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# Computing a fuzzy shortest path in a network with mixed fuzzy arc lengths using $\alpha$-cuts

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ABSTRACT

We are concerned with the design of a model and an algorithm for computing a shortest path in a network having various types of fuzzy arc lengths. First, we develop a new technique for the addition of various fuzzy numbers in a path using $\alpha$-cuts by proposing a linear least squares model to obtain membership functions for the considered additions. Then, using a recently proposed distance function for comparison of fuzzy numbers, we present a dynamic programming method for finding a shortest path in the network. Examples are worked out to illustrate the applicability of the proposed model.



## 1. Introduction

The problem of finding a shortest path from a specified source node to any other node is fundamental in graph theory, and is of continuing interest [1,2]. This problem arises from many applications including transportation, routing, communications, supply chain management or models involving agents. Let $G = (V, E)$ be a graph, where $V$ is the set of vertices (nodes) and $E$ is the set of edges (arcs). A *path* between two nodes is an alternating sequence of vertices and edges beginning with a starting node and ending with an ending node. The *distance* (cost) of a path is the sum of the weights (arc lengths) of the edges on the path. However, since there can be more than one path between any two vertices, the problem of finding a path with a minimal cost between two specified vertices of interest is the so-called shortest path problem (SPP).

Although in conventional graph theory, the weights of the edges in an SPP are assumed to be precise real numbers, for most practical applications, these parameters (i.e., costs, capacities, demands, time, etc.) are naturally imprecise. In such cases, an appropriate modeling approach may justifiably make use of fuzzy numbers, and so does the name *fuzzy shortest path problem* (FSPP) appear in the literature [1,3,4].

The FSPP, involving addition and comparison of fuzzy numbers, is quite different from the conventional SPP, which only involves crisp numbers. In an FSPP, the costs being fuzzy numbers, the task of finding a path being smaller than all the others is not straightforward, as the comparison of fuzzy numbers as an operation can be defined in a wide variety of ways.

Recently, several results have been published on the FSPP [5]. The work of Dubois and Prade [6] is one of the first on this subject and considers extensions of the classical Floyd and Ford–Moore–Bellman (FMB) algorithms. However, it was verified that the algorithm would compute the shortest path distance without identifying an existing path (see [7]) as was outlined by Klein [8] with a fuzzy dominance set. Lin and Chern [3] defined the denomination of vital arcs as being those whose removal from the path resulted in an increase of the cost. Another algorithm for this problem, presented by Okada and Gen [9,10], is a generalization of the Dijkstra algorithm. In this algorithm, the weights of the arcs are considered to be interval numbers, and are defined using a partial order between interval numbers. Okada and Soper [5] characterized a

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solution not as the shortest path, but as a fuzzy set solution, where each element of the set is a no dominated path or a Pareto optimal path with fuzzy edge weights. However, this algorithm does not provide decision-makers with any guidelines for choosing a best path according to their own viewpoints (optimistic/pessimistic, risky/conservative) [11]. Blue et al. [12] presented an algorithm which would find a cut value to limit the number of analyzed paths, and then applied a modified version of the $k$-shortest path (crisp) algorithm proposed by Eppstein [7]. Following the idea of finding a fuzzy set solution, Okada [13] introduced the concept of the degree of possibility of an arc being on a shortest path. Among the most recent work is the one by Nayeem and Pal [4] that proposes an algorithm based on the acceptance index introduced by Sengupta and Pal [14] and which gives a single fuzzy shortest path or a guideline for choosing a best fuzzy shortest path according to the decision-maker viewpoint [15].

Here, we propose a new approach and an algorithm to find a shortest path in a network with various fuzzy arc lengths. The remainder of the paper is organized as follows. In Section 2, basic concepts and definitions are given. Section 3 explains ways of computing $\alpha$-cuts for fuzzy numbers. We present our fuzzy sum operator by use of a linear least squares model in Section 4. In Section 5, using a recently proposed distance function, we present a dynamic programming algorithm for finding fuzzy shortest path in a mixed fuzzy network. We conclude in Section 6.

## 2. Definitions

We start with basic definitions of some well-known fuzzy numbers.

**Definition 1.** An $LR$ fuzzy number is represented by $\tilde{a} = (m, a, b)_{LR}$, with the membership function, $\mu_{\tilde{a}}(x)$, defined by

$$\mu_{\tilde{a}}(x) = \begin{cases} L\left[\dfrac{m-x}{a}\right] & x \le m \\ R\left[\dfrac{x-m}{b}\right] & x \ge m, \end{cases}$$

where $L$ and $R$ are non-increasing functions from $R^+$ to $[0, 1]$, $L(0) = R(0) = 1$, $m$ is the center, $a$ is the left spread and $b$ is the right spread.

Note that if $L(x) = R(x) = 1 - x$ with $0 < x < 1$, then $x$ is a triangular fuzzy number and is represented by the triplet $\tilde{a} = (m, a, b)$, with the membership function, $\mu_{\tilde{a}}(x)$, defined by

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 - \left(\dfrac{m-x}{a}\right) & x \le m \\ 1 - \left(\dfrac{x-m}{b}\right) & x \ge m. \end{cases}$$

**Definition 2.** A trapezoidal fuzzy number $\tilde{a}$ is shown by $\tilde{a} = (a_1, a_2, a_3, a_4)$, with the membership function as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} 0 & x \le a_1 \\ \dfrac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ 1 & a_2 \le x \le a_3 \\ \dfrac{a_4-x}{a_4-a_3} & a_3 \le x \le a_4 \\ 0 & a_4 \le x. \end{cases}$$

A general trapezoidal fuzzy number, along with a cut (to be explained later in Definition 4), is shown in Fig. 1. It is apparent that a triangular fuzzy number is a special trapezoidal fuzzy number with $a_2 = a_3$.

**Definition 3.** If $L(x) = R(x) = e^{-x^2}$, with $x \in \Re$, then $x$ is a normal fuzzy number that is shown by $(m, \sigma)$ and the corresponding membership function is defined to be:

$$\mu_{\tilde{a}}(x) = e^{-\left(\frac{x-m}{\sigma}\right)^2}, \quad x \in \Re,$$

where $m$ is the mean and $\sigma$ is the standard deviation. A normal fuzzy number, along with a cut (to be explained later in Definition 4), is shown in Fig. 2.

