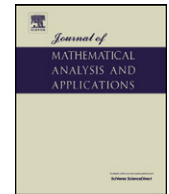Contents lists available at SciVerse ScienceDirect

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Generalizations of an integral for Legendre polynomials by Persson and Strang

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A B S T R A C T

Persson and Strang (2003) evaluated the integral over $[-1, 1]$ of a squared odd degree Legendre polynomial divided by x^2 as being equal to 2. We consider a similar integral for orthogonal polynomials with respect to a general even orthogonality measure, with Gegenbauer and Hermite polynomials as explicit special cases. Next, after a quadratic transformation, we are led to the general nonsymmetric case, with Jacobi and Laguerre polynomials as explicit special cases. Examples of indefinite summation also occur in this context. The paper concludes with a generalization of the earlier results for Hahn polynomials. There some adaptations have to be made in order to arrive at relatively nice explicit evaluations.

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1. Introduction

The idea of this article comes from an integral

$$\int_{-1}^{1} \left(\frac{P_{2n+1}(x)}{x} \right)^2 dx = 2 \tag{1.1}$$

given by Persson and Strang [11, (34)]. Here P_n is a Legendre polynomial. They prove the identity by deriving a first order recurrence for the left-hand side of (1.1). A natural question is if this integral can be generalized for other orthogonal polynomials. We consider this problem first for orthogonal polynomials with respect to an even orthogonality measure (the symmetric case), a class which includes the Legendre polynomials, but also, for instance, the Gegenbauer and Hermite polynomials. The method of finding a first order recurrence for the integral still works, but a method involving the Christoffel–Darboux formula (to some extent equivalent to the earlier method) turns out to be more powerful in the case of the Gegenbauer polynomials.

By applying a quadratic transformation to the symmetric case we arrive at the idea of a further generalization in the case of a general orthogonality measure. Kernel polynomials, defined in terms of the Christoffel–Darboux kernel, enter the integral here. Explicit examples are considered for Jacobi and Laguerre polynomials. Explicitly summable indefinite sums naturally occur here as side results.

The original Persson–Strang integral was motivated by a particular application. Related motivations may also be given for our generalizations discussed until now (see also Remark 3.1). But our last section on Hahn polynomials is driven by

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the pure mathematical question how the Persson–Strang integral and the related indefinite sum may generalize throughout the Askey scheme. For the Hahn polynomials it turns out that the most straightforward generalizations do not admit nice explicit evaluations. We are able to make there an adaptation which admits relatively nice evaluations and which still has our results for Jacobi polynomials as a limit case.

2. Preliminaries on orthogonal polynomials

Let μ be a nonzero Borel measure on $\mathbb{R}$ with infinite support (or equivalently a nondecreasing function on $\mathbb{R}$ with an infinite number of points of increase) such that $\int_{\mathbb{R}} |x|^n \, d\mu(x) < \infty$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Let $p_n(x)$ ($n \in \mathbb{Z}_{\geqslant 0}$) be a polynomial in x of degree n such that

$$\int_{\mathbb{R}} p_m(x) p_n(x) \, d\mu(x) = h_n \delta_{m,n} \quad (m, n \in \mathbb{Z}_{\geqslant 0}) \tag{2.1}$$

for certain constants h_n (necessarily positive). The polynomials p_n are called *orthogonal polynomials* with respect to the measure μ, see for instance [1, §5.2], [3] or [13]. Up to constant nonzero factors they are uniquely determined by the above properties. Let k_n be the coefficient of x^n in $p_n(x)$. We assume that $p_0(x) = k_0 = 1$.

If μ has support within some closed interval I then we say that the p_n are orthogonal polynomials with respect to μ on I. In many examples we have $d\mu(x) = w(x) \, dx$ on I with the *weight function* w a nonnegative integrable function on I. Then (2.1) becomes

$$\int_I p_m(x) p_n(x) w(x) \, dx = h_n \delta_{m,n}.$$

In many other examples μ is a discrete measure given by positive weights w_j on points x_j ($j \in J$, a countably infinite set). Then (1.1) becomes

$$\sum_{j \in J} p_m(x_j) p_n(x_j) w_j = h_n \delta_{m,n}.$$

We may take J finite, say $J = \{0, 1, \ldots, N\}$. Then the p_n are well defined by orthogonality for $n = 0, 1, \ldots, N$.

Orthogonal polynomials satisfy a three-term recurrence relation (see [1, Theorem 5.2.2])

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x), \qquad p_{-1}(x) = 0, \qquad p_0(x) = 1, \tag{2.2}$$

with A_n, B_n, C_n real and $A_{n-1} A_n C_n > 0$. If the measure μ is *even* (i.e., invariant under reflection with respect to 0) then $B_n = 0$ for all n in (2.2). If μ is even and $d\mu(x) = w(x) \, dx$ on I then $I = -I$ and $w(x) = w(-x)$ ($x \in I$).

