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Asymptotic Equivalence of Abstract Parabolic Equations with Delays

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Let A be an elliptic operator without time dependence, let $\tau > 0$ be fixed, and consider the parabolic equation with time delay

$$u'(t) = Au(t) + f(t, u(t - \tau)). \tag{1}$$

To render (1) well posed, one must specify boundary conditions and the initial value of u for $-\tau \leqslant t \leqslant 0$; such matters are considered in [8–12]. The concern here is not with questions of existence, uniqueness, and continuous dependence, but rather with the detectability of the term f in (1) for large t. That is, we ask under what restrictions on A and f will it be impossible to distinguish between bounded solutions of (1) and bounded solutions of

$$v'(t) = Av(t) \tag{2}$$

for sufficiently large time. Specifically, we seek conditions on f which will guarantee that for any bounded solution $u(t)$ of (1) there exists for large time a solution $v(t)$ of (2) such that

$$\lim_{t\to\infty} \| u(t) - v(t)\| = 0$$

for some suitable space-variable norm $\|\cdot\|$, and conversely. Such questions for ordinary differential equations have been considered in [1, 3] among others; for systems of ordinary differential equations with delays see the recent paper of Cooke [2] and references therein.

Instead of dealing directly with (1) and (2), where A is a member of a certain class of elliptic differential operators, we prefer to deal with an abstract problem on a Banach space; we shall also consider a more general time delay than that discussed above. Let then $\mathscr{X}$ be a Banach space with norm $\|\cdot\|$ and let $\tau > 0$; with no loss of generality we may conveniently consider the time scale so chosen

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that $\tau < 1$. Let $\mathscr{C}$ be the Banach space of continuous maps from the interval $[-\tau, 0]$ into $\mathscr{X}$ with the norm of $\phi \in \mathscr{C}$ given by

$$\|\phi\|_0 = \max_{-\tau \leqslant t \leqslant 0} \|\phi(t)\|.$$

If $t_0 \geqslant 0$ is a real number and x a continuous function from $[t_0 - \tau, \infty)$ into $\mathscr{X}$, then for each $t \in [t_0, \infty)$ we define the element $x_t \in \mathscr{C}$ by

$$x_t(s) = x(t+s) \qquad (-\tau \leqslant s \leqslant 0).$$

Let $A: \mathscr{X} \supset \mathscr{D}(A) \to \mathscr{X}$ be closed with dense domain. We shall assume that A generates an analytic semigroup $T_A(t)$ $(t \geqslant 0)$. This implies that, for all $x \in \mathscr{X}$, $T_A(t)\,x$ is differentiable for $t > 0$ and

$$(d/dt)\, T_A(t)\, x = A T_A(t)\, x,$$

and that there is a constant K such that [5–7]

$$\| T_A(t)\| \leqslant K, \qquad \| A T_A(t)\| \leqslant K/t.$$

With $R_+ = \{t \mid t \geqslant 0\}$, let $f: R_+ \times \mathscr{C} \to \mathscr{X}$ and $b: R_+ \to \mathscr{X}$. Then our problem can be formulated as: under what conditions on f and b is the equation

$$u'(t) = Au(t) + b(t) + f(t, u_t) \tag{3}$$

asymptotically equivalent to the equation

$$v'(t) = Av(t) + b(t), \tag{4}$$

in the sense that for any bounded solution of (3) there exists a bounded solution of (4) such that

$$\lim_{t \to \infty} \| u(t) - v(t)\| = 0,$$

and conversely. Here by a solution of (3) on $(0, T]$ we mean a function $u: [0, T] \to \mathscr{X}$ such that u is strongly continuous on $[0, T]$, strongly continuously differentiable on $(0, T]$, $u(t) \in \mathscr{D}(A)$ for $t \in (0, t]$, and $u(t)$ satisfies (3). A similar definition applies to (4).

The Abstract Theorem

The following technical lemma is well known [5, 7].

Lemma. *Let A be a closed operator in $\mathscr{X}$. Let $b \leqslant \infty$, let $c(t)$ be continuous on $[a, b)$ to $\mathscr{X}$ with $c(t) \in \mathscr{D}(A)$ and $Ac(t)$ continuous on $[a, b)$. If the improper integrals*

$$\int_a^b c(t)\, dt, \qquad \int_a^b Ac(t)\, dt$$

exist, then $\int_a^b c(t)\,dt \in \mathscr{D}(A)$ *and*

$$A \int_a^b c(t)\,dt = \int_a^b Ac(t)\,dt.$$

The following, our basic result on the asymptotic equivalence of (3) and (4), is an extension of a classical result for ordinary differential equations [1, 3]. Concrete applications to parabolic equations will be given in the last section of the paper.