**Definition 4.** The $\alpha$-cut and strong $\alpha$-cut for a fuzzy number $\tilde{a}$ are shown by $\tilde{a}_\alpha$ and $\tilde{a}^+_\alpha$, respectively, and for $\alpha \in [0, 1]$ are defined to be:

$$\tilde{a}_\alpha = \left\{x \mid \mu_{\tilde{A}}(x) \ge \alpha, x \in X\right\},$$
$$\tilde{a}^+_\alpha = \left\{x \mid \mu_{\tilde{A}}(x) > \alpha, x \in X\right\},$$

where $X$ is the universal set.

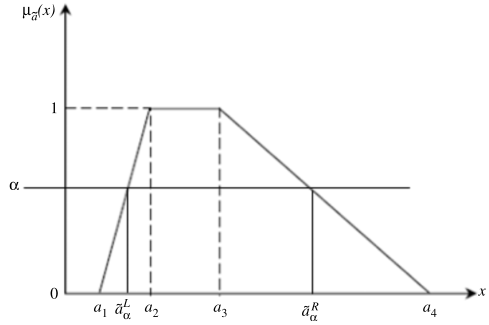


**Fig. 1.** A trapezoidal fuzzy number with an $\alpha$-cut.

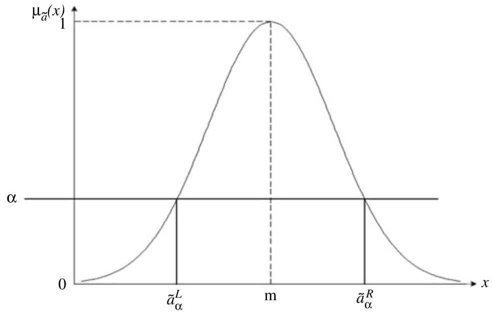


**Fig. 2.** A normal fuzzy number with an $\alpha$-cut.

Note that the upper and lower bounds for the $\alpha$-cut set ($\tilde{a}_\alpha$) are shown by $\sup \tilde{a}_\alpha$ and $\inf \tilde{a}_\alpha$, respectively. Here, we assume that the upper and lower bounds of $\alpha$-cuts are finite values and for simplicity we show $\sup \tilde{a}_\alpha$ by $\tilde{a}^R_\alpha$ and $\inf \tilde{a}_\alpha$ by $\tilde{a}^L_\alpha$ (see Figs. 1 and 2).

## 3. Computing $\alpha$-cuts for fuzzy numbers

For the *LR* fuzzy numbers with *L* and *R* invertible functions, the $\alpha$-cuts are:

$$\alpha = L\left[\frac{m-x}{a}\right] \Rightarrow \frac{m-x}{a} = L^{-1}(\alpha) \Rightarrow \tilde{a}^L_\alpha = x = m - aL^{-1}(\alpha),$$

$$\alpha = R\left[\frac{x-m}{b}\right] \Rightarrow \frac{x-m}{b} = R^{-1}(\alpha) \Rightarrow \tilde{a}^R_\alpha = x = m + bR^{-1}(\alpha).$$

For specific *L* and *R* functions, the following cases are discussed.

### 3.1. $\alpha$-cuts for trapezoidal fuzzy numbers

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number. An $\alpha$-cut for $\tilde{a}$, $\tilde{a}_\alpha$, is computed as:

$$\alpha = \frac{x-a_1}{a_2-a_1} \Rightarrow \tilde{a}^L_\alpha = x = (a_2-a_1)\alpha + a_1,$$

$$\alpha = \frac{a_4-x}{a_4-a_3} \Rightarrow \tilde{a}^R_\alpha = x = a_4 - (a_4-a_3)\alpha,$$

$$\Rightarrow \tilde{a}_\alpha = \begin{cases} \tilde{a}^L_\alpha = (a_2-a_1)\alpha + a_1 \\ \tilde{a}^R_\alpha = a_4 - (a_4-a_3)\alpha, \end{cases} \quad 0 \le \alpha \le 1, \tag{1}$$

where $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^R_\alpha]$ is the corresponding $\alpha$-cut. The $\alpha$-cuts for triangular fuzzy numbers are obtained by using the above equations considering $a_2 = a_3$.

### 3.2. $\alpha$-cuts for normal fuzzy numbers

If $\tilde{a} = (m, \sigma)$ is a normal fuzzy number, then $\tilde{a}_\alpha$ is computed as:

$$\alpha = \mathrm{e}^{-\left(\frac{m-x}{\sigma}\right)^2} \Rightarrow \sqrt{-\ln(\alpha)} = \frac{m-x}{\sigma} \Rightarrow \tilde{a}^L_\alpha = x = m - \sigma\sqrt{-\ln\alpha}$$

$$\alpha = \mathrm{e}^{-\left(\frac{x-m}{\sigma}\right)^2} \Rightarrow \sqrt{-\ln(\alpha)} = \frac{x-m}{\sigma} \Rightarrow \tilde{a}_\alpha^R = x = m + \sigma\sqrt{-\ln\alpha}$$

$$\Rightarrow \tilde{a}_\alpha = \begin{cases} \tilde{a}_\alpha^L = m - \sigma\sqrt{-\ln\alpha} \\ \tilde{a}_\alpha^R = m + \sigma\sqrt{-\ln\alpha} \end{cases}, \quad 0 < \alpha \le 1. \tag{2}$$

## 4. Fuzzy approximate sum operators

Here, we propose an approach for summing various fuzzy numbers approximately using $\alpha$-cuts. The approximation is based on fitting an appropriate model for the sum using $\alpha$-cuts of the addition as the fitness data. Let us divide the $\alpha$-interval [0, 1] into $n$ equal subintervals by letting $\alpha_0 = 0, \alpha_i = \alpha_{i-1} + \Delta\alpha_i, \Delta\alpha_i = \frac{1}{n}, i = 1, \ldots, n$. This way, we have a set of $n+1$ equidistant points. For the normal fuzzy numbers $x \in (-\infty, +\infty)$, it is improper to assume $\alpha$ being equal to zero. Therefore, in this case we consider $\alpha \in (0, 1]$, and thus use the nonzero $\alpha_i,\ 1 \le i \le n$.

