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On Positive Solutions of Some Pairs of Differential Equations, II

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This paper is a sequel to the author's paper [6], which studied positive solutions of a pair of differential equations introduced by Conway, Gardner, and Smoller [4]. Their equations were equations for population problems in biology. Their significance is discussed further in [4]. The purpose of the present paper is to obtain further results for the Conwat–Gardner–Smoller equations and to show how the methods in [6] can be modified to decide exactly when the classical predator-prey system (as in Blat and Brown [2]) has a strictly positive solution. We also show how the methods can be applied to a number of other problems. In particular, our results answer some questions in Conway [5].

Secondly, we show how the asymptotic methods developed in [6] can be used to study the existence and uniqueness of strictly positive solutions of competing species problems. Our results suggest that this is a much more complicated problem than the predator–prey problem. (We do obtain a condition which is, except for a few special cases, necessary and sufficient for the existence of a strictly positive solution. However, the condition is complicated and rather implicit.)

Thirdly, we prove the uniqueness of the strictly positive solution in the Conway–Smoller system if d is small and $n = 1$ (that is, the case of ordinary differential equations). This partially answers a question in [4]. Our proof is a local result and can be used in other situations. For example, it can be used to show that, in certain cases of "asocial" nonlinearities (where our notation follows [4, Sect. 4]), there are exactly two strictly positive solutions for small d. We also show how our method applies to more general predator–prey problems.

In Section 1, we find necessary and sufficient conditions for the existence of strictly positive solutions of predator–prey systems; in Section 2, we study competing species models, by iteration and asymptotic methods. Finally, in Section 3, we prove our uniqueness results. In an appendix, we briefly discuss how the methods of Section 3 apply to more general models.

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1. Existence of Strictly Positive Solutions of Predator–Prey Systems

Consider the system

$$\begin{aligned} -\Delta u &= u(a-bu-cv) \\ -\Delta v &= v(e+fu-gv) \end{aligned} \tag{1}$$

on a smooth bounded domain Ω with Dirichlet boundary conditions. Here b, c, d, g are positive constants and a, e are constants. This is a predator–prey system (compare Blat and Brown [2]). We obtain necessary and sufficient conditions for the system to have a strictly positive solution (that is, a solution (u, v) where u and v are non-negative on Ω and neither vanishes identically). Then (cp. [6, Sect. 1]) it follows that, if (u, v) is such a solution, $u(x)>0$ and $v(x)>0$ in Ω. As in [6, Sect. 1], let $\bar{u}$ denote the maximal non-negative solution of the first equation (for Dirichlet boundary conditions) when $v \equiv 0$. As there, $\bar{u} \equiv 0$ if $a \leqslant \lambda_1$, while $\bar{u}(x)>0$ on Ω if $a>\lambda_1$. Here λ_1 is the first eigenvalue of $-\Delta$ on Ω (for Dirichlet boundary conditions). Moreover, as in [6], $\bar{u}$ is the unique non-negative non-trivial solution of the first equation (for $v \equiv 0$) if $a>\lambda_1$. We define $\bar{v}$ analogously by considering the second equation ($\bar{v}$ is non-trivial if and only if $e>\lambda_1$). As in [6], we write $r(-\Delta^{-1}(a-c\bar{v})\, I)>1$ (<1) to mean that the spectral radius of $(-\Delta+KI)^{-1}(a-c\bar{v}+K)I$ has spectral radius greater than 1 (less than 1) for sufficiently large positive K such that $a-c\bar{v}(x)+K \geqslant 0$ in Ω. (Here $(-\Delta+KI)^{-1}$ means the inverse under Dirichlet boundary conditions.) As in [6], the above spectral radius condition is independent of K. *We will use this notation throughout this paper.*

THEOREM 1 (i) *Assume that* $\bar{v} \equiv 0$. *Then* (1) *has a strictly positive solution if and only if* (a) $\bar{u} \neq 0$ (*that is,* $a>\lambda_1$) *and* (b) $r(-\Delta^{-1}(e+f\bar{u})I)>1$.

(ii) *Assume that* $\bar{v} \neq 0$. *Then* (1) *has a strictly positive solution if and only if* (a) $\bar{u} \neq 0$ *and* (c) $r(-\Delta^{-1}(a-c\bar{v})I)>1$.

Proof. We first prove the conditions are necessary. If (u, v) is a strictly positive solution, then by the first equation

$$-\Delta u < u(a-bu), \tag{2}$$

where, as in [6], $s<t$ means that $s(x) \leqslant t(x)$ in Ω with strict inequality on a set of positive measure. (Remember that $u(x)>0$ and $v(x)>0$ in Ω.) It follows from Lemma 1(iii) in [6] that $\bar{u} \geqslant u$. In particular, $\bar{u} \neq 0$. (In fact

$\bar{u} > u$ because of the strict inequality in (2).) Thus (a) is necessary. Since $\bar{u} > u$ and $v(x) > 0$ in Ω, the second equation implies that

$$-\Delta v < v(e + f\bar{u}). \tag{3}$$

We can now argue in the necessity part of the proof of Theorem 1 in [6] to deduce that $r(-\Delta^{-1}(e + f\bar{u})I) > 1$. Thus (b) is necessary. Assume now that $\bar{v} \neq 0$. By the second equation,

$$-\Delta v > v(e - gv). \tag{4}$$

Thus v is a supersolution of the second equation when $u \equiv 0$. It follows easily that $v \geqslant \bar{v}$. To see this, we note that, by standard arguments (cp. Amann [1]), the sequence v_n defined by $v_1 = v$

$$-\Delta v_n + Kv_n = v_{n-1}(e - gv_{n-1} + K) \tag{5}$$

(where K is large) *decreases* to a non-negative solution $\tilde{v}$ of the second equation with $u \equiv 0$. It is easy to see that the condition that $e > \lambda_1$ ensures that $\tilde{v} \neq 0$. (One way is to take the scalar product of (5) with the first eigenfunction of $-\Delta$ and note that

$$v_{n-1}(e - gv_{n-1} + K) - (K + \lambda_1)\, v_n \geqslant v_n(e - gv_{n-1} + K) - (K + \lambda_1)\, v_n > 0$$

in Ω if n is large and $\tilde{v} = 0$.) Hence by uniqueness, $\tilde{v} = \bar{v}$. Hence, since $v \geqslant \tilde{v}$, $v \geqslant \bar{v}$. By the strict inequality in (4) $v \neq \bar{v}$ and hence $v > \bar{v}$. Hence, by the first equation,

$$-\Delta u < u(a - c\bar{v}).$$

As before, it follows that $r(-\Delta^{-1}(a - c\bar{v})I) > 1$. Thus (c) is necessary if $\bar{v} \neq 0$.

We now prove the sufficiency. This is only a slight modification of the sufficiency result in [6, Theorem 1]. Thus we point out the differences. (If $e = \lambda_1$, there is an extra difficulty which we return to at the end of the proof.) The result is proved by using degree theory with respect to the cone C of non-negative functions in $E = C_0(\bar{\Omega}) \oplus C_0(\bar{\Omega})$. We define a map A as in [6]. First note that by using the first equation and by considering where u has its maximum, we see that $u(x) \leqslant ab^{-1}$ in Ω. By then similarly considering the second equation, we find that $v(x) \leqslant g^{-1}(e + fab^{-1})$ in Ω. (Similar arguments appear in [6].) Using these a priori bounds, we easily see that the sum of the indices of solutions (in C) is 1. (To see this, one notes that the a priori bounds are unchanged if all the coefficients are multiplied by τ where $0 < \tau \leqslant 1$. Thus, by homotopy invariance, it suffices to prove this for τ small. By our earlier arguments, we easily see that (0, 0) is the only solution in C if τ is small. As in [6], we easily see that $(0, 0)$ has index 1 if τ is small and the result follows.) Moreover, since $r(-\Delta^{-1}(e+f\bar{u})I)>1$, we see as in [6] that $(\bar{u}, 0)$ is an isolated solution of index zero. (Note that $(-\Delta)^{-1}(e+f\bar{u})I \geqslant (-\Delta)^{-1}eI$ in the positive operator sense and thus $r((-\Delta)^{-1}(e+f\bar{u})I) \geqslant r((-\Delta)^{-1}eI)>1$ if $e>\lambda_1$.) Similarly, if $\bar{v} \neq 0$, our condition that $r(-\Delta^{-1}(a-c\bar{v})I)>1$ ensures that $(0, \bar{v})$ has index zero. As in [6], our assumption that $a>\lambda_1$ ensures that $r(A'(0,0))>1$. Thus, provided that the equation $x=A'(0,0)x$ has no solution in C, it will follow as in [6] that $(0, 0)$ also is isolated in C and has index 0. An easy calculation (similar to one in [6]) shows that this condition holds if $e \neq \lambda_1$. Thus, if all our necessary conditions hold, and if $e \neq \lambda_1$, then $(0, 0)$, $(\bar{u}, 0)$, and $(0, \bar{v})$ (when it exists) are all isolated solutions of index 0. Since the sum of the indices of solutions is 1 (in C), there must be another solution, as required.