It follows immediately from (2.2) that

$$A_n = \frac{k_{n+1}}{k_n}, \qquad C_n = \frac{k_{n-1} h_n A_n}{k_n h_{n-1}} = \frac{k_{n-1} k_{n+1} h_n}{k_n^2 h_{n-1}}. \tag{2.3}$$

Furthermore,

$$C_{2n-1} = -\frac{p_{2n}(0)}{p_{2n-2}(0)} \quad \text{if } \mu \text{ is even.} \tag{2.4}$$

We will also need the *Christoffel–Darboux formula* (see [1, Remark 5.2.2])

$$K_n(x, y) := \sum_{k=0}^{n} \frac{p_k(x) p_k(y)}{h_k} = \frac{k_n}{k_{n+1} h_n} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x - y} \quad (x \neq y). \tag{2.5}$$

Note that K_n is the kernel of the integral operator which projects onto the space of polynomials of degree $\leqslant n$. Thus

$$\int_{\mathbb{R}} p(x) K_n(x, y) \, d\mu(x) = p(y) \quad (p \text{ a polynomial of degree } \leqslant n). \tag{2.6}$$

If we let $y \to x$ in (2.5) then we obtain

$$K_n(x, x) = \sum_{k=0}^{n} \frac{p_k(x)^2}{h_k} = \frac{k_n}{k_{n+1} h_n} \left(p'_{n+1}(x) p_n(x) - p'_n(x) p_{n+1}(x) \right). \tag{2.7}$$

If the orthogonality measure μ is even then we have as a special case of (2.7):

$$K_{2n}(0, 0) = \sum_{k=0}^{n} \frac{p_{2k}(0)^2}{h_{2k}} = \frac{k_{2n}}{k_{2n+1} h_{2n}} p'_{2n+1}(0) p_{2n}(0). \tag{2.8}$$

Remark 2.1. If we are dealing with classical orthogonal polynomials p_n then p_n' is again a classical orthogonal polynomial, so (2.7) can be written very explicitly. However, if we are dealing with orthogonal polynomials p_n in the Hahn class, i.e., where the polynomials $\{\Delta p_n\}_{n\in\mathbb{Z}_{\geqslant 1}}$ are again orthogonal (with the difference operator Δ being defined by $(\Delta f)(x) := f(x+1) - f(x)$), then we may better consider (2.5) for $y = x - 1$ (see [7, formula after (2.11)] for an analogous observation in the q-case), which yields

$$
\begin{aligned}
K_n(x, x-1) &= \sum_{k=0}^{n} \frac{p_k(x) p_k(x-1)}{h_k} \\
&= \frac{k_n}{k_{n+1} h_n} \big(p_n(x)(\Delta p_{n+1})(x-1) - p_{n+1}(x)(\Delta p_n)(x-1) \big).
\end{aligned}
\tag{2.9}
$$

Formula (2.9) will take a very explicit form for polynomials of Hahn class. If we consider (2.9) for Hahn polynomials then a suitable limit case will yield formula (2.7) for Jacobi polynomials.

3. The Persson–Strang integral generalized: the symmetric case

As a generalization of the integral of Persson and Strang we want to compute the integral

$$
I_n := \int_{-a}^{a} \left(\frac{p_{2n+1}(x)}{x} \right)^2 d\mu(x),
\tag{3.1}
$$

where the p_n are orthogonal polynomials with respect to an even measure μ on the interval $[-a, a]$ or $(-\infty, \infty)$. We will discuss two methods to solve this problem. The first method, followed by Persson and Strang in the Legendre case, is by a recurrence relation for I_n. The second method uses the Christoffel–Darboux formula.

3.1. The recurrence method

Rewriting Eq. (2.2) with $B_n = 0$ as

$$
\frac{p_{2n+1}(x)}{x} = A_{2n} p_{2n}(x) - C_{2n} \frac{p_{2n-1}(x)}{x},
$$

squaring, and integrating over the interval $[-a, a]$ with respect to μ gives

$$
\int_{-a}^{a} \left(\frac{p_{2n+1}(x)}{x} \right)^2 d\mu(x) = A_{2n}^2 \int_{-a}^{a} p_{2n}(x)^2 d\mu(x) - 2 A_{2n} C_{2n} \int_{-a}^{a} p_{2n}(x) \frac{p_{2n-1}(x)}{x} d\mu(x) + C_{2n}^2 \int_{-a}^{a} \left(\frac{p_{2n-1}(x)}{x} \right)^2 d\mu(x)
$$

$$
= A_{2n}^2 h_{2n} + C_{2n}^2 \int_{-a}^{a} \left(\frac{p_{2n-1}(x)}{x} \right)^2 d\mu(x).
$$

Hence we obtain the recurrence

$$
I_n = C_{2n}^2 I_{n-1} + A_{2n}^2 h_{2n}
\tag{3.2}
$$

with starting value

$$
I_0 = \int_{-a}^{a} \left(\frac{p_1(x)}{x} \right)^2 w(x)\, dx = k_1^2 h_0.
\tag{3.3}
$$

The first order inhomogeneous linear recurrence relation (3.2) with initial value (3.3) has a unique solution. As observed by Persson and Strang, in the case of Legendre polynomials the solution is $I_n = 2$ for all n, since this solution satisfies (3.2) and (3.3).