THEOREM 1. *Let A generate the analytic semigroup $T_A(t)$, and let $P: \mathscr{X} \to \mathscr{X}$ be a bounded projection with range in the null space of A and which commutes with $T_A(t)$ for $t \geqslant 0$ ($P \equiv 0$ is allowed). Let $b(t)$ be uniformly Hölder continuous, and let $f(t, \phi)$ satisfy the following conditions:*

(a) *f is continuous on $R_+ \times \mathscr{C}$.*

(b) $\lim_{t\to\infty} \| f(t, 0)\| = 0$ *and* $\int_0^\infty \| f(t, 0)\|\,dt < \infty$.

(c) *For each $N > 0$ there exists $\gamma(s, N) \equiv \gamma(s)$ such that for $\|\phi_1\| \leqslant N$, $\|\phi_2\| \leqslant N$ we have $\| f(s, \phi_1) - f(s, \phi_2)\| \leqslant \gamma(s)\, \|\phi_1 - \phi_2\|_0$, where*

$$\lim_{t\to\infty} \gamma(t) = 0 \qquad \textit{and} \qquad \int_0^\infty \gamma(t)\,dt < \infty.$$

(d) $\| f(s, \phi) - f(t, \phi)\| \leqslant \mu(t, \|\phi\|_0)\, | s - t |^\beta$, *where $0 < \beta \leqslant 1$, $t + 1 > s > t > 0$, and μ is increasing in the second argument and continuous in the first. Then there is a one-to-one correspondence between bounded solutions of* (3) *and bounded solutions of* (4). *Moreover, if $\lim_{t\to\infty} \| T_A(t)(1 - P)x\| = 0$ for each fixed $x \in \mathscr{X}$ and if $u(t)$ is a bounded solution of* (3) *and $v(t)$ the corresponding solution of* (4), *then*

$$\lim_{t\to\infty} \| u(t) - v(t)\| = 0.$$

Proof. Let

$$\Psi_\alpha = \max\left(1, \sup_{0<\nu<1} \frac{(\nu + \tau)\ln(\nu + \tau) - \nu \ln \nu - \tau \ln \tau}{\nu^\alpha}\right);$$

straightforward application of l'Hôpital's rule shows that $\Psi_\alpha < \infty$ for $0 < \alpha < 1$. Let $y(t)$: $[t_2 - \tau, \infty) \to \mathscr{X}$ for some $t_2 \geqslant \tau$ satisfy (i) $y_t \in \mathscr{C}$ for $t \geqslant t_2$,

(ii) $\sup_{t\geqslant t_2} \| y_t \|_0 = \sup_{t\geqslant t_2 - \tau} \| y(t)\| \leqslant \rho$,

(iii) $\sup_{s>t>t_2, |s-t|<1} \| y_s - y_t \|_0 / | s - t |^\alpha \equiv M < \infty$ for some α, $0 < \alpha < 1$.

Condition (iii) states that y satisfies a sort of local Hölder continuity of exponent α with Hölder coefficient M. We have assumed that there is a K ($\geqslant 1$) such that

$\| T_A(t)\| \leqslant K$ for $t \geqslant 0$. Choose $t_1 \geqslant t_2$ so large that the following estimates are valid for $\gamma(t) = \gamma(t, 3\rho)$:

$$\Theta = K \int_{t_1}^{\infty} \gamma(t)\, dt < \min\left(\frac{1}{3}, \frac{M\tau}{72\rho}\right), \qquad \sup_{t \geqslant t_1 - \tau} \gamma(t) \leqslant \frac{M}{72K\rho\Psi_\alpha},$$

$$K \int_{t_1}^{\infty} \| f(s, 0)\|\, ds < \min\left(\rho, \frac{M\tau}{24}\right), \qquad \sup_{t \geqslant t_1 - \tau} \| f(s, 0)\| \leqslant \frac{M}{24K\Psi_\alpha}.$$

Let $\mathscr{S}_\alpha$ be the set of all functions x: $[t_1 - \tau, \infty) \to \mathscr{X}$ such that

(i) $x_t \in \mathscr{C}$ for $t \geqslant t_1$,

(ii) $\||x\|| = \sup_{t \geqslant t_1} \| x_t \|_0 = \sup_{t \geqslant t_1 - \tau} \| x(t)\| \leqslant 3\rho$,

(iii) $\sup_{s > t > t_1, s - t < 1} \| x_s - x_t \|_0 / | s - t |^\alpha \leqslant 2M$.

