A.19)

The case \( p = j \) follows immediately from (A.13) by using (A.17) and (A.18). Now assume that (A.19) holds for \( p - 1 \geq j \). The left-hand side of (A.19) at \( p - 1 \) can then be transformed as follows:

\[ \text{l.h.s.} = \sum_{\beta = j+1}^{p} \sum_{c_{\beta-1} < \alpha < c_{\beta}} x_{\beta,\alpha} + \sum_{c_p \leq \alpha \leq r_p-1} y_{p,\alpha} \]

\[ = \sum_{\beta = j+1}^{p} \sum_{c_{\beta-1} < \alpha < c_{\beta}} x_{\beta,\alpha} + \sum_{\alpha > c_p} y_{p+1,\alpha} - \sum_{\alpha > r_p-1} y_{p,\alpha} \]

\[ = \sum_{\beta = j+1}^{p} \sum_{c_{\beta-1} < \alpha < c_{\beta}} x_{\beta,\alpha} - \sum_{r_{p-1} < \alpha < r_p} x_{p,\alpha} + \sum_{c_p < \alpha \leq r_p} y_{p+1,\alpha}, \]

where the first equality follows from (A.16) at \( p \), for the second equality follows from (A.15) at \( p \), and for the last equality we used (A.17) and (A.18). Hence (A.19) holds at \( p \).

For \( p = s \), all \( y_{s+1,\alpha} \) in (A.19) can be replaced by \( x_{s+1,\alpha} \) by (A.16) since by assumption \( r_s < c_{s+1} \). Hence (A.19) with \( p = s \) is equivalent to

\[ \Delta(c) \leq \Delta(c_0, c_1, \ldots, c_{j-1}, r_j, \ldots, r_s, c_{s+1}, \ldots, c_k) \]

which is true by (5.10). This proves (A.13).
Next we prove (A.14) by induction on \( m \). By induction hypothesis we may assume that (A.14) holds for all indices greater than \( m \). Define \( s \) and \( r_p \) for \( j \leq p \leq s \) as follows. Set \( r_j = m \) and let \( r_{p-1} < r_p < c_p \) be minimal such that

\[
y_{p,r_p} = \sum_{\alpha > r_p} (y_{p+1,a} - y_{p,a})
\]

(A.20)

holds. Let \( s < k - 1 \) be minimal such that \( r_{s+1} \) does not exist; otherwise let \( s = k - 1 \). The minimality of \( r_p \) implies

\[
y_{p,m} = x_{p,m} \quad \text{for} \quad r_{p-1} < m < r_p.
\]

We prove by induction on \( p \) that (A.14) is equivalent to

\[
\sum_{p-1}^{p+1} \sum_{\beta = j}^{c_j-1} \sum_{c_j-1 < \alpha < c_p} x_{\beta,\alpha} < \sum_{c_j-1 < \alpha < c_j} x_{j,\alpha} + \sum_{r_{p-1} < \alpha < r_p} x_{p+1,\alpha} + \sum_{r_p < \alpha < c_p} y_{p+1,\alpha}.
\]

(A.21)

To see that this equivalence holds at \( p = j \), we use the induction hypothesis \( y_{j,\alpha} = x_{j,\alpha} \) for \( m < \alpha < c_j \) and (A.15). With this (A.14) reads

\[
\sum_{r_j \leq \alpha < c_j} x_{j,\alpha} < \sum_{r_j \leq \alpha < c_j} y_{j+1,\alpha}.
\]