Here, we intend to show how to add up a trapezoidal fuzzy number with a normal one. We present a numerical approach to approximate the sum and its corresponding membership function.

### 4.1. $\alpha$-cut sum

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (m, \sigma)$ be the trapezoidal and normal fuzzy numbers, respectively. Given $\alpha_i \in (0, 1]$, $1 \le i \le n$, the $\alpha$-cut sum of these fuzzy numbers using Eqs. (1) and (2) is obtained as follows:

$$\tilde{c}_{\alpha_i} = \tilde{a}_{\alpha_i} + \tilde{b}_{\alpha_i} \Rightarrow \tilde{c}_{\alpha_i} = \begin{cases} \tilde{c}_\alpha^L = (\tilde{a}_\alpha^L + \tilde{b}_\alpha^L) \\ \tilde{c}_\alpha^R = (\tilde{a}_\alpha^R + \tilde{b}_\alpha^R), \end{cases} \quad \forall i,\ 1 \le i \le n,$$

where,

$$\tilde{c}_{\alpha_i}^L = \left[(a_2 - a_1)\alpha + a_1 + m - \sigma\sqrt{-\ln\alpha}\right],$$
$$\tilde{c}_{\alpha_i}^R = \left[a_4 - (a_4 - a_3)\alpha + m + \sigma\sqrt{-\ln\alpha}\right]. \tag{3}$$

Using Eq. (3), corresponding to $\alpha_i, 1 \le i \le n, 2n$ points are obtained for $\tilde{c}$ ($n$ points for the $\tilde{c}_{\alpha_i}^L$ and $n$ points for the $\tilde{c}_{\alpha_i}^R$). Using these points, it is possible to approximate the sum of the two fuzzy numbers. An approximate membership function of the sum is computed by fitting an appropriate function using the $\alpha$-cut points. For the addition of normal and trapezoidal fuzzy numbers, the case being considered in our examples later on, we propose an exponential membership function for approximating the sum as follows (later, we will see that this choice would indeed provide a good model for the approximating sum of trapezoidal and normal fuzzy numbers). Let $x_i = \tilde{c}_{\alpha_i}^R$ and $y_i = \mu(\tilde{c}_{\alpha_i}^R)$, and using the $n$ points $(x_i, y_i),\ 1 \le i \le n$, consider the fitting model as $y = \mathrm{e}^{-\left(\frac{x-\lambda}{\beta}\right)^2}$. The unknown parameters $\lambda$ and $\beta$ appear nonlinearly. We linearize the model, by noting that for any $x_i > \lambda$ (as is the case here for the right hand model), we must have:

$$\ln y_i = -\left(\frac{x_i - \lambda}{\beta}\right)^2. \tag{4}$$

Since $0 < y_i \le 1$, then $\ln(y_i) \le 0$, and thus we can write,

$$\sqrt{-\ln y_i} = \left(\frac{x_i - \lambda}{\beta}\right), \tag{5}$$

and hence,

$$\beta\sqrt{-\ln y_i} + \lambda = x_i. \tag{6}$$

Therefore, we define a linear of least squares model for the minimization of error as follows:

$$\min E = \sum_{i=1}^{n} \left(\beta\sqrt{-\ln y_i} + \lambda - x_i\right)^2, \tag{7}$$

where $x_i = \tilde{c}_{\alpha_i}^R$ is given by (3). To solve (7), we need to have:

$$\frac{\partial E}{\partial \beta} = \sum_i \left[\left(2\sqrt{-\ln y_i}\right)\left(\beta\sqrt{-\ln y_i} + \lambda - x_i\right)\right] = 0, \tag{8}$$
$$\frac{\partial E}{\partial \lambda} = \sum_i \left[2\left(\beta\sqrt{-\ln y_i} + \lambda - x_i\right)\right] = 0.$$

Hence, we need to solve the following so-called normal equations for the unknown parameters $\beta$ and $\lambda$:

$$-\beta \sum_i \ln y_i + \lambda \sum_i \sqrt{-\ln y_i} = \sum_i x_i \sqrt{-\ln y_i}, \tag{9}$$

$$\beta \sum_i \sqrt{-\ln y_i} + n\lambda = \sum_i x_i. \tag{10}$$

Using Cramer's rule to solve (9) and (10), $\beta$ and $\lambda$ are explicitly determined to be:

$$\beta = \frac{\begin{vmatrix} \sum_i x_i \sqrt{-\ln y_i} & \sum_i \sqrt{-\ln y_i} \\ \sum_i x_i & n \end{vmatrix}}{\begin{vmatrix} -\sum_i \ln y_i & \sum_i \sqrt{-\ln y_i} \\ \sum_i \sqrt{-\ln y_i} & n \end{vmatrix}} \Rightarrow \beta = \frac{n \sum_i x_i \times \sqrt{-\ln y_i} - \sum_i \sqrt{-\ln y_i} \times \sum_i x_i}{-n \sum_i \ln y_i - \sum_i \sqrt{-\ln y_i} \times \sum_i \sqrt{-\ln y_i}}, \tag{11}$$

$$\lambda = \frac{\begin{vmatrix} -\sum_i \ln y_i & \sum_i \left(\sqrt{-\ln y_i} \times x_i\right) \\ \sum_i \sqrt{-\ln y_i} & \sum_i x_i \end{vmatrix}}{\begin{vmatrix} -\sum_i \ln y_i & \sum_i \sqrt{-\ln y_i} \\ \sum_i \sqrt{-\ln y_i} & n \end{vmatrix}} \Rightarrow \lambda = \frac{-\sum_i \ln y_i \sum_i x_i - \sum_i \left(x_i \sqrt{-\ln y_i}\right) \times \sum_i \sqrt{-\ln y_i}}{-n \sum_i \ln y_i - \sum_i \sqrt{-\ln y_i} \times \sum_i \sqrt{-\ln y_i}}. \tag{12}$$