Then $\mathscr{S}_\alpha$ is a closed subset of the Banach space of continuous functions from $[t_1 - \tau, \infty)$ to $\mathscr{X}$ with norm $\|| \cdot \||$. To see this, it is necessary only to show that the uniform limit of functions satisfying a local Hölder condition with a fixed bound on the Hölder coefficient is also locally Hölder continuous with the same bound on the Hölder coefficient. Let then $\|| x_n - x \|| \to 0$, and let $\epsilon > 0$, s, and t $(0 < | s - t | < 1)$ be given. Choose n so large that $\| x_n(t) - x(t)\| < \epsilon | s - t |^\alpha$, $\| x_n(s) - x(s)\| < \epsilon | s - t |^\alpha$. Then

$$\frac{\| x(s) - x(t)\|}{| s - t |^\alpha} \leqslant \frac{\| x(s) - x_n(s)\|}{| s - t |^\alpha} + \frac{\| x_n(s) - x_n(t)\|}{| s - t |^\alpha} + \frac{\| x_n(t) - x(t)\|}{| s - t |^\alpha}$$
$$\leqslant 2\epsilon + 2M,$$

where $\epsilon > 0$ is arbitrary. It follows that

$$\frac{\| x(s) - x(t)\|}{| s - t |^\alpha} \leqslant 2M,$$

whence

$$\frac{\| x_s - x_t \|_0}{| s - t |^\alpha} = \sup_{-\tau \leqslant \theta \leqslant 0} \frac{\| x(s + \theta) - x(t + \theta)\|}{| s - t |^\alpha} \leqslant 2M,$$

and the closedness of $\mathscr{S}_\alpha$ follows.

For $x \in \mathscr{S}_\alpha$, we now define the operators $\mathscr{T}$ and $\mathscr{S}$ by

$$\begin{aligned} (\mathscr{T}x)(t) &= y(t) + (\mathscr{S}x)(t) \\ &= y(t) + \int_{t_1}^{t} T_A(t - s)(1 - P) f(s, x_s)\, ds - \int_{t}^{\infty} P f(s, x_s)\, ds \qquad (t \geqslant t_1), \\ &= y(t) - \int_{t_1}^{\infty} P f(s, x_s)\, ds \qquad (t_1 - \tau \leqslant t < t_1). \end{aligned}$$

Our hypotheses guarantee the existence of the integrals. It follows easily that $\mathscr{T}x$ is continuous for $t \geqslant t_1 - \tau$ and that for $t \geqslant t_1$

$$\| \mathscr{T}x(t)\| \leqslant \| y(t)\| + K \int_{t_1}^{t} (\gamma(s) \| x_s \|_0 + \| f(s, 0)\|) \, ds$$

$$+ \int_{t}^{\infty} (\gamma(s) \| x_s \|_0 + \| f(s, 0)\|) \, ds$$

$$\leqslant \rho + K \sup_{t \geqslant t_1 - \tau} \| x(t)\| \int_{t_1}^{\infty} \gamma(s) \, ds + K \int_{t_1}^{\infty} \| f(s, 0)\| \, ds$$

$$\leqslant \rho + 3\rho\Theta + \rho < 3\rho;$$

the final estimate is easily seen to be valid also for $t_1 \geqslant t \geqslant t_1 - \tau$. Thus $||| \mathscr{T}x ||| \leqslant 3\rho$.

We must also show that $\mathscr{T}x$ is locally Hölder continuous with exponent α and coefficient $2M$. Suppose first that $t + 1 > s > t \geqslant t_1 + \tau$; then

$$\|(\mathscr{T}x)_s - (\mathscr{T}x)_t \|_0$$

$$= \sup_{-\tau \leqslant \theta \leqslant 0} \| \mathscr{T}x(s + \theta) - \mathscr{T}x(t + \theta)\|$$

$$\leqslant \| y_s - y_t \|_0 + \sup_{-\tau \leqslant \theta \leqslant 0} \left\| \int_{t+\theta}^{s+\theta} Pf(\sigma, x_\sigma) \, d\sigma \right\|$$

$$+ \sup_{-\tau \leqslant \theta \leqslant 0} \left\| \int_{t+\theta}^{s+\theta} T_A(s + \theta - \sigma) \, (1 - P) f(\sigma, x_\sigma) \, d\sigma \right\|$$

$$+ \sup_{-\tau \leqslant \theta \leqslant 0} \left\| \int_{t_1}^{t+\theta} [T_A(s + \theta - \sigma) - T_A(t + \theta - \sigma)] \, (1 - P) f(\sigma, x_\sigma) \, d\sigma \right\|$$

$$= J_1 + J_2 + J_3 + J_4 .$$

For J_1 we have, of course, $J_1 \leqslant M \mid s - t \mid^\alpha$. For J_2 ,

$$J_2 \leqslant | \, s - t \, | \sup_{\sigma \geqslant t_1} \| f(\sigma, x_\sigma)\|$$

$$\leqslant | \, s - t \, | \, [||| \, x \, ||| \sup_{\sigma \geqslant t_1} \gamma(\sigma) + \sup_{\sigma \geqslant t_1} \| f(\sigma, 0)\|]$$