It remains to remove the condition that $e \neq \lambda_1$. Assume $e=\lambda_1$. Then $\bar{v} \equiv 0$. If we replace e by $e-1/n$, it is easy to see that our necessary conditions for the existence of a strictly positive solution still hold. (Note that $\bar{v}$ will still be zero and that the spectral radius of a compact linear operator changes continuously under small perturbation, by Kato [11, Theorem 4.3.1 and Sect. 4.3.5].) Thus, by what we have already proved, (1) has a strictly positive solution (u_n, v_n) if e is replaced by $e-1/n$. By our earlier bounds, $\{(u_n, v_n)\}_{n=1}^{\infty}$ is bounded in $C_0(\bar{\Omega}) \oplus C_0(\bar{\Omega})$. Thus, by standard compactness arguments, we can choose a subsequence of $\{(u_n, v_n)\}_{n=1}^{\infty}$ converging to a solution $(\tilde{u}, \tilde{v})$ of (1) (for the original e). Either $(\tilde{u}, \tilde{v})$ is a strictly positive solution or $(\tilde{u}, \tilde{v})=(0,0)$ or $(\bar{u}, 0)$. We eliminate the last two possibilities. If $(u_n, v_n) \to (0,0)$ in E (or more strictly a subsequence does), then (cp. [6]) the first equation implies that $r(-\Delta^{-1}(a-bu_n-cv_n)I)=1$. (This follows because u_n is a positive eigenfunction of $(-\Delta+KI)^{-1}$ $(a-bu_n-cv_n+K)I$ corresponding to the eigenvalue 1.) By taking the limit as $n \to \infty$ and by using the continuity of the spectral radius, we see that $r(-\Delta^{-1}a)=1$. This is impossible since $a>\lambda_1$, by assumption. If $(u_n, v_n) \to (\bar{u}, 0)$, we can apply a similar argument to the second equation to deduce that $r(-\Delta^{-1}(e+f\bar{u})I)=1$. Since this contradicts our assumptions, we have completed the proof.

Remarks. (1) The argument in the last part can be used to prove that $(0, 0)$ is an isolated solution in C of index 0 even if $e=\lambda_1$. As in [6], our spectral radius assumptions are equivalent to stability conditions. For example, $r(-\Delta^{-1}(e+f\bar{u})I)<1$ if and only if every eigenvalue of the linearization of (1) at $(\bar{u}, 0)$ has negative real part.

(2) The above results and especially the necessity conditions hold for much more general problems. Suppose we replace $a-bu-cv$ by $M(u, v)$ and $e+fu-gv$ by $N(u, v)$ and assume that M, N are C^1, $M_1'(u, v)<0$,

$M_2'(u, v) < 0$, $N_2'(u, v) < 0$, and $N_1'(u, v) > 0$ for $u, v > 0$, $M(u, v) < 0$ if $u \geqslant u_0$ and $v > 0$ and $N(u, v) < 0$ if $0 < u \leqslant u_0$ and v is large. Then the natural analogue of Theorem 1 holds. (The condition that $r(-\Delta^{-1}(e + f\bar{u})I) > 1$ becomes $r(-\Delta^{-1}N(\bar{u}, 0)I) > 1$ and the other conditions are modified accordingly.) The necessity conditions hold under some weaker assumptions. If $\bar{v} \equiv 0$, we could replace the assumption that $M_2'(u, v) < 0$ by $M(u, v) < M(0, v)$ and delete the assumption that $M_1'(u, v) < 0$ provided we define $\bar{u}$ to be the maximal non-negative solution of the first equation when $v \equiv 0$. ($\bar{u}$ is obtained by a suitable iteration starting from $\tilde{u}(x) = u_0$ on Ω.) As the case of the non-linearity $M(u, v) = a(1-u)(u-b) - v$, $N(u, v) = -v + m(u - \gamma)$ discussed in [6] shows, the necessary conditions need no longer be sufficient and the existence of a strictly positive solution becomes a more complicated problem. However, our index methods can be used to get partial results. Our methods can be used to answer or partially answer many of the questions in Conway [5].

In Section 2, we will see that competing-species equations are rather more complicated.

2. Competing-Species Models

In this section, we consider the simplest competing-species model:

$$\begin{aligned} -\Delta u &= u(a - bu - cv) \\ -\Delta v &= v(e - fu - gv) \end{aligned} \tag{6}$$

on a smooth bounded domain Ω with Dirichlet boundary conditions. Here $a, b, c, e, f, g > 0$. The main purpose of this section of this section is to point out that competing-species models are much more complicated than predator–prey models. We first obtain an "almost" necessary and sufficient condition for the existence of a strictly positive solution. However, the condition we obtain seems too implicit and complicated to be useful. Secondly, we use the asymptotic methods of [6] to show that the most obvious conditions for existence do not give the complete answer. We also use this method to show that uniqueness is not at all trivial.

Firstly, we deduce $\bar{u}$ and $\bar{v}$ as in Section 1. (Thus $\bar{u}$ is the maximal non-negative solution of the first equation when $v \equiv 0$.) As in Section 1, $\bar{u} \equiv 0$ if and only if $a \leqslant \lambda_1$ and $\bar{v} \equiv 0$ if and only if $e \leqslant \lambda_1$. By similar arguments to Section 1 (cp. [6, Sect. 2]), $\bar{u} \neq 0$ and $\bar{v} \neq 0$ are necessary conditions for the existence of a strictly positive solution. *We suppose that these conditions both hold.* It is shown in [6, Sect. 2] (by similar arguments to those in Section 1 here) that (6) has a strictly positive solution if $r(-\Delta^{-1}(a - c\bar{v})I)$ and $r(-\Delta^{-1}(e - f\bar{u})I)$ are both less than 1 or both greater than 1. Thus the major case left to study is when one of these spectral radii is less than 1 and one is greater than 1. (There is also the non-generic case where one of the two spectral radii is equal to 1. We do not study this case.) Assume that $r(-\Delta^{-1}(a-c\bar{v})I)>1$ and $r(-\Delta^{-1}(e-f\bar{u})I)<1$. The other case is similar. We first obtain necessary conditions for the existence of a strictly positive solution and then prove they are sufficient. *Assume that* $(\tilde{u}, \tilde{v})$ *is a strictly positive solution.* By the first equation,

$$-\Delta\tilde{u}<\tilde{u}(a-b\tilde{u}).$$

As in Section 1, it follows that $\tilde{u}<\bar{u}$. Similarly, $\tilde{v}<\bar{v}$. Hence, by the first equation of (6)

$$-\Delta\tilde{u}>\tilde{u}(a-b\tilde{u}-c\bar{v}). \tag{7}$$

Thus $\tilde{u}$ is a supersolution for the equation

$$-\Delta u=u(a-bu-c\bar{v}) \tag{8}$$

(with Dirichlet boundary conditions). Now since $r(-\Delta^{-1}(a-c\bar{v})I)>1$, we can argue as in [6, Sect. 1 or 3] to deduce that (8) has a unique non-negative non-trivial solution u_1. Since $\tilde{u}$ is a supersolution it follows as in the proof of Theorem 1 that $u_1<\tilde{u}$. Since $\tilde{v}(x)>0$ in Ω and $\tilde{u}>u_1$, the second equation of (6) implies that

$$-\Delta\tilde{v}<\tilde{v}(e-fu_1).$$

As in the proof of Theorem 1 in [6], it follows that $r(-\Delta^{-1}(e-fu_1)I)>1$. This gives an extra necessary condition for the existence of a strictly positive solution. Since $\tilde{u}>u_1$, the second equation implies that

$$-\Delta\tilde{v}<\tilde{v}(e-fu_1-g\tilde{v}).$$

Hence $\tilde{v}$ is a subsolution of the equation

$$-\Delta v=v(e-fu_1-gv). \tag{9}$$

Hence $\tilde{v}<v_1$ where v_1 is the unique non-trivial non-negative solution of (9). By similar arguments as before, $v_1<\bar{v}$. We now continue the same process inductively. Since $v_1<\bar{v}$,

$$(-\Delta+KI)^{-1}(a+K-cv_1)I\geqslant(-\Delta+KI)^{-1}(a+K-c\bar{v})I$$

in the positive operator sense for large K. Thus, by a standard result for compact positive linear operators (proved by combining [13, Theorem 2.5] and [18, p. 265])