Now, similarly let $\bar{x}_i = \tilde{c}^L_{\alpha_i}$ and $\bar{y}_i = \mu(\tilde{c}^L_{\alpha_i})$, and consider the model $y = \mathrm{e}^{-\left(\frac{\lambda' - x}{\beta'}\right)^2}$. Using the above approach, we have:

$$\ln y_i = -\left(\frac{\lambda' - x_i}{\beta'}\right)^2. \tag{13}$$

Or,

$$-\beta' \sqrt{-\ln y_i} + \lambda' = x_i.$$

Thus, solving

$$\min E' = \sum_{i=1}^{n} \left(-\beta' \sqrt{-\ln y_i} + \lambda' - x_i\right)^2,$$

similarly leads to:

$$\beta' = \frac{-n \sum_i x_i \sqrt{-\ln y_i} + \sum_i x_i \times \sum_i \sqrt{-\ln y_i}}{-n \sum_i \ln y_i - \sum_i \sqrt{-\ln y_i} \times \sum_i \sqrt{-\ln y_i}}, \tag{14}$$

$$\lambda' = \frac{-\sum_i \ln y_i \times \sum_i x_i - \sum_i \left(x_i \times \sqrt{-\ln y_i}\right) \times \sum_i \sqrt{-\ln y_i}}{-n \sum_i \ln y_i - \sum_i \sqrt{-\ln y_i} \times \sum_i \sqrt{-\ln y_i}}. \tag{15}$$

Thus, the membership function is determined to be:

$$\mu_{\tilde{c}}(x) = \begin{cases} \mathrm{e}^{-\left(\frac{\lambda' - x}{\beta'}\right)^2} & x < \lambda' \\ 1 & \lambda' \le x \le \lambda \\ \mathrm{e}^{-\left(\frac{x - \lambda}{\beta}\right)^2} & x > \lambda, \end{cases} \tag{16}$$

with $\lambda$, $\beta$, $\lambda'$ and $\beta'$ as defined by (11), (12), (14) and (15), respectively.

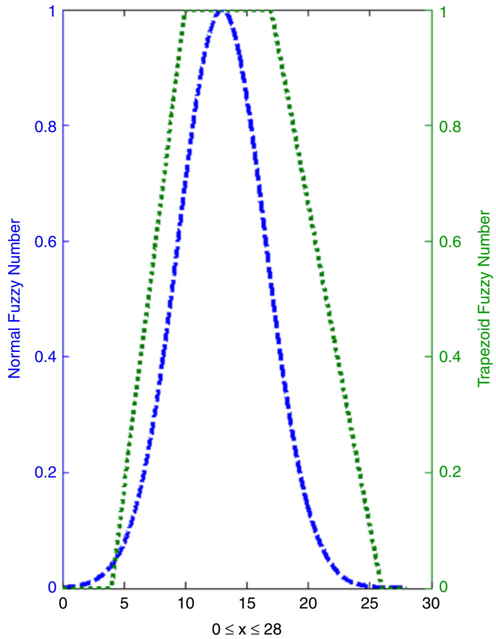


**Fig. 3a.** The $\alpha$-cut points of fuzzy numbers.

**Remarks.** An equivalent approach for fitting the least squares model is to consider the basis functions $\left\{\sqrt{-\ln y},\, 1\right\}$, for the fitting relation (6). This would lead to the minimization model,

$$\min_{\theta} \|A\theta - x\|_2\,, \tag{17}$$

with

$$A = \begin{bmatrix} \sqrt{-\ln y_1} & 1 \\ \vdots & \vdots \\ \sqrt{-\ln y_n} & 1 \end{bmatrix}, \qquad \theta = \begin{bmatrix} \beta \\ \lambda \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

The solution of (17) is obtained by solving the normal equations, $A^T A\theta = A^T x$, yielding the same results as (11) and (12). Conveniently, this general data fitting approach can be used to consider other membership functions in cases using various types of fuzzy numbers. The key element is thus to decide the appropriate basic functions.

Now, we provide a numerical illustration for our proposed model for addition of trapezoidal and normal fuzzy numbers.

**Example 1.** Consider the following two fuzzy numbers, one being normal and the other being trapezoidal:

$$\tilde{a} = (13, 5), \qquad \tilde{b} = (4, 10, 17, 26).$$

The diagrams of the numbers at the $\alpha$-cuts and their sum, $(\tilde{a} + \tilde{b})$, using Eq. (3), are shown in Figs. 3a and 3b, respectively. As indicated in the diagram, the result is neither trapezoidal nor normal.

Using the points obtained from the $\alpha$-cuts considering $n = 100$, the values of $(\lambda, \beta)$ and $(\lambda', \beta')$ are obtained by (9)–(15). For this example, we obtain:

$$\mu_{\tilde{c}}(x) = \begin{cases} e^{-\left(\frac{23-x}{8.559}\right)^2} & x < 23 \\ 1 & 23 \le x \le 30 \\ e^{-\left(\frac{x-30}{10.34}\right)^2} & x > 30. \end{cases}$$

## 5. An algorithm for fuzzy shortest path in a network

### 5.1. Distance between fuzzy numbers

Knowing that we can obtain a good approximation for the addition of various fuzzy numbers by use of $\alpha$-cuts, we compute the distance between two fuzzy numbers using the resulting points from the $\alpha$-cuts. Assume that $\tilde{a}$ and $\tilde{b}$ are two fuzzy

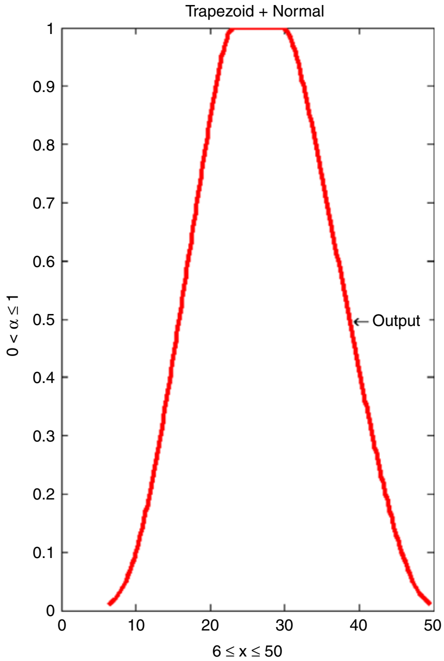


**Fig. 3b.** The sum of mixed fuzzy numbers using $\alpha$-cuts.

numbers. We apply a fuzzy ranking method for fuzzy numbers. We have used this ranking method effectively in a recent work [16].