$$< \tfrac{1}{3} M \mid s - t \mid \leqslant \tfrac{1}{3} M \mid s - t \mid^\alpha$$

since K, ρ, $\Psi_\alpha \geqslant 1$. For J_3 we have

$$J_3 \leqslant K \sup_{-\tau\leqslant\theta\leqslant 0} \int_{t+\theta}^{s+\theta} (\gamma(\sigma) \| x_\sigma \|_0 + \| f(\sigma, 0)\|)\, d\sigma$$

$$\leqslant | s - t | \, [K ||| x ||| \sup_{\sigma\geqslant t_1} \gamma(\sigma) + K \sup_{\sigma\geqslant t_1} \| f(\sigma, 0)\|]$$

$$\leqslant \tfrac{1}{3} M \, | s - t |^\alpha.$$

J_4 can be estimated as follows.

$$J_4 \leqslant \sup_{-\tau\leqslant\theta\leqslant 0} \left\| \int_t^{t+\theta-\tau} [T_A(s+\theta-\sigma) - T_A(t+\theta-\sigma)]\,(1-P) f(\sigma, x_\sigma)\, d\sigma \right\|$$

$$+ \sup_{-\tau\leqslant\theta\leqslant 0} \left\| \int_{t+\theta-\tau}^{t+\theta} [T_A(s+\theta-\sigma) - T_A(t+\theta-\sigma)]\,(1-P) f(\sigma, x_\sigma)\, d\sigma \right\|$$

$$\equiv I_1 + I_2 .$$

Since A generates an analytic semigroup, we have $\| A T_A(t)\| \leqslant K/t$ and $(d/dt)\, T_A(t)\, x = A T_A(t)\, x$; it follows that for $s > t > 0$

$$\|[T_A(s) - T_A(t)]\, x \| = \left\| \int_t^s A T_A(\sigma)\, x \, d\sigma \right\| \leqslant \frac{K}{t} \, | s - t | \, \| x \| .$$

For I_1 we thus get the estimate

$$I_1 \leqslant \sup_{-\tau\leqslant\theta\leqslant 0} \int_{t_1}^{t+\theta-\tau} \frac{K}{\tau} \, | s - t | \, \| f(\sigma, x_\sigma)\| \, d\sigma$$

$$\leqslant | s - t | \, \frac{K}{\tau} \left[3\rho \int_{t_1}^{\infty} \gamma(\sigma)\, d\sigma + \int_{t_1}^{\infty} \| f(\sigma, 0)\| \, d\sigma \right] \leqslant \frac{1}{12} M \, | s - t | .$$

For I_2 we have

$$I_2 \leqslant \sup_{-\tau\leqslant\theta\leqslant 0} \int_{t+\theta-\tau}^{t+\theta} \int_{t+\theta-\sigma}^{s+\theta-\sigma} \| A T_A(\mu)\| \, d\mu \, \| f(\sigma, x_\sigma)\| \, d\sigma$$

$$\leqslant K \sup_{\sigma\geqslant t_1-\tau} \{\gamma(\sigma) \| x_\sigma \|_0 + \| f(\sigma, 0)\|\} \sup_{-\tau\leqslant\theta\leqslant 0} \int_{t+\theta-\tau}^{t+\theta} \int_{t+\theta-\sigma}^{s+\theta-\sigma} \frac{1}{\mu} \, d\mu \, d\sigma$$

$$\leqslant K \sup_{\sigma\geqslant t_1-\tau} \{3\rho\gamma(\sigma) + \| f(\sigma, 0)\|\} \{(s - t + \tau) \ln(s - t + \tau)$$

$$- (s - t) \ln(s - t) - \tau \ln \tau\}$$

$$\leqslant K\Psi_\alpha \, | s - t |^\alpha \{3\rho \sup_{\sigma\geqslant t_1-\tau} \gamma(\sigma) + \sup_{\sigma\geqslant t_1-\tau} \| f(\sigma, 0)\|\} \leqslant (M/12) \, | s - t |^\alpha.$$

We conclude that, for $t + 1 > s > t \geqslant t_1 + \tau$,

$$\|(\mathscr{T}x)_s - (\mathscr{T}x)_t\|_0 \leqslant 2M \mid s - t\mid^\alpha.$$

By similar arguments, we can dispose of the easier cases where $t + 1 > s > t > t_1 - \tau$ but where $s > t > t_1 + \tau$ does not hold, to conclude that $\mathscr{T}x$ satisfies the Hölder condition with coefficient $2M$ if x does. It follows that $\mathscr{T}$ maps $\mathscr{S}_\alpha$ into itself.

Note. We have made no use in the above argument of the assumption that x itself is Hölder-continuous. Thus $\mathscr{T}$ actually maps functions satisfying (i), (ii) into $\mathscr{S}_\alpha$.