$$r((-\Delta+KI)^{-1}(a+K-cv_1)I)\geqslant r((-\Delta+KI)^{-1}(a+K-c\bar{v})I)>1$$

by our earlier assumption. (Note that, as we commented earlier, whether the spectral radius is greater than 1 is independent of K for large K.) Hence, as before the equation

$$-\Delta u = u(a - bu - cv_1)$$

has a unique non-negative non-trivial solution u_2. Since

$$-\Delta u_1 = u_1(a - bu_1 - c\bar{v}) < u_1(a - bu_1 - cv_1),$$

u_1 is a subsolution and thus, as before, $u_1 < u_2$. Similarly, $\tilde{u}$ is a supersolution and hence $u_2 < \tilde{u}$. Since $u_2 < \tilde{u}$, we see by the equation for $\tilde{v}$ that

$$-\Delta \tilde{v} < \tilde{v}(e - fu_2).$$

As before, it follows that $r(-\Delta^{-1}(e - fu_2)I) > 1$. This gives a second necessary condition for the existence of a strictly positive solution. We define v_2 to be the unique non-negative non-trivial solution of

$$-\Delta v = v(e - fu_2 - gv).$$

Since $\tilde{v}$ is a subsolution and v_1 is a supersolution, v_2 exists and $\tilde{v} < v_2 < v_1$. We continue inductively. Define inductively u_n to be the unique non-negative non-trivial solution of

$$-\Delta u = u(a - bu - cv_{n-1}).$$

As before, we find that $u_{n-1} \leqslant u_n \leqslant \tilde{u}$ and that $r(-\Delta^{-1}(e - fu_n)I) > 1$ is a necessary condition for the existence of a strictly positive solution. If this holds, we define v_n to be the unique non-negative non-trivial solution of

$$-\Delta v = v(e - fu_n - gv).$$

As before, we find that $\tilde{v} \leqslant v_n \leqslant v_{n-1}$. Thus we can continue the process and find sequences $u_1 < u_2 < u_3 \cdots < u_n \cdots < \tilde{u}$ and $v_1 > v_2 \cdots > v_n > \cdots > \tilde{v}$ *such that*

$$r(-\Delta^{-1}(e - fu_n)I) > 1 \tag{10}$$

for all n is necessary condition for the existence of a strictly positive solution. Note that we have nowhere used $r(-\Delta^{-1}(e - f\tilde{u})I) < 1$. (In fact, if $r(-\Delta^{-1}(e - f\tilde{u})I) \geqslant 1$, it is easy to see that (10) holds automatically.) Now the construction of the sequence (u_n, v_n) is independent of the existence of $(\tilde{u}, \tilde{v})$. Thus, by our earlier argument, provided that (10) holds, we construct a sequence (u_n, v_n) such that $0 < u_n < u_{n+1} < \bar{u}$ and $\bar{v} > v_n > v_{n+1} > 0$ for all n. Since $\{(u_n, v_n)\}_{n=1}^{\infty}$ is bounded in $C(\bar{\Omega}) \oplus C(\bar{\Omega})$, it follows easily from the equation for (u_n, v_n) that the sequence is bounded in $W^{2,p}(\Omega)$ for all p. Thus by standard arguments, we easily find that $u_n \to \hat{u}$ and $v_n \to \hat{v}$ in $C(\bar{\Omega})$ strongly and in $W^{2,p}(\Omega)$ weakly. By passing to the limit as $n \to \infty$ in the equation for (u_n, v_n), we find that $(\hat{u}, \hat{v})$ is a solution of (6). We want to prove that $(\hat{u}, \hat{v})$ is a strictly positive solution. Since $0 < u_n < u_{n+1}$, the only way this can fail is if $\hat{v} = 0$. Hence, by the equations, $\hat{u} = \bar{u}$. Now, by the equation for v_n, $r(-\Delta^{-1}(e - fu_n - gv_n)I) = 1$. By using the continuity of the spectral radius and by passing to the limits as $n \to \infty$, we see that $r(-\Delta^{-1}(e - f\bar{u})I) = 1$. Since this contradicts our assumptions, we have completed the proof. *Hence we see that*, if $r(-\Delta^{-1}(a - c\bar{v})I) > 1$ *and if* $r(-\Delta^{-1}(e - f\bar{u})I) < 1$, *then* (6) *has a strictly positive solution if and only if* $r(-\Delta^{-1}(e - fu_i)I) > 1$ for all i. It is a consequence of our work below that these last conditions sometimes hold and sometimes do not.

The above argument has one other useful consequence. Assume that $r(-\Delta^{-1}(e - f\bar{u})I) \neq 1$. If a strictly positive solution exists, then there is one $(\hat{u}, \hat{v})$ such that $\tilde{u} > \hat{u}$ and $\tilde{v} < \hat{v}$ for any other strictly positive solution $(\tilde{u}, \tilde{v})$. Here $(\hat{u}, \hat{v})$ is the limit of (u_n, v_n). The result follows because we showed in the previous paragraph that $u_n < \tilde{u}$ and $v_n > \tilde{v}$. This result may be helpful for proving uniqueness. Note that it is not surprising because it can be shown that our equation can be written as an increasing map (for $u, v \geq 0$) for the order generated by the cone $\{(u, v) \in C_0(\bar{\Omega}) \oplus C_0(\bar{\Omega}) : u \geq 0, v \leq 0\}$. Note also that our iteration differs from the one in Pao [16] but is similar to one in Leung [19].

Our iteration and comparison theorems for scalar parabolic equations can also be used to prove that if $r(-\Delta^{-1}(a - c\bar{v})I)$ and $r(-\Delta^{-1}(e - f\bar{u})I)$ are both less than 1, and if the strictly positive solution is unique, then a solution of the corresponding parabolic system approaches the strictly positive solution as $t \to \infty$ if its initial value $(\tilde{u}_0, \tilde{v}_0)$ is non-negative and neither component vanishes identically. Moreover, if Ω is a disk we see by similar arguments that it suffices to prove that the *radially symmetric* strictly positive solution is unique.

Secondly for this section, we study asymptotic problems. We consider

$$\begin{aligned} -\Delta u &= u(a - bu - cv) \\ -\Delta v &= d^{-1}v(e - fu - gv), \end{aligned} \tag{11}$$

where $d > 0$ and Ω, a, b, c, e, f, g, and the boundary conditions are as before. (In other words, we look at (6) where e, f, g are replaced by $d^{-1}e$, $d^{-1}f$, $d^{-1}g$, respectively.) We study limiting behaviour as $d \to 0$. Note that, by the same arguments as before, $\tilde{u}(x) \leq b^{-1}a$ and $\tilde{v}(x) \leq g^{-1}e$ for any non-negative solution $(\tilde{u}, \tilde{v})$ of (11). We also need to consider the limit equation

$$-\Delta u = uk(u) \tag{12}$$

with Dirichlet boundary condition, where

$$\begin{aligned} k(y) &= a - by - cg^{-1}(e - fy)^{+} \\ &= (a - cg^{-1}e) - (b - cg^{-1}f)\,y \qquad \text{if } y \leqslant ef^{-1} \\ &= a - by \qquad \text{if } y \geqslant ef^{-1}. \end{aligned}$$

Let $h(y) = yk(y)$. The limit equation occurs as in Section 3 of [6]. More precisely, if (u_n, v_n) are strictly positive solutions of (11) for $d = d_n$, if $d_n \to 0$ as $n \to \infty$, and if $p < \infty$, then $\{(u_n, v_n)\}_{n=1}^{\infty}$ is precompact in $C_0(\bar{\Omega}) \oplus L^p(\Omega)$ and the set of possible limit points (as $n \to \infty$) is contained in

$$\{(w, g^{-1}(e - fw)^{+}: w \text{ is a non-negative solution of } (12)\}.$$

The proof of this is an easy modification of the proof of Theorem 2 in [6]. If $a - ceg^{-1} \neq \lambda_1$, any such limit point must have $w \neq 0$. To see this, note that, if $(u_n, v_n) \to (0, g^{-1}e)$ in $C_0(\bar{\Omega}) \oplus L^p(\Omega)$, then, by the first equation of (11), $r(-\Delta^{-1}(a - bu_n - cv_n)I) = 1$. Hence, by the continuity of the spectral radius, we see that in the limit $r(-\Delta^{-1}(a - cg^{-1}e)I) = 1$. This contradicts our assumption that $a - cg^{-1}e \neq \lambda_1$. (We have to be slightly more careful to check the convergence of the operators since v_n only converges to $g^{-1}e$ in $L^p(\Omega)$ for all $p < \infty$. However, this causes no difficulties since Δ^{-1} is a continuous map of $L^p(\Omega)$ into $C(\bar{\Omega})$ if $p > \frac{1}{2}n$ by [12, Sect. 2B4].) This proves our claim that $w \neq 0$ if $a - cg^{-1}e \neq \lambda_1$. Note that, unlike Section 3 of (6), our result cannot be improved to show that $v_n \to g^{-1}(e - fw)^{+}$ in $C(\bar{\Omega})$ (since $v_n = 0$ on $\partial\Omega$ while $g^{-1}(e - fw) = g^{-1}e > 0$ on $\partial\Omega$). However, the methods in [6] could be used to prove uniform convergence on compact subsets on Ω and upper uniform convergence (that is, $v_n \leqslant g^{-1}(e - fw)^{+} + \varepsilon$ for large n).