Let us consider fuzzy *min* operations as follows:

$$\text{Min value } (\tilde{a}, \tilde{b}) = (\min(a_1, b_1), \min(a_2, b_2), \min(a_3, b_3), \min(a_4, b_4)). \tag{18}$$

It is evident that, for non-comparable fuzzy numbers $\tilde{a}$ and $\tilde{b}$, the fuzzy *min* operation results in a fuzzy number different from both of them. For example, for $\tilde{a} = (5, 10, 13, 19)$ and $\tilde{b} = (6, 9, 15, 20)$, we get from (18) a fuzzy $M\tilde{V} = \text{Min value}(\tilde{a}, \tilde{b}) = (5, 9, 13, 19)$, which is different from both $\tilde{a}$ and $\tilde{b}$. To alleviate this drawback, we use a method based on the distance between fuzzy numbers. We use the distance function introduced in [17]. The main advantages of this distance function are the generality of its usage on various fuzzy numbers, and its reliability in distinguishing unequal fuzzy numbers. Indeed, the usage of the distance function below worked out to be quite appropriate for our approach.

The $D_{p,q}$-distance, indexed by parameters $1 < p < \infty$ and $0 < q < 1$, between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ is a nonnegative function given by:

$$D_{p,q}(\tilde{a}, \tilde{b}) = \begin{cases} \left[ (1-q) \displaystyle\int_0^1 \left| a_\alpha^- - b_\alpha^- \right|^p \mathrm{d}\alpha + q \int_0^1 \left| a_\alpha^+ - b_\alpha^+ \right|^p \mathrm{d}\alpha \right]^{\frac{1}{p}}, & p < \infty, \\ (1-q) \displaystyle\sup_{0<\alpha\le 1} \left| a_\alpha^- - b_\alpha^- \right| + q \inf_{0<\alpha\le 1} \left| a_\alpha^+ - b_\alpha^+ \right|, & p = \infty. \end{cases} \tag{19}$$

The analytical properties of $D_{p,q}$ depend on the first parameter $p$, while the second parameter $q$ of $D_{p,q}$ characterizes the subjective weight attributed to the end points of the support; i.e., the $a_\alpha^+$ and $a_\alpha^-$ of the fuzzy numbers. If there is no reason for distinguishing any side of the fuzzy numbers, then $D_{p,\frac{1}{2}}$ is recommended. Having $q$ close to 1 results in considering the right side of the support of the fuzzy numbers more favorably. Since the significance of the end points of the support of the fuzzy numbers is assumed to be the same, then we consider $q = \frac{1}{2}$.

For two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ with corresponding $\alpha_i$-cuts, the $D_{p,q}$ distance is approximately proportional to:

$$D_{p,q}(\tilde{a}, \tilde{b}) = \left[ (1-q) \sum_{i=1}^{n} \left| a_{\alpha_i}^- - b_{\alpha_i}^- \right|^p + q \sum_{i=1}^{n} \left| a_{\alpha_i}^+ - b_{\alpha_i}^+ \right|^p \right]^{\frac{1}{p}}. \tag{20}$$

If $q = \frac{1}{2}$, $p = 2$, then the above equation turns into:

$$D_{2,\frac{1}{2}}(\tilde{a}, \tilde{b}) = \sqrt{\frac{1}{2} \sum_{i=1}^{n} (a_{\alpha_i}^- - b_{\alpha_i}^-)^2 + \frac{1}{2} \sum_{i=1}^{n} (a_{\alpha_i}^+ - b_{\alpha_i}^+)^2}. \tag{21}$$

To compare two fuzzy arc lengths $\tilde{a}$ and $\tilde{b}$ with $\alpha_i$-cuts as their approximations, since they are supposed to represent positive values, we compare them with $M\tilde{V} = (0, 0, \ldots, 0)$. In fact, we use formula (21) to compute $D_{2,\frac{1}{2}}(\tilde{a}, M\tilde{V})$ and $D_{2,\frac{1}{2}}(\tilde{b}, M\tilde{V})$ and then use these values for comparison of the two numbers. Here, we consider $\tilde{a} = (6, 9, 15, 20)$ and $\tilde{b} = (5, 10, 13, 19)$ with $n = 10$. Then, the $\alpha$-cuts for $\tilde{a}$ are obtained to be

$$a^+_{\alpha_i} = \{19.5, 19, 18.5, 18, 17.5, 17, 16.5, 16, 15.5, 15\}.$$

$$a^-_{\alpha_i} = \{9, 8.7, 8.4, 8.1, 7.8, 7.5, 7.2, 6.9, 6.6, 6.3\}$$

and for $\tilde{b}$ we have

$$b^+_{\alpha_i} = \{18.4, 17.8, 17.2, 16.6, 16, 15.4, 14.8, 14.2, 13.6, 13\}.$$

$$b^-_{\alpha_i} = \{10, 9.5, 9, 8.5, 8, 7.5, 7, 6.5, 6, 5.5\}.$$

As a result, the distances are

$$D_{2,\frac{1}{2}}(\tilde{a}, M\tilde{V}) = 42.36.$$

$$D_{2,\frac{1}{2}}(\tilde{b}, M\tilde{V}) = 39.47.$$

Therefore, $\tilde{a} > \tilde{b}$.

### 5.2. An algorithm for computing a shortest path

The following dynamic programming algorithm is for computing the shortest path in a network. The algorithm is based on Floyd's dynamic programming method to find a shortest path, if it exists, between every pair of nodes $i$ and $j$ in the network [18].