3.2. Using the Christoffel–Darboux formula

Putting $y = 0$ in the Christoffel–Darboux formula (2.5) gives

$$K_{2n+1}(x,0) = \sum_{k=0}^{2n+1} \frac{p_k(x)p_k(0)}{h_k} = K_{2n}(x,0)$$

$$= \sum_{k=0}^{n} \frac{p_{2k}(x)p_{2k}(0)}{h_{2k}} = \frac{k_{2n}p_{2n}(0)}{k_{2n+1}h_{2n}} \frac{p_{2n+1}(x)}{x}. \tag{3.4}$$

Remark 3.1. Combination of (3.4) with (2.6) yields

$$\frac{k_{2n}p_{2n}(0)}{k_{2n+1}h_{2n}} \int_{-a}^{a} p(x) \frac{p_{2n+1}(x)}{x} \, d\mu(x) = p(0) \quad (p \text{ a polynomial of degree} \leqslant 2n+1). \tag{3.5}$$

Thus the linear functional $\lambda: p \mapsto p(0)$ on the finite Hilbert space of real-valued polynomials of degree $\leqslant 2n+1$ with respect to the inner product $\langle p, q \rangle := \int_{-a}^{a} p(x)q(x)\,d\mu(x)$ gives $\lambda(p)$ as a constant times the inner product of p with the polynomial $x \mapsto p_{2n+1}(x)/x$. Therefore, the square of the norm $\|\lambda\|^2$ of the linear functional λ equals a constant times the integral I_n given by (3.1). This gives a motivation for trying to compute I_n explicitly.

Another motivation considers the left-hand side of (3.5) with p being a white noise signal on $[-a,a]$. Then this expression equals the projection of p on the subspace of polynomials of degree $\leqslant 2n+1$, evaluated at 0. This value is a random variable. The expectation of the square of this value equals a constant times I_n. This is related to the motivation in Persson and Strang [11, §4].

Now square the two sides of the last equality in (3.4) and integrate over the orthogonality interval with respect to μ, where we use the orthogonality property. As a result we obtain

$$I_n = \int_{-a}^{a} \left(\frac{p_{2n+1}}{x} \right)^2 d\mu(x) = \left(\frac{k_{2n+1}h_{2n}}{k_{2n}p_{2n}(0)} \right)^2 \sum_{k=0}^{n} \frac{p_{2k}(0)^2}{h_{2k}}. \tag{3.6}$$

Then a very simple expression for I_n can be obtained by substitution of (2.8) in (3.6):

$$\int_{-a}^{a} \left(\frac{p_{2n+1}(x)}{x} \right)^2 d\mu(x) = \frac{k_{2n+1}h_{2n}p'_{2n+1}(0)}{k_{2n}p_{2n}(0)}. \tag{3.7}$$

Remark 3.2. The sum (3.6) is equivalent to the recurrence

$$\left(\frac{k_{2n}p_{2n}(0)}{k_{2n+1}h_{2n}} \right)^2 I_n = \left(\frac{k_{2n-2}p_{2n-2}(0)}{k_{2n-1}h_{2n-2}} \right)^2 I_{n-1} + \frac{p_{2n}(0)^2}{h_{2n}}$$

with starting value (3.3). The recurrence can be rewritten as

$$I_n = \left(\frac{k_{2n-2}k_{2n+1}h_{2n}p_{2n-2}(0)}{k_{2n-1}k_{2n}h_{2n-2}p_{2n}(0)} \right)^2 I_{n-1} + \left(\frac{k_{2n+1}}{k_{2n}} \right)^2 h_{2n}. \tag{3.8}$$

In view of (2.3) and (2.4), the recurrences (3.8) and (3.2) are the same.

3.3. Example: Gegenbauer polynomials

With the notation of Section 3.2 let $\alpha > -1$ and take orthogonality measure $d\mu(x) := (1-x^2)^\alpha \, dx$ on the interval $[-1,1]$. Then the p_n are *Gegenbauer polynomials* which we write as special Jacobi polynomials (see [4, 10.9(4)]):

$$p_n(x) = P_n^{(\alpha,\alpha)}(x) = \frac{(\alpha+1)_n}{(2\alpha+1)_n} C_n^{\alpha+\frac{1}{2}}(x).$$

For the evaluation of the right-hand side of (3.7) in this case we need (see [4, §10.8, 10.9]):

$$k_n = \frac{(n+2\alpha+1)_n}{2^n n!}, \qquad h_n = \frac{2^{2\alpha+1}\Gamma(n+\alpha+1)^2}{(2n+2\alpha+1)\Gamma(n+2\alpha+1)n!},$$

$$p_{2n}(0) = \frac{(\alpha+1)_{2n}}{(2\alpha+1)_{2n}} C_{2n}^{\alpha+\frac{1}{2}}(0) = (-1)^n \frac{(\alpha+n+1)_n}{2^{2n}n!},$$

$$p'_{2n+1}(0) = (n+\alpha+1) P_{2n}^{(\alpha+1,\alpha+1)}(0) = (-1)^n \frac{(\alpha+n+2)_n}{2^{2n}n!}.$$

Then (3.7) yields

$$I_n = \int\limits_{-1}^{1} \left(\frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{x} \right)^2 \left(1 - x^2\right)^\alpha dx = \frac{2^{2\alpha+1} \Gamma(2n+\alpha+2)^2}{\Gamma(2n+2\alpha+2)(2n+1)!}. \tag{3.9}$$

As special cases of (3.9) we note:

- $\alpha = 0$, Legendre polynomials $P_n := P_n^{(0,0)}$. Then $I_n = 2$ and we recover (1.1).
- $\alpha = -\frac{1}{2}$, Chebyshev polynomials of the first kind $T_n := \frac{n!}{(1/2)_n} P_n^{(-1/2,-1/2)}$. Then

$$I_n = \int\limits_{-1}^{1} \left(\frac{T_{2n+1}(x)}{x} \right)^2 \left(1 - x^2\right)^{-1/2} dx = (2n+1)\pi.$$

This corresponds to (3.2) and (3.3) becoming $I_n = I_{n-1} + 2\pi$, $I_0 = \pi$.
- $\alpha = \frac{1}{2}$, Chebyshev polynomials of the second kind $U_n := \frac{(n+1)!}{(3/2)_n} P_n^{(1/2,1/2)}$. Then

$$I_n = \int\limits_{-1}^{1} \left(\frac{U_{2n+1}(x)}{x} \right)^2 \left(1 - x^2\right)^{1/2} dx = (2n+2)\pi.$$

This corresponds to (3.2) and (3.3) becoming $I_n = I_{n-1} + 2\pi$, $I_0 = 2\pi$.