Thus we see that the study of non-negative solutions of (12) is important for the study of strictly positive solutions of (11) for small d. Hence we discuss the non-negative solution of (12). Note that $k(y) < 0$ if $y > b^{-1}a$ and thus any non-negative solution u of (12) satisfies $u(x) \leqslant b^{-1}a$ on Ω. Moreover, as earlier, (12) must have a maximal solution, possibly trivial. If $cg^{-1}f < b$, $k(y)$ is strictly decreasing in y for $y \geqslant 0$. It follows easily (cp. [6, Sect. 3]) that (12) has no non-trivial non-negative solution if $k(0) \leqslant \lambda_1$ and a unique non-negative non-trivial solution if $k(0) > \lambda_1$. If $cg^{-1}f > b$, more complicated behaviour occurs. We multiply each of the coefficients a, b, c, e, f, g by τ, where $\tau > 0$. It is easy to see that with this change (12) should be replaced by

$$-\Delta u = \tau h(u) \tag{13}$$

on Ω, with Dirichlet boundary conditions. *First suppose that* $k(0)>0$. By standard bifurcation theorems (cp. [8, Theorem 2]), a branch $\tilde{C}$ of positive solutions of (13) will branch off at $(0, \tau^*)$ where $\tau^*=\lambda_1(k(0))^{-1}$ and $\tilde{C}$ is unbounded in $C_0(\bar{\Omega})\times[0,\infty)$. By our a priori bound, $\tilde{C}$ must become unbounded by τ becoming unbounded. Since k is increasing for small y, it is easy to prove that this branch moves to the left (that is, τ decreases at first.) Hence, if $\tau_1\equiv\inf\{\tau>0\colon$ (13) has non-trivial non-negative solution$\}$, then $\tau_1<\tau^*$. It is also easy to prove that $\tau_1>0$. It is easy to use the method of sub- and supersolutions to prove that (13) has at least 2 non-trivial non-negative solutions for each $\tau\in(\tau_1,\tau^*)$ and at least one such solution for $\tau\geqslant\tau^*$. (Similar arguments appear in Lions [14]. The existence of the second solution can also be proved by using degree theory in cones, as in [8]. This method has the advantage of applying if Δ is replaced by a non-self-adjoint operator or if Ω does not have a smooth boundary.) If $n=1$, a much better result can be obtained. The methods of [4, Sect. 4 and Appendix] can be easily used to show that there is exactly one non-negative non-trivial solution if $\tau=\tau_1$ or $\tau\geqslant\tau^*$ while there are exactly 2 for $\tau_1<\tau<\tau^*$. Diagrammatically, we have the picture in Fig. 1.

Moreover, all these non-trivial solutions are non-degenerate (that is, the formal linearization is invertible) for $\tau<\tau_1$. *We now consider the case where* $k(0)\leqslant 0$ (and a general n again). If $k(y)\leqslant 0$ for all $y>0$, that is, $a\leqslant bef^{-1}$, it is easy to see that there is no non-trivial non-negative solution for any $\tau>0$. If $a>bef^{-1}$, if $\int_0^{ab^{-1}} h(y)\,dy\leqslant 0$, and if Ω is star-shaped, Pokozhaev's identity [17] easily implies that there is no non-trivial non-negative solution. On the other hand, if $a>bef^{-1}$ and $\int_0^{ab^{-1}} h(y)\,dy>0$, one can apply a theorem of Hess [10] to deduce that there is a non-trivial non-negative solution for τ sufficiently large. In particular, if Ω is star-shaped, the methods in [14] imply that there is a $\tau_1>0$ such that there are no non-trivial non-negative solutions for $\tau<\tau_1$, at least one if $\tau=\tau_1$, and at least two for $\tau>\tau_1$. As before, if $n=1$, the methods in [4] imply that there is exactly one solution if $\tau=\tau_1$ and exactly two for $\tau>\tau_1$. Moreover, all solutions for $\tau>\tau_1$ are non-degenerate. The solutions are sketched in the diagram below (Fig. 2).

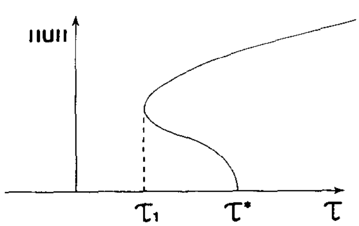

FIGURE 1

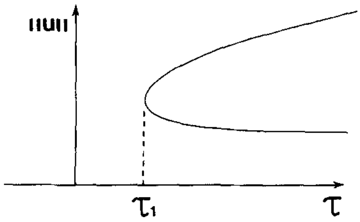

FIGURE 2

We have only sketched the proofs of the above results because the proofs are easy modifications of known results.

We now consider how these results can be used to study Eq. (11). We first see that, *if* (12) *has no non-trivial non-negative solution and if* $a - cg^{-1}e \neq \lambda_1$, *then* (11) *has no strictly positive solution for all sufficiently small positive d.* This follows immediately from the asymptotic results we discussed a little earlier in the section. To proceed further, we need the following proposition. Let K, C denote the cone of non-negative functions in $C_0(\bar{\Omega})$, $C_0(\bar{\Omega}) \oplus C_0(\bar{\Omega})$, respectively.

PROPOSITION 1. *Assume that w is a non-trivial non-negative solution of* (12) *which is isolated in K and has non-zero index k* (*relative to K*) *and assume that $p > 1$. Then, for sufficiently small positive d, there is a strictly positive solution of* (11) *near* $(w, g^{-1}(e - fw)^+)$ *in* $C_0(\bar{\Omega}) \oplus L^p(\Omega)$. *Moreover, the sum of the indices of such solutions* (*calculated relative to C*) *is k.*

We will prove this proposition in a moment. It is the analogue of a result stated but not proved in [6]. Note that, to be completely formal, we should specify the maps for which the indices are defined. In fact, the maps we take are the natural ones. For example, the index of the strictly positive solutions is calculated for the map $A_d(\,, 1)$ which we will define in the proof of Proposition 1 below. It is possible to prove an analogue of Proposition 1 for an isolated set of solutions of (12) (rather than just an isolated solution). This could be used to prove weaker versions of some of the results below for the case where $n > 1$. (Most of the results below require that $n = 1$.)

Suppose now that $n = 1$ and we are in the situation where (12) *has* 2 *non-trivial non-negative solutions* w_1, w_2 where w_1 is the maximal solution. By our earlier results, these must be non-degenerate. It turns out that, in $C_0(\bar{\Omega})$, the formal linearization at w_i is in fact the Fréchet derivative at w_i. (We will prove this a little later.) Thus, by [7, Theorem 1; 6, Lemma 2], the indices of w_i are ± 1. (In fact, it can be shown that w_1 has index $+1$ and w_2 index -1 but we do not need this. Note, however, by deforming to the case τ is small, we see that the sum of the indices at w_1 and w_2 is 0.) By Proposition 1, *we see that* (11) *has at least two strictly positive solutions for small positive d.* One is near $(w_1, g^{-1}(e-fw_1)^+)$ and one near $(w_2, g^{-1}(e-fw_2)^+)$. Moreover, by Proposition 1 and our earlier comments, it follows that the sum of the indices of all the strictly positive solutions near $(w_1, g^{-1}(e-fw_1)^+)$ and $(w_2, g^{-1}(e-fw_2)^+)$ is $1-1=0$. Since our earlier asymptotics imply that all strictly positive solutions lie near one of the above for small d, it follows that, for small d, the sum of the indices of the strictly positive solutions of (11) is zero. This suggests that one of $r(-\Delta^{-1}(a-c\bar{v}_d)I)$ and $r(-\Delta^{-1}(d^{-1}e-d^{-1}f\bar{u})I)$ is greater than 1 and one is less than 1 for small d. Here $\bar{v}_d$ is the maximal solution of the second equation in (11) when $u \equiv 0$. (Indeed, it would follow from our index calculations in [6, Sect. 2] if we knew that neither was equal to 1.) In fact, the first of these spectral radii is less than 1 while the second is greater than 1. To see this, we note that by similar (but easier) arguments to that in the proof of Theorem 2 in [6], $\bar{v}_d \to eg^{-1}$ in $L^p(\Omega)$ as $d \to 0$. Thus, as before