We make use of the following optimal value function $f_k(i, j)$ and the corresponding labeling function $P_k(i, j)$:

$f_k(i, j)$ = length of the shortest path from node $i$ to node $j$ when the path is considered to use only the nodes from the set of nodes $\{1, \ldots, k\}$,

$P_k(i, j)$ = the last intermediate node on the shortest path from node $i$ to node $j$ using $\{1, \ldots, k\}$ as intermediate node,

where, $i$ is a source node, $j$ is the end node and $k$ refers to an intermediate node.

The dynamic updating for the optimal path length and its corresponding labeling are:

$$f_k(i, j) = \min\{f_{k-1}(i, j), f_{k-1}(i, k) + f_{k-1}(k, j)\},$$

$$P_k(i, j) = \begin{cases} P_{k-1}(i, j) & \text{if } k \text{ is not on shortest path from } i \text{ to } j \text{ using } \{1, \ldots, k\} \\ P_{k-1}(k, j) & \text{otherwise.} \end{cases}$$

We are now ready to give the steps of the algorithm.

**Algorithm.** A dynamic programming method for computing a shortest path in a fuzzy network $G = (V, A)$, where $V$ is the set of nodes with $|V| = N$, and $A$ is the set of arcs. The value $\tilde{d}_{ij}$ is the fuzzy arc distance for arc $(i, j)$, if it exists. Below, $\tilde{f}_k(i, j)$ the shortest path length is set to $\infty$, when there is no arc.

*Step* 1: Let $k = 0$ and $\tilde{f}_k(i, j) = \tilde{d}_{ij}$, for all $(i, j) \in A$, $\tilde{f}_k = (i, j) = \infty$, for all $(i, j) \notin A$. **If** an arc exists from node $i$ to node $j$ **then** let $P_k(i, j) = i$.

*Step* 2: Let $k = k + 1$.

Do the following steps for $i = 1, 2, 3, \ldots, N$, $j = 1, 2, 3, \ldots, N$, $i \neq j$.

2.1 Compute the value of $\tilde{f}_k(i, j) = \min\left[\tilde{f}_{k-1}(i, j), \tilde{f}_{k-1}(i, k) + \tilde{f}_{k-1}(k, j)\right]$ (for the addition, our proposed method discussed in Section 4.1 and for comparison of fuzzy numbers the $D_{p,q}$ distance function (21) of Section 5.1 are applied).

2.2 **If** node $k$ is not on the shortest path using the nodes $\{1, 2, \ldots, k\}$ as intermediate nodes, **then** let $P_k(i, j) = P_{k-1}(i, j)$ **else** let $P_k(i, j) = P_{k-1}(k, j)$.

*Step* 3: **If** $k < N$ **then go to Step** 2.

*Step* 4: Obtain the shortest path using the $P_k(i, j)$ values. If $f_N(i, j) = \infty$, then there is no path between $i$ and $j$. The shortest path from node $i$ to $j$, if it exists, is identified backwards and read by the nodes: $j$, $P_N(i, j) = k$ followed by $P_N(i, k), \ldots, P_N(i, l) = i$, where $l$ is the node immediately after $i$ in the path.

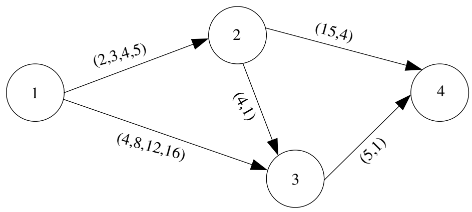


**Fig. 4.** A small sized network having mixed fuzzy arc lengths.

**Table 1**
The $\tilde{f}_0(i,j)$ matrix for $k = 0$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | (2, 3, 4, 5) | (4, 8, 12, 16) | 2 |
| 2 | – | – | (4, 1) | (15, 4) |
| 3 | – | – | – | (5, 1) |
| 4 | – | – | – | – |

**Table 2**
The $P_0(i,j)$ matrix for $k = 0$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | 1 | 1 | – |
| 2 | – | – | 2 | 2 |
| 3 | – | – | – | 3 |
| 4 | – | – | – | – |

**Table 3**
The $\tilde{f}_1(i,j)$ matrix for $k = 1$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | (2, 3, 4, 5) | (4, 8, 12, 16) | – |
| 2 | – | – | (4, 1) | (15, 4) |
| 3 | – | – | – | (5, 1) |
| 4 | – | – | – | – |

**Table 4**
The $P_1(i,j)$ matrix for $k = 1$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | 1 | 1 | – |
| 2 | – | – | 2 | 2 |
| 3 | – | – | – | 3 |
| 4 | – | – | – | – |

### 5.3. *Termination and complexity of the algorithm*

The proposed algorithm terminates after $N$ outer iterations corresponding to $k$. A total of $N(N-1)^2$ additions and comparisons are needed for every $k$. For each addition, $n$ fuzzy additions for the $\alpha_i$-cuts should be performed resulting in $2n(N)(N-1)^2$ additions. For comparisons, we have $(2n+1)N(N-1)^2$ additions and $(2n+1)\,N(N-1)^2$ multiplications using (7). Therefore, the total needed operations are $(6n+2)\,N(N-1)^2$ additions and multiplications, with $N(N-1)^2$ comparisons.

**Example 2.** Consider the mixed fuzzy network in Fig. 4 with four nodes and five arcs having two trapezoidal and three normal arc lengths as specified in Table 1.

*Step* 1: We set $\tilde{f}_k(i,j) = \tilde{d}_{ij}$, for $k = 0$, as specified in Table 1.
Therefore, with $P_k(i,j) = i$, Table 2 is obtained.