3.4. Example: Hermite polynomials

With the notation of Section 3.2 take $d\mu(x) := e^{-x^2} dx$ on the interval $(-\infty, \infty)$. Then the p_n are *Hermite polynomials*,

$$p_n(x) = H_n(x),$$

for which we have (see [4, §10.13]):

$$h_n = \pi^{1/2} 2^n n!, \qquad k_n = 2^n,$$
$$p_{2n}(0) = (-1)^n 2^{2n} \left(\frac{1}{2} \right)_n, \qquad p'_{2n+1}(0) = 2(2n+1) p_{2n}(0).$$

Then (3.7) yields

$$I_n = \int\limits_{-\infty}^{\infty} \left(\frac{H_{2n+1}(x)}{x} \right)^2 e^{-x^2} dx = \pi^{1/2} 2^{2n+2}(2n+1)!.$$

4. Persson–Strang type integrals for general measures

4.1. Quadratic transformation

Let the polynomials p_n be orthogonal with respect to an even weight function w on the interval $(-a, a)$. For nonzero constants C_n define polynomials q_n of degree n by

$$p_{2n+1}(x) = c_n x q_n\left(x^2\right).$$

Then the polynomials q_n are orthogonal with respect to the measure $x^{1/2} w(x^{1/2}) dx$ on the interval $[0, \sqrt{a})$, see Chihara [3, Ch. I, §8]. We can also rewrite the integral (3.1) in terms of the polynomials q_n:

$$\int\limits_{-a}^{a} \left(\frac{p_{2n+1}(x)}{x} \right)^2 w(x) \, dx = c_n^2 \int\limits_{0}^{\sqrt{a}} q_n(x)^2 x^{-1/2} w\left(x^{1/2}\right) dx.$$

This suggests that, for an orthogonality measure μ on an interval $[0, a]$ and for orthogonal polynomials q_n with respect to the measure $x \, d\mu(x)$ on $[0, a]$ the integral

$$\int\limits_0^a q_n(x)^2 \, d\mu(x)$$

may have a nice evaluation. Moreover, we recognize the polynomials q_n as *kernel polynomials* corresponding to the orthogonal polynomials on $[0, a]$ with respect to measure μ.

4.2. Using kernel polynomials

Let $\{p_n\}$ be a system of orthogonal polynomials with respect to an orthogonality measure μ with support within $(-\infty, a]$. Let k_n be the coefficient of x^n in $p_n(x)$. Let $x_0 \geqslant a$. For certain nonzero constants c_n put

$$q_n(x) := c_n K_n(x_0, x) = c_n \sum_{k=0}^n \frac{p_k(x_0) p_k(x)}{h_k}. \tag{4.1}$$

Then $\{q_n\}$ is a system of orthogonal polynomials with respect to the orthogonality measure $(x_0 - x) \, d\mu(x)$. These polynomials are called *kernel polynomials* (see [1, §5.6]). Let k_n' be the coefficient of x^n in $q_n(x)$. Then

$$c_n = \frac{k_n' h_n}{k_n p_n(x_0)}.$$

Note as a special case of (4.1):

$$c_n \sum_{k=0}^n \frac{p_k(x_0)^2}{h_k} = q_n(x_0). \tag{4.2}$$

The kernel polynomial property of q_n follows by combination of (2.6) and (4.1):

$$\int\limits_{\mathbb{R}} q_n(x) p(x) \, d\mu(x) = c_n p(x_0) \quad (p \text{ a polynomial of degree} \leqslant n). \tag{4.3}$$

In particular, with $p = q_n$,

$$\int\limits_{\mathbb{R}} q_n(x)^2 \, d\mu(x) = c_n q_n(x_0). \tag{4.4}$$

If, for special choices of μ and x_0, we can explicitly evaluate $p_n(x_0)$, $q_n(x_0)$, h_n and c_n, then (4.4) and (4.2) yield possibly interesting explicit evaluations of an integral and a finite sum, respectively.

Note that all formulas in this subsection remain unchanged if μ has support within $[a, \infty)$ and if $x_0 \leqslant a$. Then $\{q_n\}$ is a system of orthogonal polynomials with respect to the orthogonality measure $(x - x_0) \, d\mu(x)$. For x_0 in the interior of the orthogonality interval for $\{p_n\}$, the orthogonality property of $\{q_n\}$ persists, but then the orthogonality measure is no longer positive.