$$r((-\Delta+\tilde{K}I)^{-1}(a-c\bar{v}_d+\tilde{K})I) \to r((-\Delta+\tilde{K}I)^{-1}(a-ceg^{-1}+\tilde{K})I)$$
$$=(\lambda_1+\tilde{K})^{-1}(a-ceg^{-1}+\tilde{K})$$

as $d \to 0$. Since by our earlier results $a-ceg^{-1}<\lambda_1$ if there are two non-trivial non-negative solution, the first equality follows. The second can be proved by a similar argument to that in the proof of Proposition 1(v) in [6]. The above argument shows that *there are cases where there is a strictly positive solution of* (11) *and where one of* $r(-\Delta^{-1}(a-c\bar{v})I)$ *and* $r(-\Delta^{-1}(e-f\bar{u})I)$ *is greater than* 1 *and one is less than* 1. By multiplying all the coefficients by τ where $\tau<\tau_1$, we get *a similar example where there is no strictly positive solution* (since there is now only the trivial solution of (12)). *This gives examples showing that the iteration at the start of the section may or may not converge to a strictly positive solution of* (11). In fact it shows that the iteration sometimes converges to a strictly positive solution and sometimes does not on a single component of $\{(a, b, c, e, f, g)\colon a>\lambda_1, e>\lambda_1, r(-\Delta^{-1}(a-c\bar{v})I)>1, r(-\Delta^{-1}(e-f\bar{u})I)<1\}$. In addition, the above arguments imply that *there is an open set of* $\{(a, b, c, e, f, g)\colon a>\lambda_1, e>\lambda_1\}$ *for which* (11) *has more than* 1 *strictly positive solution.* This provides a rather large (and quite different) range of parameters where there is non-uniqueness to the examples in [2] where non-uniqueness occurs because the two equations effectively reduce to a single equation. Note that we cannot use the above arguments if $n>1$ because we do not even know if solutions of (12) are isolated!

We still need to prove that the map $u \to -\Delta^{-1}h(u)$ is Fréchet differentiable at w (as a map of $C(\bar{\Omega})$ into itself) if w is non-trivial solution of (12). It obviously suffices to prove that the map $u \to k(u)$ is Fréchet differentiable at w (as a map of $C(\bar{\Omega})$ to $L^p(\bar{\Omega})$). It is easy to see that it suffices to prove that the map $u \to k_1(u)$ is differentiable where $k_1(y) = (e - fy)^+$. There is a difficulty because k_1 is not differentiable at $f^{-1}e$ (as a map of R to R). Suppose $\varepsilon > 0$. It is easy to prove that $\|w\|_\infty < b^{-1}a$ and hence that $w''(x) \neq 0$ when $w(x) = f^{-1}e$. Hence we easily see that $T = \{x \in \Omega: w(x) = f^{-1}e\}$ has measure zero. Hence we can find a $\delta > 0$ such that $T_\delta = \{x \in \Omega: |w(x) - f^{-1}e| \leqslant \delta\}$ has measure at most ε. If $\|\tilde{h}\|_\infty < f^{-1}\delta$ and $x \notin T_\delta$, then $e - fw(x)$ and $e - f(w(x) + \tilde{h}(x))$ have the same sign. Hence, by a simple calculation,

$$k_1(w(x) + \tilde{h}(x)) - k_1(w(x)) - k_1'(w(x))\,\tilde{h}(x) = 0.$$

Thus

$$\tilde{A} \equiv \|k_1(w + \tilde{h}) - k_1(w) - k_1'(w)\tilde{h}\|_p = \|k_1(w + \tilde{h}) - k_1(w) - k_1'(w)\tilde{h}\|_{p,\delta}$$

where $\| \ \|_{p,\delta}$ denotes the L^P norm on T_δ. Note that $k'(w)\tilde{h}$ makes sense in L^p because it is defined almost everywhere. Since k_1 is Lipschitz continuous with Lipschitz constant f, it follows that

$$\tilde{A} \leqslant 2f\,\|\tilde{h}\|_{p,\delta} \leqslant 2f\,\|\tilde{h}\|_\infty m(T_\delta)^{1/p} \leqslant 2f\,\|\tilde{h}\|_\infty \varepsilon^{1/p},$$

where m denotes Lebésgue measure. The result follows easily from this. The above result can be easily modified to apply when $n > 1$. The above result is well known but there does not seem to be a good reference in the literature.

Proof of Proposition 1. Choose $\tilde{K} > 0$ such that

$$a - bx - cy + \tilde{K} > 0 \qquad \text{and} \qquad \tilde{K} + e - fx - gy > 0$$

if $x \leqslant 2b^{-1}a$ and $y \leqslant 2e^{-1}g$. Suppose V is an open neighborhood of w in $K \subseteq C_0(\bar{\Omega})$ such that $0 \notin V$ and $\|x\|_\infty \leqslant b^{-1}a$ if $x \in V$ and such that w is the only solution of (12) in $\bar{V}$. Let $T = B_{2e^{-1}g} - \bar{B}_\varepsilon$, where B_s denotes $\{x \in K: \|x\|_\infty < s\}$ and ε is chosen such that $\varepsilon < e^{-1}g$. Define $A_d: \bar{V} \times \bar{T} \times [0, 1] \to C$ by

$$\begin{aligned} A_d(u, v, t) = ((-\Delta + \tilde{K}I)^{-1}u[\tilde{K} + a - bu - ctv - (1-t)\,cg^{-1}(e - fu)^+], \\ (-d\Delta + \tilde{K}I)^{-1}v[\tilde{K} + e - fu - gv]). \end{aligned}$$

It is easy to see that A_d is completely continuous and maps into C for fixed $d > 0$. If $A_d(u, v, t) = (u, v)$, then

$$\begin{aligned} -\Delta u &= u(a - by - tcv - (1-t)\,cg^{-1}(e - fu)^+) \\ -d\Delta v &= v(e - fu - gv). \end{aligned} \tag{14}$$

We prove that, if d is small, $A_d(\ ,\ , t)$ has no fixed points on $\partial_C(V \times T)$. To see this, suppose by way of contradiction that $(u_n, v_n) = A_{d_n}(u_n, v_n, t_n)$ where $(u_n, v_n) \in \partial_C(V \times T)$ and $d_n \to 0$ as $n \to \infty$. By our earlier a priori bounds, $\|v_n\|_\infty \leqslant e^{-1}g$. Thus $u_n \in \partial_K V$ or $\|v_n\|_\infty = \varepsilon$. By an easy modification of the proof of Theorem 2 in [6], we can, by choosing a subsequence, arrange that $u_n \to \tilde{u}$ as $n \to \infty$ weakly in $W^{2,p}(\Omega)$ and strongly in $C(\bar{\Omega})$ and $v_n \to g^{-1}(e - f\tilde{u})$ in $L^p(\Omega)$ for all $p < \infty$. Thus $\|v_n\|_\infty > \varepsilon$ for large n and hence $u_n \in \partial_K V$. By passing to the limit in the first equation of (11), we see that $\tilde{u}$ is a solution of (12). Since $u_n \in \partial_K V$, $\tilde{u} \in \partial_K V$. Hence we have a contradiction. Hence, by the homotopy invariance of the degree

$$\deg_C(I - A_d(\ , 1), V \times T) = \deg_C(I - A_d(\ , 0), V \times T)$$

for d small. Note that $I - A_d(\ , 1)$ is the natural map on C for studying (11) while

$$A_d(u, v, 0) = ((-\Delta + \tilde{K}I)^{-1} u(\tilde{K} + a - bu - cg^{-1}(e - fu)^+),$$
$$(-d\Delta + \tilde{K}I)^{-1} v(\tilde{K} + e - fu - gv)).$$

The first component of $A_d(\ ,\ , 0)$ depends only on u. Thus, by the strong form for the product theorem for the degree (cp. Brown [3, Theorem 9.4]) we see that

$$\deg_C(I - A_d(\ , 0), V \times T) = \deg_K(I - A_1, V) \deg_K(I - A_d^2, T),$$

where

$$A_1(u) = (-\Delta + \tilde{K}I)^{-1} u(\tilde{K} + a - bu - cg^{-1}(e - fu)^+)$$

and

$$A_d^2(v) = (-d\Delta + \tilde{K}I)^{-1} v(\tilde{K} + e - fw - gv).$$

The first degree is simply k. (A_1 is the natural map for studying (12). Remember that, as before, homotopy invariance ensures that $\deg_K(I - A_1, V)$ is independent of $\tilde{K}$ if $\tilde{K} \geqslant b^{-1}a$.) To calculate the second degree, we note that A_d^2 has a unique fixed point z in T and $r((A_d^2)'(z)) < 1$. (Similar arguments appear early in Section 1 of [6].) Hence, by Theorem 1 of [7], $\deg_K(I - A_d^2, T) = 1$. By combining the above results, we find that $\deg_C(I - A_d(\ , 1), V \times T) = k$ if d is sufficiently small. Since $k \neq 0$, it follows that $A_d(\ , 1)$ has a fixed point in $V \times T$ and thus (11) has a solution $(\tilde{u}_d, \tilde{v}_d)$ in $V \times T$. The result now follows from what we have proved above and our earlier results on the asymptotic behaviour of strictly positive solutions as $d \to 0$.