*Step* 2: Here, we consider $k = 1$ and compute the value of $f_k(i,j) = \min\,[f_{k-1}(i,j), f_{k-1}(i,k) + f_{k-1}(k,j)]$. The result is shown in Table 3.
Therefore, for $P_k(i,j) = i$, Table 4 is obtained.
If node $k$ is not on the shortest path using $\{1, 2, \ldots, k\}$ as intermediate nodes, then we consider $P_k(i,j) = P_{k-1}(i,j)$, otherwise we let $P_k(i,j) = P_{k-1}(i,k)$. We now report the results obtained for other values of $k$ in Tables 5–10. Note that, the

**Table 5**
The $\tilde{f}_2(i, j)$ matrix for $k = 2$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | (2, 3, 4, 5) | $V_1$ | $W_1$ |
| 2 | – | – | (4, 1) | (15, 4) |
| 3 | – | – | – | (5, 1) |
| 4 | – | – | – | – |

**Table 6**
The $P_2(i, j)$ matrix for $k = 2$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | 1 | 2 | 2 |
| 2 | – | – | 2 | 2 |
| 3 | – | – | – | 3 |
| 4 | – | – | – | – |

**Table 7**
The $\tilde{f}_3(i, j)$ matrix for $k = 3$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | (2, 3, 4, 5) | $V_2$ | $W_2$ |
| 2 | – | – | (4, 1) | (9, 2) |
| 3 | – | – | – | (5, 1) |
| 4 | – | – | – | – |

**Table 8**
The $P_3(i, j)$ matrix for $k = 3$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | 1 | 2 | 3 |
| 2 | – | – | 2 | 3 |
| 3 | – | – | – | 3 |
| 4 | – | – | – | – |

**Table 9**
The $\tilde{f}_4(i, j)$ matrix for $k = 4$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | (2, 3, 4, 5) | $V_3$ | $W_3$ |
| 2 | – | – | (4, 1) | (9, 2) |
| 3 | – | – | – | (5, 1) |
| 4 | – | – | – | – |

**Table 10**
The $P_4(i, j)$ matrix for $k = 4$.

| $i/j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | – | 1 | 2 | 3 |
| 2 | – | – | 2 | 3 |
| 3 | – | – | – | 3 |
| 4 | – | – | – | – |

sets $V_i$ and $W_i$ are the points obtained by $\alpha$-cut additions, where the $V$ and $W$ values are obtained by the $\alpha_i$-cuts considering $n = 10$. It includes 10 points for the $a^-_{\alpha_i}$ and 10 points for the $a^+_{\alpha_i}$:

$V_1 = \{(4.58257, 10.4174), (4.93136, 10.0686), (5.20274, 9.79726), (5.44277, 9.55723), (5.66745, 9.33255), (5.88528, 9.11472), (6.10278, 8.89722), (6.32762, 8.67238), (6.57541, 8.42459), (7, 8)\}$

$W_1 = \{(11.0303, 25.9697), (12.1255, 24.8745), (12.911, 24.089), (13.5711, 23.4289), (14.1698, 22.8302), (14.7411, 22.2589), (15.3111, 21.6889), (15.9105, 21.0895), (16.6016, 20.3984), (18, 19)\}$

$V_2 = \{(4.58257, 10.4174), (4.93136, 10.0686), (5.20274, 9.79726), (5.44277, 9.55723), (5.66745, 9.33255), (5.88528, 9.11472), (6.10278, 8.89722), (6.32762, 8.67238), (6.57541, 8.42459), (7, 8)\}$

$W_2 = \{(8.06515, 16.9349), (8.66273, 16.3373), (9.10549, 15.8945), (9.48554, 15.5145), (9.83489, 15.1651), (10.1706, 14.8294), (10.5056, 14.4944), (10.8552, 14.1448), (11.2508, 13.7492), (12, 13)\}$

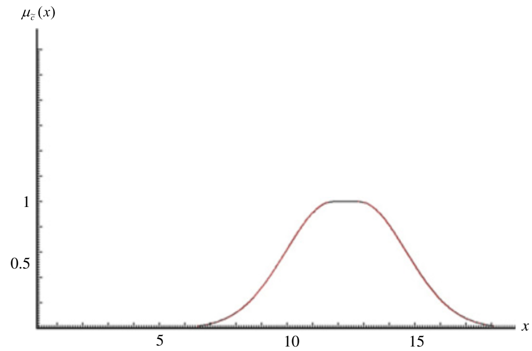


**Fig. 5.** The membership function.

**Table 11**
The analysis of parameter variation.

| $n$ | Path | $\beta'$ | $\beta$ | $\lambda'$ | $\lambda$ |
|---|---|---|---|---|---|
| 10 | 1–2–3–4 | 2.659 | 2.6591 | 12.06 | 12.93 |
| 20 | 1–2–3–4 | 2.6508 | 2.6508 | 12.064 | 12.935 |
| 25 | 1–2–3–4 | 2.646875 | 2.6468508 | 12.063175 | 12.936845 |
| 50 | 1–2–3–4 | 2.63426 | 2.6342363 | 12.056546 | 12.943473 |
| 100 | 1–2–3–4 | 2.6167371 | 2.616722 | 12.044909 | 12.955093 |
| 200 | 1–2–3–4 | 2.6167371 | 2.616722 | 12.044909 | 12.955093 |
| 250 | 1–2–3–4 | 2.614963 | 2.6149583 | 12.043631 | 12.956366 |
| 500 | 1–2–3–4 | 2.6108975 | 2.6109 | 12.040661 | 12.959343 |

**Table 12**
The parameters of non-optimal path with their distance values for Example 2.

| $n$ | Path | $\beta'$ | $\beta$ | $\lambda'$ | $\lambda$ | $D_{2,\frac{1}{2}}$ |
|---|---|---|---|---|---|---|
| 100 | 1–2–4 | 4.6590285 | 4.6590285 | 18.06286 | 18.937162 | 49.06 |
| 100 | 1–3–4 | 3.636081 | 1.6360606 | 13.251405 | 16.748611 | 46.71 |

$V_3 = \{(4.58257,\ 10.4174), (4.93136,\ 10.0686), (5.20274,\ 9.79726), (5.44277,\ 9.55723), (5.66745,\ 9.33255), (5.88528,\ 9.11472), (6.10278, 8.89722), (6.32762, 8.67238), (6.57541, 8.42459), (7, 8)\}$

$W_3 = \{(8.06515,\ 16.9349), (8.66273,\ 16.3373), (9.10549,\ 15.8945), (9.48554,\ 15.5145), (9.83489,\ 15.1651), (10.1706,\ 14.8294), (10.5056, 14.4944), (10.8552, 14.1448), (11.2508, 13.7492), (12, 13)\}$.