More generally than (4.2) and (4.4) we will consider later in this paper specializations for Hahn polynomials of the identities

$$\sum_{k=0}^n \frac{p_k(x_0) p_k(x_1)}{h_k} = K_n(x_0, x_1) = \int\limits_{\mathbb{R}} K_n(x_0, x) K_n(x_1, x) \, d\mu(x). \tag{4.5}$$

4.3. Example: Jacobi polynomials

Let $\alpha, \beta > -1$ and take for the orthogonality measure $d\mu(x) := (1-x)^\alpha (1+x)^\beta \, dx$ with support $[-1, 1]$. Then the p_n are *Jacobi polynomials* (see [4, §10.8])

$$p_n(x) = P_n^{(\alpha, \beta)}(x),$$

which are normalized by their value at $x_0 := 1$:

$$p_n(1) = \frac{(\alpha + 1)_n}{n!}.$$

Then

$$h_n = \frac{2^{\alpha + \beta + 1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{\Gamma(n + \alpha + \beta + 1)n!}, \qquad k_n = \frac{(n + \alpha + \beta + 1)_n}{2^n n!}.$$

Furthermore, the q_n are Jacobi polynomials

$$q_n(x) = P_n^{(\alpha+1,\beta)}(x),$$

for which

$$q_n(1) = \frac{(\alpha+2)_n}{n!}, \qquad k'_n = \frac{(n+\alpha+\beta+2)_n}{2^n n!},$$

hence

$$c_n = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)}.$$

Substitution of the expressions for c_n and $q_n(1)$ in (4.4) yields

$$\int_{-1}^{1} P_n^{(\alpha+1,\beta)}(x)^2 (1-x)^\alpha (1+x)^\beta \, dx = \frac{2^{\alpha+\beta+1}}{\alpha+1} \frac{\Gamma(n+\alpha+2)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)n!}. \tag{4.6}$$

Formula (4.6) is given without proof in [5, p. 285, formula (6)]. After substitution of the explicit values of $p_k(1)$, h_k, $q_n(1)$ and c_n, formula (4.2) can be written as

$$\sum_{k=0}^{n} \frac{((\alpha+\beta+3)/2)_k (\alpha+\beta+1)_k (\alpha+1)_k}{((\alpha+\beta+1)/2)_k (\beta+1)_k k!} = \frac{(\alpha+2)_n (\alpha+\beta+2)_n}{(\beta+1)_n n!}. \tag{4.7}$$

The left-hand side of (4.7) can be written as a terminating very well poised hypergeometric series, by which (4.7) takes the form

$$_5F_4\left(\begin{matrix} \alpha+\beta+1, 1+\frac{1}{2}(\alpha+\beta+1), \alpha+1, n+\alpha+\beta+2, -n \\ \frac{1}{2}(\alpha+\beta+1), \beta+1, -n, n+\alpha+\beta+2 \end{matrix} ; 1 \right) = \frac{(\alpha+2)_n (\alpha+\beta+2)_n}{(\beta+1)_n n!}. \tag{4.8}$$

Formula (4.8), which we derived here from (4.2), is also a special case of [1, Corollary 3.4.3] (which, in its turn is a terminating version of a degenerate case of Dougall's evaluation of a terminating 2-balanced very well poised $_7F_6(1)$, see (2.2.9) and (2.2.10) in [1]).

Remark 4.1. The left-hand side of (4.7) is an example of an *indefinite sum*: a sum $\sum_{k=0}^{n} c_k$, where c_k is a *hypergeometric term* (i.e., c_{k+1}/c_k is a rational function of k) which does not depend on the upper limit n of the sum. Moreover, by (4.7) the sum can be evaluated for each n as a hypergeometric term s_n (i.e., s_{n+1}/s_n is a rational function of n). In general, *Gosper's algorithm* can test whether an indefinite sum of hypergeometric terms is summable with a hypergeometric term as sum, and it explicitly gives this sum if it exists (see [12, Ch. 5]). See (5.4) for a more involved example of such an indefinite sum (which has (4.7) as a limit case). Of course, as soon as we have an explicit indefinite summation $\sum_{k=0}^{n} c_k = s_n$ then an a posteriori proof can be immediately given by checking $c_0 = s_0$ and $s_n - s_{n-1} = c_n$.

4.4. Example: Laguerre polynomials

Let $\alpha > -1$ and take for the orthogonality measure $d\mu(x) := x^\alpha e^{-x} \, dx$ with support $[0, \infty)$. Then the p_n are *Laguerre polynomials* (see [4, §10.12])

$$p_n(x) = L_n^\alpha(x),$$

which are normalized by their value at $x_0 := 0$:

$$p_n(0) = \frac{(\alpha+1)_n}{n!}.$$

Then

$$h_n = \frac{\Gamma(n+\alpha+1)}{n!}, \qquad k_n = \frac{(-1)^n}{n!}.$$

Furthermore, the q_n are Laguerre polynomials

$$q_n(x) = L_n^{\alpha+1}(x),$$

for which

$$q_n(0) = \frac{(\alpha+2)_n}{n!}, \qquad k'_n = \frac{(-1)^n}{n!}, \quad \text{hence } c_n = \Gamma(\alpha+1).$$

Substitution of the expressions for c_n and $q_n(1)$ in (4.4) yields

$$\int_0^\infty L_n^{\alpha+1}(x)^2 x^\alpha e^{-x} \, dx = \Gamma(\alpha+1)\frac{(\alpha+2)_n}{n!}. \tag{4.9}$$