It is easily seen that $\mathscr{T}$ is contracting on $\mathscr{S}_\alpha$. Indeed, let $x, \hat{x} \in \mathscr{S}_\alpha$; then for $t \geqslant t_1$,

$$\|(\mathscr{T}x)(t) - (\mathscr{T}\hat{x})(t)\| \leqslant K \int_{t_1}^{t} \|f(s, x_s) - f(s, \hat{x}_s)\| \, ds + \int_{t}^{\infty} \|f(s, x_s) - f(s, \hat{x}_s)\| \, ds$$

$$\leqslant K \int_{t_1}^{\infty} \gamma(s) \| x_s - \hat{x}_s \| \, ds \leqslant \Theta \||\, x - \hat{x} \,\||.$$

For $t_1 - \tau \leqslant t \leqslant t_1$,

$$\begin{aligned}\|(Tx)(t) - (T\hat{x})(t)\| &< \int_{t_1}^{\infty} \|f(s, x_s) - f(s, \hat{x}_s)\| \, ds \\ &\leqslant \int_{t_1}^{\infty} \gamma(s) \| x_s - \hat{x}_s \|_0 \, ds \leqslant \||\, x - \hat{x} \,\|| \int_{t_1}^{\infty} \gamma(s) \, ds \\ &= (\Theta/K) \||\, x - \hat{x} \,\|| \leqslant \Theta \||\, x - \hat{x} \,\||.\end{aligned}$$

Thus $\||\, \mathscr{T}x - \mathscr{T}\hat{x} \,\|| \leqslant \Theta \||\, x - \hat{x} \,\||$.

Now let y be a bounded solution of (4) defined for $t \geqslant t_1$. Then for $t > t_1$

$$y(t) = T_A(t - t_1)\, y(t_1) + \int_{t_1}^{t} T_A(t - s)\, b(s) \, ds$$

is differentiable and

$$y'(t) = AT_A(t - t_1)\, y(t_1) + b(t) + \int_{t_1}^{t} AT_A(t - s)\, b(s) \, ds$$

[7]. Let $t \geqslant t_1 + \tau$ and let the Hölder condition for b have the form $\| b(t) - b(s)\| \leqslant \bar{M} \mid t - s \mid_\nu$. Then

$$\| y'(t)\| \leqslant \frac{K}{\tau} \| y(t_1)\| + \| b(t)\| + \int_{t_1}^{t} \frac{K}{t-s} \| b(s) - b(t)\| \, ds$$
$$+ \left\| \int_{t_1}^{t} AT_A(t-s)\, b(t)\, ds \right\|$$
$$\leqslant \frac{K}{\tau} \| y(t_1)\| + (2 + K) \| b(t)\| + \bar{M}K \int_{t_1}^{t} (t-s)^{\nu - 1}\, ds,$$

since

$$-\int_{t_1}^{t} AT_A(t-s)\, b(t)\, ds = T_A(0)\, b(t) - T_A(t - t_1)\, b(t).$$

Thus $\| y'(t)\|$ is bounded uniformly on compact subsets of $[t_1 + \tau, \infty)$. From

$$\| y(s+\theta) - y(t+\theta)\| \leqslant | s - t | \sup_{t \leqslant \sigma \leqslant s} \| y'(\sigma + \theta)\| ,$$

it follows that y is locally Hölder continuous for $t > t_1 + \tau$ for any exponent α, $0 < \alpha \leqslant 1$.

Define x as the fixed point of $\mathscr{T}$; then $x \in \mathscr{S}_\alpha$. We show now that $x \in \mathscr{D}(A)$ for $t \geqslant t_1$. Since $y \in \mathscr{D}(A)$ and the range of P is in the null space of A, it is enough to show that

$$\int_{t_1}^{t} T_A(t-s)\, (1 - P) f(s, x_s)\, ds \in \mathscr{D}(A),$$

and by the lemma it suffices to show that the improper integral

$$\int_{t_1}^{t} AT_A(t-s)\, (1 - P) f(s, x_s)\, ds$$

converges. Since the only difficulty occurs at $s = t$, it is enough to show that

$$L(\epsilon) \equiv \int_{t-\tau}^{t-\epsilon} AT_A(t-s)\, (1 - P) f(s, x_s)\, ds$$

converges as $\epsilon \to 0+$. To that end we write

$$L(\epsilon) = \int_{t-\tau}^{t-\epsilon} AT_A(t-s)\, (1 - P)\, [f(s, x_s) - f(s, x_t)]\, ds$$
$$+ \int_{t-\tau}^{t-\epsilon} AT_A(t-s)\, (1 - P)\, [f(s, x_t) - f(t, x_t)]\, ds$$
$$+ \int_{t-\tau}^{t-\epsilon} AT_A(t-s)\, (1 - P) f(t, x_t)\, ds = L_1(\epsilon) + L_2(\epsilon) + L_3(\epsilon)$$