Remarks. 1. The above ideas could be applied to the general predator–prey model of Section 1 but it seems that nothing interesting results.

2. The above asymptotic ideas apply to much more general equations. In (11) we could replace $a-bu-cv$ and $e-fu-gv$ by $M(u, v)$ and $N(u, v)$, respectively, where M and N are C^1-functions on $\{(u, v) \in R^2: u \geqslant 0, v \geqslant 0\}$ such that $M(u, v) < 0$ if $u \geqslant u_0$ and $v \geqslant 0$, $N(u, v) < 0$ if $v \geqslant v_0$, and $0 \leqslant u \leqslant u_0$ and $N_2'(u, v) < 0$ if $0 \leqslant u \leqslant u_0$, $0 \leqslant v \leqslant v_0$. The proof of Theorem 2 in [6] needs to be modified somewhat to avoid any use of weak convergence but instead to proceed by constructing more sub- and supersolutions. In this case, $v \to s(w)$ as $d \to 0$, where $s(u) = 0$ if $N(u, 0) \leqslant 0$ while $s(u)$ is the unique solution of $N(u, v) = 0$ in $(0, v_0)$ if $N(u, 0) > 0$. The analogue of Eq. (12) is

$$-\Delta u = uM(u, s(u)). \tag{15}$$

An analogue of Proposition 1 is easy to prove. Of course, we still have the problem of understanding the solutions of (15).

3. Uniqueness of the Conway–Gardner–Smoller System for Small d

In this section, we prove the uniqueness of the strictly positive solution of the main Conway–Gardner–Smoller equation if $n = 1$ and d is small. Our technique applies to a few other problems. For example, it implies that in the case of an "asocial" nonlinearity we can prove that, in many cases, the Conway–Gardner–Smoller equation has exactly two strictly positive solutions for small d. In fact, our argument could be used to prove a local uniqueness result for some more general non-linearities. We discuss some further generalizations in the Appendix. Note that the results in Fife [9] do not apply to (16).

We consider the system

$$\begin{aligned} -u'' &= u(f(u) - v) \\ -dv'' &= v(-v + m(u - \gamma)) \end{aligned} \tag{16}$$

with Dirichlet boundary conditions on $[-L, L]$, *where* $f(y) = a(1-y)$, $0 < \gamma < 1$, $a, m > 0$, $\|\bar{u}\|_\infty > \gamma$. We also consider the limiting equation (for $d \to 0$)

$$-u'' = h(u) \tag{17}$$

with Dirichlet boundary conditions on $[-L, L]$, where $h(y) = yf(y) - my(y-\gamma)^+$. It is shown in [6, Sect. 3] that (17) has a unique solution w

with $\|w\|_\infty > \gamma$. This solution must have index 1 relative to the cone K because one sees as in [6] that the total index of solutions is 1 while zero has index 0. (Technically, the indices are calculated for the map $I - A_1$, where $A_1(u) = (-\Delta + \tilde{K}I)^{-1}(\tilde{K}u + h(u))$ for $\tilde{K}$ large.) As in Section 2, the Fréchet derivative $A_1'(w)$ exists. Since $h'(y) < y^{-1}h(y)$ for $y \neq \gamma$ as is easily checked, a simple comparison argument like those in Section 1 of [6] implies that $r(A_1'(w)) < 1$.

THEOREM 2. *For sufficiently small positive d,* (16) *has a unique strictly positive solution.*

It is shown in [6, Sect. 3] that, if (u_n, v_n) are strictly positive solutions for $d = d_n$ and $d_n \to 0$ as $n \to \infty$, then $(u_n, v_n) \to (w, m(w-\gamma)^+)$ in $C^1[-L, L] \oplus C[-L, L]$. (This is a slightly stronger result for this equation than the one in Section 2 for (11).) Moreover, the analogue of Proposition 1 of Section 2 shows that, for small d, the sum of the indices (relative to C) of the strictly positive solution of (16) is $\mathrm{index}_K(A_1, w) = 1$. Here the indices are calculated for the map $I - \mathscr{A}_d$ where $\mathscr{A}_d$ is defined by

$$\mathscr{A}_d(u, v) = ((-\Delta + \tilde{K}I)^{-1}u(f(u) - v + \tilde{K}),$$
$$(-d\Delta + \tilde{K}I)^{-1}v(-v + m(u-\gamma) + \tilde{K})).$$

The proof of the above result is very much the same as the proof of Proposition 1. Hence, we see that, if we can prove that, for small d, any strictly positive solution (u, v) has the property that $I - \mathscr{A}_d'(u, v)$ is invertible and

$$\mathrm{index}_E(I - \mathscr{A}_d'(u, v), 0) = \mathrm{index}_{C[-L,L]}(I - A_1'(w), 0) = 1,$$

Theorem 2 will follow. (Remember that, in this case,

$$\mathrm{index}_C(I - A_d(u, v)) = \mathrm{index}_E(I - \mathscr{A}_d'(u, v), 0).$$

Here we are using Theorem 1 in [6] and note that (u, v) is demiinterior to C.) Hence our proof reduces to showing this. We first need a better estimate for the v component of strictly positive solutions.

LEMMA 1. *Suppose* $0 < \tilde{a} < 1$. *There is a* $d_0 > 0$ *such that, if* $0 < d < d_0$ *and if* (u, v) *is a strictly positive solution of* (16), *then* $v \geq \tilde{a}m(u-\gamma)$.

Proof. Note that the inequality is trivial when $u(x) \leq \gamma$. Moreover, we know that, as $d \to 0$, $u \to w$ and $v \to m(w-\gamma)^+$ uniformly. Hence we see that we need only prove the inequality for points near where $w(x) = \gamma$. By using the first integral of (17), we easily see that $w'(x) \neq 0$ when $w(x) < \|w\|_\infty$. In particular, $w(x) = \gamma$ at only two points x_1, x_2 and

$w'(x_i) \neq 0$. (Note that $\|w\|_\infty > \gamma$. Otherwise, w would be a solution of $-u'' = uf(u)$ and our assumption that $\|\bar{u}\|_\infty > \gamma$ excludes this.) It obviously suffices to prove our result for x near x_1. We assume $w'(x_1) > 0$. (The other case is similar.) Since $w'(x_1) > 0$ and u is near w in $C^1[-L, L]$, we see that there exist $\varepsilon > 0$ and $M_1, M_2 > 0$ such that $M_1 \leqslant u'(x) \leqslant M_2$ on $[x_1 - \varepsilon, x_1 + \varepsilon]$ if (u, v) is strictly a positive solution of (16) and d is small. In particular $u(x) - \gamma$ will have a unique zero $\tilde{x}$ near x_1 if d is small and $\tilde{x}$ will lie in $(x_1 - \frac{1}{2}\varepsilon, x_1 + \frac{1}{2}\varepsilon)$. We now construct a subsolution $z(x)$ of the second equation in (16) in the form $z(x) = \alpha(x - \tilde{x})$ for $\tilde{x} \leqslant x \leqslant x_1 + \varepsilon$. Choose $\alpha \in (\tilde{a}mw'(x_1), mw'(x_1))$. It follows easily that $\tilde{a}mu'(x) + \delta_2 < z'(x) < mu'(x) - \delta_2$ on $(\tilde{x}, \tilde{x} + \delta_1)$ if d is small. (Recall that u' converges to w' uniformly.) Hence

$$\tilde{a}m(u(x) - \gamma) + \delta_2(x - \tilde{x}) \leqslant z(x) \leqslant m(u(x) - \gamma) - \delta_2(x - \tilde{x}) \tag{18}$$