Adding

\[
\sum_{c_j-1 < \alpha < r_j} x_{j,\alpha} + \sum_{r_j < \alpha < c_{j+1}} x_{j+1,\alpha} = \sum_{c_j-1 < \alpha < c_j} x_{j,\alpha} + \sum_{c_j < \alpha < r_{j+1}} y_{j+1,\alpha}
\]

to both sides yields (A.21) at \( p = j \). Now assume that (A.21) holds for \( p - 1 \geq j \). The last term in this inequality can be rewritten as

\[
\sum_{r_{p-1} < \alpha < c_p} y_{p,\alpha} = \sum_{r_{p-1} < \alpha < r_p} y_{p,\alpha} + \sum_{\alpha > r_p} (y_{p+1,\alpha} - y_{p,\alpha}) + \sum_{r_p < \alpha < c_p} y_{p,\alpha}
\]

\[
= \sum_{r_{p-1} < \alpha < r_p} x_{p,\alpha} + \sum_{\alpha > r_p} y_{p+1,\alpha} - \sum_{\alpha > c_p} y_{p,\alpha}
\]

\[
= \sum_{r_{p-1} < \alpha < r_p} x_{p,\alpha} + \sum_{r_p < \alpha < c_p} y_{p+1,\alpha}.
\]

Using this and adding \( \sum_{c_p < \alpha < c_{p+1}} x_{p+1,\alpha} = \sum_{c_p < \alpha < c_{p+1}} y_{p+1,\alpha} \) to both sides of (A.21) at \( p - 1 \) yields (A.21) at \( p \).

At \( p = s \) the last term in (A.21) can be replaced by \( \sum_{r_s < \alpha < c_{s+1}} x_{s+1,\alpha} \). For if \( s < k - 1 \) then there is no \( r_{s+1} < c_{s+1} \) such that (A.20) holds, and if \( s = k - 1 \) then by (A.2), \( x_{k,\alpha} = y_{k,\alpha} \) for all \( 1 \leq \alpha \leq n \). Hence (A.21) at \( p = s \) is equivalent to
\[ \Delta(c) < \Delta(c_0, \ldots, c_{j-1}, r_j, \ldots, r_k, c_{j+1}, \ldots, c_k) \]

which is true by (5.11) since \( c_{j-1} < r_j < c_j \).

### A.3. Equivalence of the two methods for \( \tilde{e}_0 \)

Fix \( b \in B_k^\ell \) and let \( x = x(b) \). Let \( c \) be defined as in (5.10) and (5.11). Let \( y \) be defined by (A.2). By Section A.2, \( y \) satisfies (A.9) and (A.10). We show that the KKMMNN and promotion methods of computing \( \tilde{e}_0 b \), are equivalent.

First, we show the equivalence of the conditions given in Sections 5.3 and A.1, that \( \tilde{e}_0 b \neq 0 \). Explicitly, we show that \( c \neq c_{\min} \) or \( x_{k,k} > 0 \), if and only if \( y_{11} > 0 \).

Recall that \( y_1 \) has length \( \ell - x_{k,n+1} = \sum_{i \geq 1} y_{1i} \). Applying (A.9) and (A.10) repeatedly, we have

\[ \ell - x_{k,n+1} - y_{11} = \sum_{i > 1} y_{1i} \]
\[ = \sum_{c_0 < i < c_1} x_{1i} + \sum_{i > c_1} y_{2i} \]
\[ = \sum_{c_0 < i < c_1} x_{1i} + \sum_{c_1 < i < c_2} x_{2i} + \sum_{i > c_2} y_{2i} \]
\[ = \cdots = \Delta(c). \] (A.22)

By the definition of \( c_{\min} \), \( \Delta(c_{\min}) = \sum_{i=k+1}^n x_{ki} \). The \( k \)th row of \( x \) has length \( \ell \). By semistandardness it can only have elements in the set \( \{ k, k+1, \ldots, n+1 \} \). Therefore, \( \Delta(c_{\min}) = \ell - x_{kk} - x_{k,n+1} \). By (A.22),

\[ y_{11} = \ell - x_{k,n+1} - \Delta(c) = x_{kk} + \Delta(c_{\min}) - \Delta(c). \] (A.23)

In light of (5.11) the desired equivalence is evident.