for arbitrary $\epsilon > 0$. For L_1 we have

$$\| L_1(\epsilon) \| \leqslant K \int_{t-\tau}^{t-\epsilon} \frac{1}{t-s} \gamma(s) \| x_s - x_t \|_0 \, ds;$$

the right-hand side converges as $\epsilon \to 0+$ because of the Hölder continuity of x. Similarly

$$\| L_2(\epsilon) \| \leqslant K \int_{t-\tau}^{t-\epsilon} \frac{1}{t-s} \mu(s, 3\rho) \mid t - s \mid^{\beta} ds,$$

which also converges as $\epsilon \to 0+$. Finally,

$$\begin{aligned} L_3(\epsilon) &= - \int_{t-\tau}^{t-\epsilon} \frac{d}{ds} \{ T_A(t-s)(1-P) f(t, x_t) \} \, ds \\ &= T_A(\tau)(1-P) f(t, x_t) - T_A(\epsilon)(1-P) f(t, x_t), \end{aligned}$$

which certainly converges as $\epsilon \to 0$. We have thus shown that $\mathscr{T}x \in \mathscr{D}(A)$ and that

$$(A\mathscr{T}x)(t) = Ay(t) + \int_{t_1}^{t} AT_A(t-s)(1-P) f(s, x_s) \, ds$$

for $t \geqslant t_1$.

We show now that x is differentiable and satisfies the differential equation (3) for $t > t_1$. It suffices to show that $\mathscr{S}x$ is differentiable for $t > t_1$. Consider first

$$\frac{1}{h} \left\{ \int_{t+h}^{\infty} Pf(s, x_s) \, ds - \int_{t}^{\infty} Pf(s, x_s) \, ds \right\} = \frac{1}{h} \int_{t}^{t+h} Pf(s, x_s) \, ds,$$

which converges to $Pf(t, x_t)$ as $h \to 0$. Also,

$$\begin{aligned} &\frac{1}{h} \left\{ \int_{t_1}^{t+h} T_A(t+h-s)(1-P) f(s, x_s) \, ds - \int_{t_1}^{t} T_A(t-s)(1-P) f(s, x_s) \, ds \right\} \\ &= \frac{1}{h} \left\{ \int_{t}^{t+h} T_A(t+h-s)(1-P) f(s, x_s) \, ds \right. \\ &\quad \left. + \int_{t_1}^{t} [T_A(t+h-s) - T_A(t-s)](1-P) f(s, x_s) \, ds \right\}. \end{aligned}$$

For the second of these integrals we have

$$\frac{1}{h}\int_{t_1}^{t}[T_A(t+h-s)-T_A(t-s)]\,(1-P)f(s,x_s)\,ds$$

$$=\frac{1}{h}[T_A(h)-I]\int_{t_1}^{t}T_A(t-s)\,(1-P)f(s,x_s)\,ds$$

$$\to A\int_{t_1}^{t}T_A(t-s)\,(1-P)f(s,x_s)\,ds$$

$$=\int_{t_1}^{t}AT_A(t-s)\,(1-P)f(s,x_s)\,ds,$$

as shown above. We show now that the first integral converges to

$$(1-P)f(t,x_t)=\frac{1}{h}\int_{t}^{t+h}(1-P)f(t,x_t)\,ds.$$

To prove this, we show that the difference, which can be written as

$$\frac{1}{h}\int_{t}^{t+h}T_A(t+h-s)\,(1-P)\,[f(s,x_s)-f(s,x_t)]\,ds$$

$$+\frac{1}{h}\int_{t}^{t+h}T_A(t+h-s)\,(1-P)\,[f(s,x_t)-f(t,x_t)]\,ds$$

$$+\frac{1}{h}\int_{t}^{t+h}[T_A(t+h-s)-I]\,(1-P)f(t,x_t)\,ds$$

$$=K_1+K_2+K_3,$$

converges to zero as $h\to 0$. For the norm of K_1, we have the estimate

$$\|K_1\|\leqslant\frac{K}{h}\int_{t}^{t+h}\gamma(s)\,\|x_s-x_t\|_0\,ds\leqslant\frac{2MK}{h}\int_{t}^{t+h}\gamma(s)\,(s-t)^{\alpha}\,ds,$$

which tends to zero as $h\to 0$. Similarly, for K_2 we have the bound

$$\|K_2\|\leqslant\frac{K}{h}\int_{t}^{t+h}\mu(t,3\rho)\,|s-t|^{\beta}\,ds,$$

which again tends to zero with h. Finally, with the change of variable $z=t+h-s$, K_3 can be written as

$$\frac{1}{h}\int_{0}^{h}[T_A(z)-I]\,(1-P)f(t,x_t)\,dz,$$

which is well-known to converge to $[T_A(0)-I]\,(1-P)f(t,x_t)=0$.