A more general version of formula (4.9), but still a specialization of (4.3), was earlier obtained by Carlitz [2, p. 340] (however with an erroneous factor $(-1)^n$ and without a side condition that $m \leqslant n$). Yet earlier, Mayr [10, §3] evaluated the integral $\int_0^\infty e^{-\lambda x} L_r^a(\alpha x) L_s^b(\beta x) x^{\sigma-1} \, dx$ as an Appell F_2 hypergeometric function. Specialization of his formula puts the left-hand side of (4.9) equal to

$$\Gamma(\alpha+1)\left(\frac{(\alpha+2)_n}{n!}\right)^2 F_2(\alpha+1, -n, -n, \alpha+2, \alpha+2; 1, 1)$$

$$= \Gamma(\alpha+1)\left(\frac{(\alpha+2)_n}{n!}\right)^2 \sum_{m=0}^n \frac{(\alpha+1)_m(-n)_m}{m!(\alpha+2)_m} {}_2F_1\left(\begin{matrix} \alpha+m+1, -n \\ \alpha+2 \end{matrix}; 1\right)$$

$$= \Gamma(\alpha+1)\left(\frac{(\alpha+2)_n}{n!}\right)^2 \sum_{m=0}^n \frac{(\alpha+1)_m(-n)_m}{m!(\alpha+2)_m} \frac{(1-m)_n}{(\alpha+2)_n},$$

which equals the right-hand side of (4.9). In the last equality we used the Chu–Vandermonde formula [1, Corollary 2.2.3].

After substitution of the explicit values of $p_k(0)$, h_k, $q_n(0)$ and c_n, formula (4.2) can be written as

$$\sum_{k=0}^n \frac{(\alpha+1)_k}{k!} = \frac{(\alpha+2)_n}{n!}. \tag{4.10}$$

The left-hand side can be rewritten as the terminating hypergeometric series

$$_2F_1(-n, \alpha+1; -n; 1).$$

Hence (4.10) is a special case of the Chu–Vandermonde formula. Of course, (4.10) can also be immediately checked (see end of Remark 4.1).

5. Further generalization in the case of Hahn polynomials

In this section we use *Hahn polynomials*

$$p_n(x) = Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}; 1\right) \quad (n = 0, 1, \ldots, N) \tag{5.1}$$

(see [8, §1.5]) for a pilot study in order to see how the general theory of Section 4.2 can be made concrete for families of orthogonal polynomials higher up in the Askey scheme. We will take $x_0 = N$, but then it will turn out that the right-hand sides of (4.2) and (4.4) cannot be made explicit in a simple form. Instead we will therefore consider (4.5) with $x_0 = N$, $x_1 = N-1$.

Hahn polynomials satisfy the orthogonality relation

$$\sum_{x=0}^N p_m(x) p_n(x) w_x = h_n \delta_{m,n} \quad (m, n = 0, 1, \ldots, N)$$

with

$$w_x = \frac{(\alpha+1)_x}{x!} \frac{(\beta+1)_{N-x}}{(N-x)!}$$

and

$$h_n = \frac{(\alpha+\beta+2)_N}{N!} \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} \frac{(-1)^n n!}{(-N)_n} \frac{(\beta+1)_n}{(\alpha+1)_n} \frac{(N+\alpha+\beta+2)_n}{(\alpha+\beta+2)_n}.$$

We have also (with notation of Section 4.2 and with $x_0 = N$):

$$k_n = \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(-N)_n}, \qquad p_n(N) = \frac{(-1)^n(\beta+1)_n}{(\alpha+1)_n},$$

$$q_n(x) = Q_n(x; \alpha, \beta+1, N-1), \qquad k_n' = \frac{(n+\alpha+\beta+2)_n}{(\alpha+1)_n(-N+1)_n}.$$

Hence

$$c_n = \frac{(\alpha + \beta + 2)_N}{N!} \frac{(N + \alpha + \beta + 2)_n}{(\alpha + \beta + 2)_n} \frac{n!}{(-N+1)_n}.$$

Then (4.1) takes the form

$$Q_n(x; \alpha, \beta + 1, N - 1)$$
$$= \frac{(N + \alpha + \beta + 2)_n}{(\alpha + \beta + 2)_n} \frac{n!}{(-N+1)_n} \sum_{k=0}^{n} \frac{2k + \alpha + \beta + 1}{k + \alpha + \beta + 1} \frac{(\alpha + \beta + 2)_k}{(N + \alpha + \beta + 2)_k} \frac{(-N)_k}{k!} Q_k(x; \alpha, \beta, N). \tag{5.2}$$

Instead of putting $x = N$ in (5.2) (like we obtained (4.2) from (4.1)), we can better put $x = N - 1$ in (5.2). Indeed, $Q_n(N; \alpha, \beta + 1, N - 1)$ does not have a simple explicit expression, but there is a simple expression

$$q_n(N - 1) = Q_n(N - 1; \alpha, \beta + 1, N - 1) = \frac{(-1)^n (\beta + 2)_n}{(\alpha + 1)_n},$$

while

$$p_n(N - 1) = Q_n(N - 1; \alpha, \beta, N) = \frac{(-1)^n (\beta + 1)_n}{(\alpha + 1)_n} \left(1 - \frac{n(n + \alpha + \beta + 1)}{(\beta + 1)N}\right),$$

as follows easily from (5.1). Thus we will specialize (4.5) for Hahn polynomials with $x_0 = N$ and $x_1 = N - 1$. Then, by recalling that $q_n(x) = c_n K_n(N, x)$ and by putting

$$r_n(x) := c_n K_n(N - 1, x),$$

we can rewrite (4.5) as

$$c_n \sum_{k=0}^{n} \frac{p_k(N) p_k(N - 1)}{h_k} = q_n(N - 1) = \frac{1}{c_n} \sum_{x=0}^{N} q_n(x) r_n(x) w_x. \tag{5.3}$$

We will examine the two identities in (5.3) more closely in the next two subsections.