on $(\tilde{x}, \tilde{x} + \delta_1)$. Now $dz''(x) = 0$ while $z(-z + m(u - \gamma)) \geqslant 0$ on $[\tilde{x}, \tilde{x} + \delta_1]$. Hence z is a subsolution of the second equation of (16) on $[\tilde{x}, \tilde{x} + \delta_1]$. Now it is easy but tedious to extend $z(x)$ so that $m(w(x) - \gamma) - z(x) \geqslant \mu > 0$ on $[\tilde{x} + \delta_1, x_1 + \varepsilon]$, $z(x) > 0$ on $[\tilde{x} + \delta_1, x_1 + \varepsilon)$, $z(x_1 + \varepsilon) = 0$, z is linear on $[x_1 + \frac{3}{4}\varepsilon, x_1 + \varepsilon]$, $z'(x_1 + \varepsilon) < -1$, and z'' is uniformly bounded. (Note that $z(\tilde{x}_1 + \delta_1) \leqslant m(u(\tilde{x}_1 + \delta_1) - \gamma) - \delta_2\delta_1$.) It follows by an easy calculation that z is a subsolution of the second equation of (16) on $[\tilde{x}, x_1 + \varepsilon]$ if d is small. (Note that u is uniformly close to w on $[\tilde{x} + \delta_1, x_1 + \varepsilon]$.) Now define z to be zero outside of $[\tilde{x}, x_1 + \varepsilon]$. As in [20, Lemma I.1], z is a subsolution of the second equation of (16) if d is small. Since v is the unique non-trivial non-negative solution of the second equation of (16), it follows that $v \geqslant z$. In particular, by (18), $v(x) \geqslant \tilde{a}m(u(x) - \gamma)$ on $(\tilde{x}, \tilde{x} + \delta_1)$. Since δ_1 is independent of d and the particular solution (u, v) the result follows.

It turns out that this is really the key lemma. Note that it is largely independent of the particular form of the first equation in (16). By our earlier comments, the following lemma will complete the proof of Theorem 2.

THEOREM 2. *There is a $d_1 > 0$ such that if $0 < d \leqslant d_1$ and (u, v) is a strictly positive solution of (16), then $I - \mathscr{A}'_d(u, v)$ is invertible and* $\operatorname{index}_E(I - \mathscr{A}'_d(u, v), 0) = \operatorname{index}_{\tilde{E}}(I - A'_1(w), 0) = 1$. *(Here $\tilde{E} = C[-L, L]$.)*

Proof. Suppose by way of contradiction that (u_n, v_n) are strictly positive solutions for $d = d_n$ such that $d_n \to 0$ as $n \to \infty$ and $\mathscr{A}'_d(u_n, v_n)(h_n, k_n) = (h_n, k_n)$ where $(h_n, k_n) \neq 0$. It is convenient to normalize (h_n, k_n) such that $\|h_n\|_p + \|k_n\|_p = 1$ where $1 < p < \infty$. Now, by the definition of $\mathscr{A}_d$,

$$\begin{aligned} -h_n'' &= (f(u_n) + u_n f'(u_n) - v_n)\, h_n - u_n k_n \\ -d_n k_n'' &= (-2v_n + m(u_n - \gamma))\, k_n + m v_n h_n \end{aligned} \tag{19}$$

with Dirichlet boundary conditions. Since u_n and v_n are uniformly bounded in $C[-L, L]$, the first equation implies that h_n'' is bounded in $L^p[-L, L]$. It follows easily that h_n is uniformly bounded in $C(-L, L]$. By Lemma 1,

$$-2v_n + m(u_n - \gamma) \leqslant -\tfrac{1}{2}v_n \tag{20}$$

on $[-L, L]$ if n is large. If k_n has a positive maximum at x_n, then $k_n''(x_n) \leqslant 0$ and thus, by the second equation,

$$(-2v_n(x_n) + m(u_n(x_n) - \gamma))\, k_n(x_n) + mv_n(x_n)\, h_n(x_n) \geqslant 0.$$

By (20) and a simple calculation, we see that $k_n(x_n) \leqslant 2mh_n(x_n) \leqslant 2m \|h_n\|_\infty$. Since we could use a similar argument at a negative minimum, it follows that $\|k_n\|_\infty \leqslant 2m \|h_n\|_\infty$. This implies that k_n is uniformly bounded in $C[-L, L]$ and also that $\|h_n\|_\infty$ does not tend to zero with n. (If $\|h_n\|_\infty \to 0$ as $n \to \infty$, then $\|k_n\|_\infty \to 0$ as $n \to \infty$. This is impossible since $\|h_n\|_p + \|k_n\|_p = 1$.) Since h_n is bounded in $W^{2,p}[-L, L]$ and k_n is bounded in $L^p[-L, L]$, we can, by choosing a subsequence if necessary, assume that $k_n \to k$ weakly in $L^p[-L, L]$ and $h_n \to \tilde{h}$ weakly in $W^{2,p}[-L, L]$ and strongly in $C[-L, L]$. Since $\|h_n\|_\infty$ does not tend to zero with n, we can choose the subsequence such that $\tilde{h} \neq 0$. (This is vital.) Choose ϕ a C^2 function of compact support in $[-L, L]$. If we multiply the second equation of (19) by ϕ and integrate by parts, we find that

$$-d_n(k_n, \phi'') = (-2v_n + m(u_n - \gamma), k_n\phi) + (mv_n, h_n\phi).$$

Passing to the limit as $n \to \infty$, we find that

$$0 = (-m\,|w - \gamma|\, k, \phi) + (m^2(w - \gamma)^+ \tilde{h}, \phi).$$

Recall that $v_n \to m(w - \gamma)^+$ as $n \to \infty$. Since such C^2 functions ϕ are dense in $L^q[-L, L]$, we have that

$$-m\,|w - \gamma|\, k + m^2(w - \gamma)^+ \tilde{h} = 0.$$

Since $w(x) = \gamma$ only on a set of measure zero, $k = m\,\mathrm{sgn}(w - \gamma)\,\tilde{h}$. If we pass to the limit in the first equation of (19), we see that

$$-\tilde{h}'' = (f(w) + wf'(w) - m(w - \gamma)^+)\,\tilde{h} - wk.$$

Hence, by our formula above for k,

$$\begin{aligned} -\tilde{h}'' &= (f(w) + wf'(w) - m(w - \gamma)^+ - mw\,\mathrm{sgn}(w - \gamma))\,\tilde{h} \\ &= h'(w)\,\tilde{h}. \end{aligned}$$

Since $\tilde{h}$ satisfies the boundary condition and $\tilde{h}$ is non-trivial, this means that $I-A_1'$ fails to be invertible. This contradicts our earlier comments. Hence we see that $I-\mathscr{A}_d'(u,v)$ is invertible if (u,v) is a strictly positive solution of (16) and d is small.

To complete the proof, we still have to evaluate $\operatorname{index}_E(I-\mathscr{A}_d'(u,v),0)$. We can use the same argument as above to show that there is a $d_2>0$ such that the equations

$$\begin{aligned} -\tilde{h}'' &= (f(u)+uf'(u)-v)\tilde{h}-tu_nk-(1-t)\,u_nm\operatorname{sgn}(w-\gamma)\,\tilde{h} \\ -dk'' &= (-2v+m(u-\gamma))k+mv\tilde{h} \end{aligned} \tag{21}$$

$\tilde{h}(-L)=\tilde{h}(L)=k(-L)=k(L)=0$ has no non-trivial solutions if $0\leqslant t\leqslant 1$, if d is sufficiently small and if (u,v) is a strictly positive solution of (16). (The point is that we obtain the same limit equation as before.) Hence we see that the maps $I-B_t$ are all invertible for $0\leqslant t\leqslant 1$ if d is small, where

$$B_t(\tilde{h},k)=((-\Delta+\tilde{K}I)^{-1}(\tilde{K}\tilde{h}+a_t\tilde{h}+b_tk),(-d\Delta+\tilde{K}I)^{-1}(\tilde{K}k+\tilde{c}\tilde{h}+\tilde{d}k)).$$

Here $a_t\tilde{h}+b_tk$ and $\tilde{c}\tilde{h}+\tilde{d}k$ denote the right-hand sides of the first and second equations in (21), respectively. Hence

$$\operatorname{index}_E(I-\mathscr{A}_d'(u,v),0)=\operatorname{index}_E(I-B_1,0)=\operatorname{index}_E(I-B_0,0).$$

Since $b_0=0$, the first component of B_0 depends only on $\tilde{h}$. Thus, by the strong form of the product theorem for the degree (cp. Brown [3, Theorem 9.41]),

$$\operatorname{index}_E(I-\mathscr{A}_d'(u,v),0)=\operatorname{index}_{\tilde{E}}(I-\tilde{B}_1,0)\operatorname{index}_{\tilde{E}}(I-\tilde{B}_2,0), \tag{22}$$

where

$$\tilde{B}_1\tilde{h}=(-\Delta+\tilde{K}I)^{-1}\,a_0\tilde{h}$$

and

$$\tilde{B}_2k=(-d\Delta+\tilde{K}I)^{-1}(\tilde{d}+\tilde{K})k.$$

As $d\to 0$, $\tilde{B}_1\to(-\Delta+\tilde{K}I)^{-1}(h'(w)+\tilde{K})I=A_1'(w)$ (since $u\to w$, $v\to m(w-\gamma)^+$). Thus the first index on the right-hand side of (22) is 1 (since $r(A_1'(w))<1$). For the second, we note that $\tilde{B}_2y<(-d\Delta+\tilde{K}I)^{-1}(-v+m(u-\gamma)+\tilde{K})y$ for each $y\in K$. Since the second operator has spectral radius 1 (with eigenvector v), we can argue as in Section 1 of [6] to deduce that $r(\tilde{B}_2)<1$. Thus the second index in (22) is 1. Thus $\operatorname{index}_E(I-\mathscr{A}_d'(u,v),0)=1$ if d is small and the result follows.