We are now left to show that both methods agree provided that \( \tilde{e}_0 b \neq 0 \). Assume \( \tilde{e}_0 b \neq 0 \). Let \( c' \in C \) be the sequence defined by the promotion method. Then \( y \) (which is defined only in terms of \( x \)) satisfies (A.9) and (A.10) for both \( c \) and \( c' \). It follows that \( c = c' \) so that the two methods agree.

### A.4. Proof of (5.13) and (5.15)

Equation (5.13) follows immediately from (A.22), (A.4), and the equivalence of the KKMMNN and promotion methods for \( \tilde{e}_0 \).

Equation (2.1) implies

\[ \psi_0(b) - \varepsilon_0(b) = \langle h_0, \text{wt } b \rangle = x_{k,n+1} - x_{11}. \] (A.24)
Equation (5.15) follows from this and (5.13). Note that although different sequences \( c \in C_{\text{min}}^k \) are used in the definition of \( \tilde{e}_0 \) and \( f_0 \), they both attain the same minimum value of the function \( \Delta \).

A.5. Equivalence of the two methods for \( \tilde{f}_0 \)

The promotion definition of \( \tilde{f}_0 \) is essentially the inverse of that of \( \tilde{e}_0 \). However, the KKMMNN definition of \( \tilde{f}_0 \) is not defined so as to be obviously equal to the inverse of \( \tilde{e}_0 \).

Let \( b' \in B^k \) be such that \( \tilde{f}_0 b' = b \neq 0 \). Then of course \( b' = \tilde{e}_0 b \). Write \( x' = x(b') \) and \( x = x(b) \). Let \( c \) be the unique \( \subseteq \)-minimum element of \( C_{\text{min}}^k \). Then \( x' \) and \( x \) are related as in (5.12).

It must be shown that the KKMMNN method for \( \tilde{f}_0 \) sends \( b' \) to \( b \). Let \( c' \) and \( c'' \) be the unique \( \subseteq \)-maximum and minimum elements of \( C_{\text{min}}^{k'} \), respectively; in particular \( \Delta_{x'}(c') = \Delta_{x'}(c'') \). By (5.14) it must be shown that \( c' = c \).

To prove (1) for \( \tilde{f}_0 b' \), note that \( \ell \) is the length of the first row of \( b' \) and that \( x'_i = 0 \) for \( i > k' + 1 \) by semistandardness. The definition of \( c_{\text{max}} \) gives

\[
\ell = \sum_{i=1}^{k'+1} x'_i = x_{11} + \Delta_{x'}(c_{\text{max}}) + x'_{1,k'+1}.
\]

By (5.15) we have

\[
\varphi_0(b') = x'_{1,k'+1} + \Delta_{x'}(c_{\text{max}}) - \Delta_{x'}(c').
\]

From this and the definition of \( c' \in C_{\text{min}}^{k'} \) it is seen that the conditions for \( \tilde{f}_0 b' = 0 \) given by the two methods coincide.

By (5.13) applied to both \( b \) and \( b' \) we have

\[
\Delta_x(c) = -\varepsilon_0(b) + \ell - x_{k,n+1} = -\varepsilon_0(b') + 1 + \ell - (x'_{k,n+1} - 1) = \Delta_{x'}(c') = \Delta_{x'}(c'').
\]

(A.25)

By (5.12) it follows that for all \( m \in C \)

\[
\Delta_{x'}(m) = \Delta_x(m) + \sum_{j=1}^{k} \left( \chi(m_{j-1} < c_j < m_j) - \chi(m_{j-1} < c_{j-1} < m_j) \right). \quad (A.26)
\]

Evaluating (A.26) at \( m = c \) and applying (A.25) we have \( \Delta_{x'}(c) = \Delta_{x'}(c') \). Evaluating (A.26) at \( m \supseteq c \) such that \( m \neq c \), we see that \( \Delta_{x'}(m) \geq \Delta_x(m) > \Delta_x(c) \) by (5.11). In other words, \( c \) is the unique \( \subseteq \)-maximum element of \( C_{\text{min}}^{k'} \), that is, \( c = c' \).
References