5.1. The first identity: terminating very well poised $_6F_5(-1)$

The first identity in (5.3) can be rewritten more explicitly as follows

$$\frac{(N + \alpha + \beta + 2)_n}{(\alpha + \beta + 2)_n} \frac{n!}{(-N+1)_n} \sum_{k=0}^{n} \frac{2k + \alpha + \beta + 1}{k + \alpha + \beta + 1} \frac{(\alpha + \beta + 2)_k}{(N + \alpha + \beta + 2)_k} \frac{(-N)_k}{k!}$$
$$\times \frac{(-1)^k (\beta + 1)_k}{(\alpha + 1)_k} \left(1 - \frac{k(k + \alpha + \beta + 1)}{(\beta + 1)N}\right)$$
$$= \frac{(-1)^n (\beta + 2)_n}{(\alpha + 1)_n}. \tag{5.4}$$

The left-hand side of (5.4) can be written as a linear combination of two terminating very well poised hypergeometric series of argument -1, by which (5.4) takes the form

$$_6F_5 \left(\begin{matrix} \alpha + \beta + 1, 1 + \frac{1}{2}(\alpha + \beta + 1), \beta + 1, -N, n + \alpha + \beta + 2, -n \\ \frac{1}{2}(\alpha + \beta + 1), \alpha + 1, N + \alpha + \beta + 2, -n, n + \alpha + \beta + 2 \end{matrix}; -1\right)$$
$$- {}_6F_5 \left(\begin{matrix} \alpha + \beta + 3, 1 + \frac{1}{2}(\alpha + \beta + 3), \beta + 2, -N + 1, n + \alpha + \beta + 3, -n + 1 \\ \frac{1}{2}(\alpha + \beta + 3), \alpha + 2, N + \alpha + \beta + 3, -n + 1, n + \alpha + \beta + 3 \end{matrix}; -1\right) \frac{(\alpha + \beta + 2)(\alpha + \beta + 3)}{(N + \alpha + \beta + 2)(\alpha + 1)}$$
$$= \frac{(\beta + 2)_n (\alpha + \beta + 2)_n}{(\alpha + 1)_n n!} \frac{(-1)^n (-N + 1)_n}{(N + \alpha + \beta + 2)_n}. \tag{5.5}$$

We can also write the left-hand side of (5.5) as one single terminating very well poised $_8F_7$ of argument -1:

$$_8F_7 \left(\begin{matrix} a, 1 + \frac{1}{2}a, c + 1, a - c + 1, \beta + 1, -N, n + a + 1, -n \\ \frac{1}{2}a, c, a - c, \alpha + 1, N + a + 1, -n, n + a + 1 \end{matrix}; -1\right)$$
$$= \frac{(\beta + 2)_n (\alpha + \beta + 2)_n}{(\alpha + 1)_n n!} \frac{(-1)^n (-N + 1)_n}{(N + \alpha + \beta + 2)_n}, \tag{5.6}$$

where

$$a = \alpha + \beta + 1, \qquad c = \left((\beta + 1)N + \frac{1}{4}(\alpha + \beta + 1)^2 \right)^{1/2}.$$

Note that (4.8) (with α and β interchanged) can be obtained as the limit of (5.5) or (5.6) for $N \to \infty$.

Formula (5.5) can also be derived by combining a few identities for hypergeometric functions given in the literature. First apply Whipple's formula [14, (6.3)], [1, Theorem 3.4.6] to the two ${}_6F_5(-1)$'s in order to rewrite the left-hand side of (5.5) as a linear combination of two 0-balanced ${}_3F_2(1)$'s:

$$\frac{(-1)^n (\alpha + \beta + 2)_n}{n!} \left({}_3F_2 \left(\begin{matrix} N + \alpha + 1, n + \alpha + \beta + 2, -n \\ N + \alpha + \beta + 2, \alpha + 1 \end{matrix} ; 1 \right) \right.$$
$$\left. + \frac{n(n + \alpha + \beta + 2)}{(N + \alpha + \beta + 2)(\alpha + 1)} {}_3F_2 \left(\begin{matrix} N + \alpha + 1, n + \alpha + \beta + 3, -n + 1 \\ N + \alpha + \beta + 3, \alpha + 2 \end{matrix} ; 1 \right) \right). \tag{5.7}$$

By the contiguous relation (see [6, (3.11)])

$${}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} ; z \right) = {}_3F_2 \left(\begin{matrix} a, b, c - 1 \\ d, e \end{matrix} ; z \right) + \frac{abz}{de} {}_3F_2 \left(\begin{matrix} a + 1, b + 1, c \\ d + 1, e + 1 \end{matrix} ; z \right), \tag{5.8}$$

which can be proved in a straightforward way, expression (5.7) simplifies to

$$\frac{(-1)^n (\alpha + \beta + 2)_n}{n!} {}_3F_2 \left(\begin{matrix} N + \alpha, n + \alpha + \beta + 2, -n \\ N + \alpha + \beta + 2, \alpha + 1 \end{matrix} ; 1 \right). \tag{5.9}$$

By the Pfaff–Saalschütz formula [1, (2.2.8)] expression (5.9) is equal to the right-hand side of (5.5).