Putting the results of these computations together, we have shown that $x = y + \mathscr{S}x$ is differentiable and that

$$\begin{aligned}\frac{dx}{dt}(t) &= \frac{dy}{dt}(t) + f(t, x_t) + \int_{t_1}^{t} AT_A(t-s)(1-P)f(s, x_s)\,ds \\ &= Ax(t) + b(t) + f(t, x_t)\end{aligned}$$

for $t > t_1$, as required.

To show that the map $y \to x$ defined above is one-to-one and onto, let $x(t)$ be a bounded solution of (3) and define y by $y = x - \mathscr{S}x$. Using the above analysis it is easy to show that y is a bounded solution of (4).

Suppose now that $T_A(t)(1-P)x \to 0$ as $t \to \infty$ for any $x \in \mathscr{X}$. We must show that

$$\int_{t_1}^{t} T_A(t-s)(1-P)f(s, x_s)\,ds \to 0 \qquad \text{as} \qquad t \to \infty.$$

For any $T > t_1$ we have

$$\int_{t_1}^{T} T_A(t-s)(1-P)f(s, x_s)\,ds = T_A(t-T)(1-P)\int_{t_1}^{T} T_A(T-s)f(s, x_s)\,ds,$$

which converges to zero as $t \to \infty$; we have here used the commutativity of T_A and P. Finally,

$$\begin{aligned}\left\| \int_{T}^{t} T_A(t-s)(1-P)f(s, x_s)\,ds \right\| &\leqslant K \int_{T}^{t} \|f(s, x_s)\|\,ds \\ &\leqslant K \int_{T}^{\infty} \{\gamma(s)\,|||\,x\,|||+ \|f(s, 0)\|\}\,ds,\end{aligned}$$

which can be made arbitrarily small by choosing T sufficiently large. This completes the proof of the theorem.

Remarks. The hypotheses of the theorem become simpler if $f(t, \phi)$ can be factored as $g(t)\,h(\phi)$. Three sorts of factoring are possible: g scalar-valued and $h: \mathscr{C} \to \mathscr{X}$; $g: [0, \infty) \to \mathscr{X}$ and h scalar-valued; or if $\mathscr{X}$ is a Banach algebra, both g and h taking values in $\mathscr{X}$. In any of these cases the following hypotheses imply (a)–(d) of Theorem 1:

(a) g, h, are continuous,

(b) $g(t) \to 0$ as $t \to \infty$ and $\int_0^\infty |g(t)|\,dt < \infty$,

(c) for any $N > 0$ and $\phi_1, \phi_2 \in \mathscr{C}$ satisfying $\|\phi_1\| \leqslant N$, $\|\phi_2\| \leqslant N$, we have

$$|h(\phi_1) - h(\phi_2)| \leqslant \text{const.}\,\|\phi_1 - \phi_2\|_0\,,$$

(d) for $|s-t| \leqslant 1$ we have $|g(s)-g(t)| \leqslant \mu(t)\,|s-t|^{\beta}$ for μ defined and continuous on $[0,\infty)$ and $0<\beta\leqslant 1$.
Here $|\cdot|$ denotes either $\|\cdot\|$ or $|\cdot|$, as appropriate. Only (d) of the theorem perhaps requires demonstration; we have

$$\begin{aligned}\|f(s,\phi)-f(t,\phi)\| &\leqslant |g(s)-g(t)|\,[|h(\phi)-h(0)|+|h(0)|]\\ &\leqslant \mu(t)\,[\text{const.}\,\|\phi\|_0+|h(0)|]\,|s-t|^{\beta}.\end{aligned}$$

Also, a sum of terms each a product satisfying (a)–(d) above satisfies the hypotheses of Theorem 1.

We also observe that the last hypothesis of Theorem 1 is satisfied if A has a bounded inverse. Indeed, in this case we have

$$\begin{aligned}\|T_A(t)(1-P)x\| &= \|T_A(t)x\| = \|T_A(t)AA^{-1}x\|\\ &\leqslant \|AT_A(t)\|\,\|A^{-1}x\| \leqslant (K/t)\,\|A^{-1}\|,\end{aligned}$$

using the fact that A and P commute with $T_A(t)$.

Applications

Here we shall give some applications of Theorem 1, without striving for maximum generality.

1. Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary, let $\mathscr{X}$ be the Hilbert space $\mathscr{L}^2(\Omega)$, and take $P=0$. Let for $x\in\Omega$ the linear differential operator $B(x,D)$ be defined by

$$B(x,D)\,u(x)=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\,\frac{\partial u(x)}{\partial x_i}\right)-c(x)\,u(x),$$

where $c\geqslant 0$; we suppose that B is formally self-adjoint, uniformly strongly elliptic, and has smooth coefficients. Define the operator A by

$$Au=B(x,D)\,u$$

for $u\in\mathscr{D}(A)=H^2(\Omega)\cap H_0^1(\Omega)$ (standard notation). Then $-A$ is a self-adjoint and nonnegative operator, and so generates an analytic semigroup $T_A(t)$ [6]. Moreover, by Theorem 1 of [4, Chap. 6],

$$\|T_A(t)\,x\|_{L_2(\Omega)}\to 0$$

as $t\to\infty$.

The condition $u\in H_0^1(\Omega)$ is a generalized form of the boundary condition

$u|_{\partial\Omega} = 0$; cf. [5]. Since we allow nonhomogeneous equations, there is no loss of generality in assuming homogeneous boundary conditions.

Let $b(t, x)$ be in $\mathscr{L}^2(\Omega)$ for each $t \geqslant 0$ and be uniformly Hölder continuous in t. Let $f(t, x, \phi) \in \mathscr{L}^2(\Omega)$ for each $t \geqslant 0$ and $\phi \in \mathscr{C}$, and let f satisfy hypotheses (a)–(d) of Theorem 1. We conclude the following. For every (generalized) solution of

$$\frac{\partial}{\partial t} v_t(t, x) = B(x, D)\, v(t, x) + b(t, x),$$

$$v|_{\partial\Omega} = 0$$

which has bounded $\mathscr{L}^2(\Omega)$ norm there exists for all sufficiently large t a (generalized) solution $u(t, x)$ of

$$\frac{\partial}{\partial t} u_t(t, x) = B(x, D)\, u(t, x) + b(t, x) + f(t, x, u_t),$$

$$u|_{\partial\Omega} = 0$$

such that $\| u - v \|_{\mathscr{L}^2(\Omega)} \to 0$ as $t \to \infty$, and vice versa.

2. The projection P plays a nontrivial role when A is an elliptic partial differential operator without constant term and $\mathscr{X}$ contains constant functions; for example, the heat equation on a bounded region Ω in $\mathbb{R}^n$ with boundary conditions $\partial u/\partial \nu|_{\partial\Omega} = 0$, where $\partial/\partial\nu$ denotes the derivative along the normal to the boundary of Ω. Here we give a simpler example of a nontrivial P: the Cauchy problem for the one-dimensional heat equation.

Let $\mathscr{X}_1$ be the set of functions h defined and continuous on $(-\infty, \infty)$ and satisfying $\lim_{|x|\to\infty} h(x) = 0$. Let C denote the constant functions on $(-\infty, \infty)$, and let $\mathscr{X} = \mathscr{X}_1 + C$ with the sup norm. Then $\mathscr{X}$ is a Banach space. Let $A = \partial^2/\partial x^2$ on $\mathscr{D}(A) = \{h \in \mathscr{X}: h'' \in \mathscr{X}\}$. Let P denote projection onto C, the null space of A. It is well known that A generates the analytic semigroup T_A given by

$$[T_A(t)\, h]\,(x) = \frac{1}{2(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-s)^2/(4\pi t)} h(s)\, ds.$$

An elementary integration shows that $T_A(t)\, 1 \equiv 1$, hence $h \in C$ implies that $T_A(t)\, h \in C$, i.e., P commutes with T_A. Let now h be in the range of $I - P$, that is, $h \in \mathscr{X}_1$. Let $\epsilon > 0$ be given, and choose M so large that $s \geqslant M$ or $s \leqslant -M$ implies $|h(s)| < \epsilon/4$. Then

$$\left| \frac{1}{2(\pi t)^{1/2}} \int_{M}^{\infty} e^{-(x-s)^2/(4\pi t)} h(s)\, ds \right|, \left| \frac{1}{2(\pi t)^{1/2}} \int_{-\infty}^{-M} e^{-(x-s)^2/(4\pi t)} h(s)\, ds \right| < \frac{\epsilon}{4}.$$

It follows that for all $t > 0$

$$|[T_A(t)\,h]\,(x)| < \frac{\epsilon}{2} + \frac{1}{2(\pi t)^{1/2}} \int_{-M}^{M} e^{-(x-s)^2/(4\pi t)}\,|\,h(s)|\,ds$$

$$\leqslant \frac{\epsilon}{2} + \frac{M}{(\pi t)^{1/2}} \sup_s |\,h(s)|\,,$$

which can be made less than ϵ by choosing t sufficiently large. We have thus shown that $\|\,T_A(t)\,(1 - P)\,g\,\| \to 0$ as $t \to \infty$ for any $g \in \mathscr{X}$. Theorem 1 can now be applied for any suitable f and b.

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