Remarks. The above argument is really a local argument. More precisely, we could replace f by any C^1 function with $f(0)\geqslant 0$. Then, if w is a non-degenerate, non-negative, non-trivial solution of (17), with $\|w\|_\infty > \gamma$, there is a unique strictly positive solution (u, v) near $(w, m(w-\gamma)^+)$ in C for all sufficiently small d. In particular, if every non-negative, nontrivial solution of (17) is non-degenerate and none has sup norm equal to γ, we obtain an exact count of the strictly positive solutions of (16) for small positive d. In particular, this applies to the "asocial" non-linearity $f = a(1-y)(y-b)$ of [4]. If $n=1$ and (17) has two distinct solutions w_1, w_2 (where w_1 is the maximal solution), then the results in [4] imply that these are both non-degenerate. Hence we *see that, if* $\|w_2\|_\infty < \gamma < \|w_1\|_\infty$, *there is a unique strictly positive solution for small positive d, while if* $\gamma < \|w_2\|_\infty$, *there are exactly two strictly positive solutions for small positive d.* (It is relevant to note that, if w is a solution of (17) with $\|w\|_\infty \leqslant \gamma$, then $(w, 0)$ is a solution of (16).)

2. It seems likely that our results are true if $n > 1$. The difficulty in proving this by our method is in generalizing Lemma 1 and, in particular, the construction of z near where $u(x) = \gamma$. (The rest of the proof generalizes easily.) It does not seem clear how to do this. However, if Ω is a ball in R^n, our methods can be used to prove the uniqueness of the *radially symmetric* strictly positive solution for small d. Here we construct $z(r) = \alpha(r-\tilde{r}) + \beta(r-\tilde{r})^2 + \tilde{\gamma}(r-\tilde{r})^3$ (where $u(\tilde{r}) = \gamma$) such that $|\Delta z(r)| = o(r-\tilde{r})^2$ for r near $\tilde{r}$.

3. Our methods can be generalized to the equations of Section 1 or 2 for $n=1$. A difficulty occurs because, in general, v does not converge uniformly near $\pm L$ as $d \to 0$. We discuss this briefly in the Appendix.

Appendix

We give a brief sketch of a proof that the natural analogue of Theorem 2 of Section 3 holds for the equation of Section 2. More precisely, if w is a solution of (12) such that $\|w\|_\infty \neq ef^{-1}$ and $-\Delta - h'(w)I$ is invertible (for Dirichlet boundary condition), then (11) has a unique solution near $(w, g^{-1}(e-fw)^+)$ in $C(-L, L] \oplus L^p(-L, L)$ for d sufficiently small. The condition that $\|w\|_\infty \neq ef^{-1}$ can probably be avoided by proving a variant of Lemma 1.

The natural analogue of Lemma 1 holds on any compact subset on $(-L, L)$ with essentially the same proof. As in Section 3, the proof now reduces to proving the natural analogue of Lemma 2 there. We follow the proof there and the notation there. Choose $p > 2$. The proof there only runs into difficulties if $\|h_n\|_\infty \to 0$ as $n \to \infty$, where (h_n, k_n) is a solution of the linearized equation such that $\|h_n\|_p + \|k_n\|_p = 1$, h_n converges weakly in $W^{2,p}(-L, L)$, and k_n converges weakly in $L^p(-L, L)$. By passing to the limit in the second linearized equation, we see that $k_n \rightharpoonup 0$ weakly in $L^p(-L, L)$ (since $h_n \to 0$ in $C[-L, L]$). Moreover, by considering (as there) where k_n has a maximum, we see that a local maximum x_n of $|k_n|$, then $|k_n(x_n)| \leqslant K_1 |h_n(x_n)|$ unless x_n is near an end point and $|v_n(x_n)| \leqslant \frac{2}{3} g^{-1}e$. Thus $|k_n(x_n)|$ is small unless x_n is near one of the ends and $|v_n(x_n)| \leqslant \frac{2}{3} g^{-1}e$. Since $k_n \rightharpoonup 0$ weakly in $L^p(-L, L)$, it follows easily that k_n is uniformly small except near $\pm L$. Now $\|k_n\|_p \to 1$ as $n \to \infty$. Hence $\|k_n\|_\infty \to \infty$ as $n \to \infty$. We consider the case where k_n is large near $-L$. The other case is similar. By constructing subsolutions of the second equation (as in the proof of Lemma 1 here and of Theorem 2 in [6]), we easily find that, if $0 < \tilde{a} < 1$, there exist ε, $M > 0$ such that $v_n(x) \geqslant \tilde{a} g^{-1} e$ for $-L + M d_n^{-1/2} \leqslant x \leqslant \varepsilon - L$. In particular, it follows that a large maximum of $|k_n|$ must be within $M d_n^{-1/2}$ of $-L$ (if we choose $\tilde{a} \geqslant \frac{2}{3}$). We now use a new variable $X = d_n^{-1/2}(x + L)$. Define $\tilde{v}_n(X) = v_n(d_n^{-1/2}(x + L))$. $\tilde{u}_n$, etc., are defined analogously. With this new variable the equation for v_n becomes

$$-\tilde{v}_n'' = \tilde{v}_n(e + o(1) - g\tilde{v}_n)$$

if $|X| \leqslant d_n^{-1/3} M_2$. (Since u_n' is bounded in the old variables, $|\tilde{u}_n(X)| \leqslant K d_n^{1/2} |X|$.) Since $\tilde{v}_n(X) \geqslant \tilde{a} g^{-1} e$ if $M \leqslant X \leqslant \varepsilon d_n^{-1/2}$ and $\|v_n\|_\infty$ is bounded, we easily see by continuous dependence that on compact sets of $[0, \infty)$, $\tilde{v}_n$ is close to v_0 where

$$-v_0'' = v_0(e - g v_0), \tag{23}$$

$v_0(0) = 0$, $v_0(\infty) = g^{-1} e$. (Note that v_0 is the only non-trivial solution of (23) which vanishes at 0, and is bounded and non-negative on $[0, \infty)$.) In the new variables, we find that

$$\tilde{k}_n'' = (e - 2v_0 + o(1))\, \tilde{k}_n + o(1)$$

on bounded X intervals. (Once again recall that h_n' is uniformly bounded.) Since $\tilde{k}_n$ must be large before $X = M$, and $\tilde{k}_n(0) = 0$. it follows that $\tilde{k}_n'(0)$ is large. Hence we easily see from continuous dependence that, on compact sets, $\tilde{k}_n$ is near k_0^n, where k_0^n is a solution of

$$-z'' = (e - 2v_0(X))z \tag{24}$$

$z(0) = 0$ (and $(k_0^n)'(0) = \mu_n$ where μ_n is large). Now v_0' is a solution of (24) which is positive on $(0, \infty)$. Hence by the Sturm comparison theorem, k_0^n has no positive zeros. By using Wronskians (or by using asymptotic theory), we prove that $|k_0^n(X)| \to \infty$ as $X \to \infty$. It follows that there exist $x_n \geqslant 2M d_n^{-1/2}$ (but near $-L$) such that $|k_n(x_n)|$ is large and increasing. It is easy to see that this contradicts our earlier results on k_n. This completes the proof.

It is clear that our methods could be used for rather more general nonlinearities. In particular, they could be used for rather more general predator–prey equations. Note that once again our arguments are local arguments and hence can be used when the limit equation has more than one solution.

It is possible to prove that the solution we obtained above is stable for the full parabolic equations if w is a stable solution of the one-dimensional parabolic equation. This is proved by combining our ideas with positive operator theory. Note that our proof of this does not generalize to apply to (16).

It follows by combining our results above with those in Section 2 that we obtain complete information on the solutions of the corresponding parabolic equation in some cases. (Note that our methods apply if $n > 1$, Ω is a disk, and we look for radially symmetric solutions. In particular, by combining the above results with the ideas in Section 2, we also obtain conplete information on the solutions of the corresponding parabolic equations in some cases where $n > 1$ and Ω is a disk.)

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