Remark 5.1. Formula (5.8) can be extended to the more general contiguous relation

$${}_rF_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; z \right) = {}_rF_s \left(\begin{matrix} a_1, \ldots, a_{r-1}, a_r - 1 \\ b_1, \ldots, b_s \end{matrix} ; z \right) + \frac{a_1 \ldots a_{r-1} z}{b_1 \ldots b_s} {}_rF_s \left(\begin{matrix} a_1 + 1, \ldots, a_{r-1} + 1, a_r \\ b_1 + 1, \ldots, b_s + 1 \end{matrix} ; z \right). \tag{5.10}$$

For the proof just write the summand for the power series expansion in z of the left-hand side as

$$\frac{(a_1)_k \ldots (a_{r-1})_k (a_r)_{k-1} (a_r - 1)}{(b_1)_k \ldots (b_s)_k k!} z^k + \frac{(a_1)_k \ldots (a_{r-1})_k (a_r)_{k-1} k}{(b_1)_k \ldots (b_s)_k k!} z^k.$$

The q-analogue of (5.10) is given by Krattenthaler [9, (2.2)].

5.2. The second identity: the kernel polynomials $K_n(N - 1, x)$

In the second identity of (5.3) the only still unexplicit expression is the polynomial $r_n(x) = K_n(N - 1, x)$. As a kernel polynomial for the point $N - 1$ it satisfies the property that

$$\sum_{x=0}^{N} r_n(x) p(x) w_x = c_n p(N - 1) \quad (p \text{ a polynomial of degree} \leqslant n).$$

For the evaluation in terms of Hahn polynomials of $r_n(x)$ we derive the following

$$r_n(x) = c_n \sum_{k=0}^{n} \frac{p_k(N - 1) p_k(x)}{h_k} = \sum_{k=0}^{n} \left(1 - \frac{k(k + \alpha + \beta + 1)}{(\beta + 1)N} \right) \frac{p_k(N) p_k(x)}{h_k}$$
$$= q_n(x) - \frac{1}{(\beta + 1)N} (\Lambda q_n)(x),$$

where (see [8, (1.5.5)])

$$(\Lambda f)(x) := (x + \alpha + 1)(x - N)(\Delta f)(x) + x(x - \beta - N - 1)(\Delta f)(x - 1).$$

Now use (see [8, (1.5.5), (1.5.7)]) that

$$(\Lambda q_n)(x) = n(n + \alpha + \beta + 2) q_n(x) - (x + \alpha + 1)(\Delta q_n)(x)$$
$$= n(n + \alpha + \beta + 2) Q_n(x; \alpha, \beta + 1, N - 1) + \frac{n(n + \alpha + \beta + 2)}{(\alpha + 1)(N - 1)} Q_{n-1}(x; \alpha + 1, \beta + 2, N - 2).$$

So we obtain that

$$
\begin{aligned}
r_n(x) = {} & \left(1 - \frac{n(n+\alpha+\beta+2)}{(\beta+1)N}\right) Q_n(x; \alpha, \beta+1, N-1) \\
& - \frac{n(n+\alpha+\beta+2)}{N(N-1)(\alpha+1)(\beta+1)} (x+\alpha+1) Q_{n-1}(x; \alpha+1, \beta+2, N-2).
\end{aligned}
\tag{5.11}
$$

Thus the second identity in (5.3) becomes

$$
\begin{aligned}
& \left(1 - \frac{n(n+\alpha+\beta+2)}{(\beta+1)N}\right) \sum_{x=0}^{N} \left(Q_n(x; \alpha, \beta+1, N-1)\right)^2 w_x - \frac{n+\alpha+\beta+2}{(\alpha+1)(\beta+1)} \\
& \quad \times \frac{n}{N(N-1)} \sum_{x=0}^{N} Q_n(x; \alpha, \beta+1, N-1) Q_{n-1}(x; \alpha+1, \beta+2, N-2)(x+\alpha+1) w_x \\
& = \frac{(\alpha+\beta+2)_N}{N!} \frac{(-1)^n (\beta+2)_n}{(\alpha+1)_n} \frac{(N+\alpha+\beta+2)_n}{(\alpha+\beta+2)_n} \frac{n!}{(-N+1)_n}.
\end{aligned}
\tag{5.12}
$$

Remark 5.2. In (5.12) replace x by NX, divide both sides by $N^{\alpha+\beta+1}$ and let $N \to \infty$. Then we obtain, at least formally, as a limit case of (5.12) the identity

$$
\int_{-1}^{1} P_n^{(\alpha, \beta+1)}(x)^2 (1-x)^\alpha (1+x)^\beta \, dx = \frac{2^{\alpha+\beta+1}}{\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+2)}{\Gamma(n+\alpha+\beta+2)n!},
\tag{5.13}
$$

which becomes (4.6) after an easy rewriting. For the limit transition from (5.12) to (5.13) use that

$$
\lim_{N \to \infty} Q_n(Nx; \alpha, \beta, N) = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha, \beta)}(1-2x)
$$

(see [8, (2.5.1)]) and

$$
\lim_{k \to \infty} \frac{\Gamma(k+a)}{\Gamma(k+b)} k^{b-a} = 1
$$

(see [1, (1.4.3)]). Note that the second sum on the left-hand side of (5.12) is killed in the limit process.

Acknowledgments

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