Finally, when $k = N$, we identify the shortest path as follows:

Shortest path from 1 to 4: 1–2–3–4.

Shortest path length from 1 to 4:

(8.06515, 16.9349), (8.66273, 16.3373), (9.10549, 15.8945), (9.48554, 15.5145), (9.83489, 15.1651), (10.1706, 14.8294), (10.5056, 14.4944), (10.8552, 14.1448), (11.2508, 13.7492), (12, 13).

Here, we obtain the membership function as shown in Fig. 5.

To investigate the variation of $\beta$, $\beta'$, $\lambda$, and $\lambda'$ with respect to $n$, for the membership function (16), we solve the least squares problem for different values of $n$ and obtain the left and right membership function parameters. The results are reported in Table 11. Note that, the path does not change for different sizes of $n$ and after $n = 100$, the variations in membership function parameters are negligible.

Also, for this optimal path $D_{2,\frac{1}{2}} = 42.53$. Moreover, the costs of other paths with $n = 100$ are obtained as shown in Table 12.

Next, we consider a problem with a larger number of nodes to specify the general input and output structures using the proposed algorithm.

**Example 3.** We consider a larger network as shown in Fig. 6.

We consider the network having mixed arc lengths (a combination of normal and trapezoidal fuzzy numbers) and use our dynamic algorithm to find the shortest paths. The arc lengths are specified in Table 13.

Using the distance function $D_{p,q}$ (for $q = 1/2$ and $p = 2$), the shortest path from the source node 1 to the destination node 23 is determined to be: $1 \to 5 \to 12 \to 15 \to 18 \to 23$.

To find an optimal path, Table 14 is used. For instance, to find an optimal path from node 1 to node 23, according to $P_{23}(i, j)$ matrix, we have $P_{23}(1, 23) \neq 1$ and $P_{23}(1, 23) = 18$. From $P_{23}(1, 23) = 18$, the path $1 \to 18 \to 23$ is obtained.

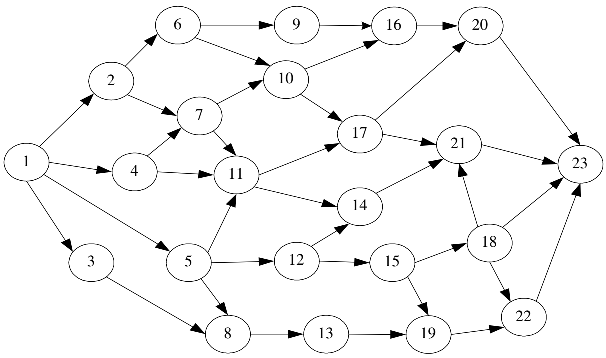


**Fig. 6.** A network.

**Table 13**
The arc lengths.

| Arc | Fuzzy number | Arc | Fuzzy number | Arc | Fuzzy number |
|---|---|---|---|---|---|
| (1, 2) | (12, 13, 15, 17) | (1, 3) | (40, 11) | (1, 4) | (8, 10, 12, 13) |
| (1, 5) | (7, 8, 9, 10) | (2, 6) | (35, 10) | (2, 7) | (6, 11, 11, 13) |
| (3, 8) | (40, 11) | (4, 7) | (17, 20, 22, 24) | (4, 11) | (6, 10, 13, 14) |
| (5, 8) | (29, 9) | (5, 11) | (7, 10, 13, 14) | (5, 12) | (10, 13, 15, 17) |
| (6, 9) | (6, 8, 10, 11) | (6, 10) | (35, 11) | (7, 10) | (9, 10, 12, 13) |
| (7, 11) | (6, 7, 8, 9) | (8, 12) | (5, 8, 9, 10) | (8, 13) | (20, 5) |
| (9, 16) | (6, 7, 9, 10) | (10, 16) | (40, 13) | (10, 17) | (15, 19, 20, 21) |
| (11, 14) | (8, 9, 11, 13) | (11, 17) | (28, 9) | (12, 14) | (13, 14, 16, 18) |
| (12, 15) | (12, 14, 15, 16) | (13, 15) | (37, 12) | (13, 19) | (17, 18, 19, 20) |
| (14, 21) | (11, 12, 13, 14) | (15, 18) | (8, 9, 11, 13) | (15, 19) | (25, 7) |
| (16, 20) | (38, 12) | (17, 20) | (7, 10, 11, 12) | (17, 21) | (6, 7, 8, 10) |
| (18, 21) | (15, 17, 18, 19) | (18, 22) | (16, 5) | (18, 23) | (15,5) |
| (19, 22) | (15, 16, 17, 19) | (20, 23) | (13, 14, 16, 17) | (21, 23) | (12, 15, 17, 18) |
| (22, 23) | (20, 5) | | | | |

**Table 14**
$P_{23}(i, j)$ matrix.

| $k/j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | – | 1 | 1 | 1 | 1 | 2 | 2 | 5 | 6 | 7 | 5 | 5 | 8 | 11 | 12 | 9 | 11 | 15 | 15 | 17 | 14 | 18 | 18 |
| 2 | – | – | – | – | – | 2 | 2 | – | 6 | 7 | 7 | – | – | 11 | – | 9 | 10 | – | – | 17 | 14 | – | 21 |
| 3 | – | – | – | – | – | – | – | 3 | – | – | – | 8 | 8 | 12 | 12 | – | – | 15 | 13 | – | 14 | 18 | 18 |
| 4 | – | – | – | – | – | – | 4 | – | – | 7 | 4 | – | – | 11 | – | 10 | 11 | – | – | 17 | 14 | – | 21 |
| 5 | – | – | – | – | – | – | – | 5 | – | – | 5 | 5 | 8 | 11 | 12 | – | 11 | 15 | 15 | 17 | 14 | 18 | 18 |
| 6 | – | – | – | – | – | – | – | – | 6 | 6 | – | – | – | – | – | 9 | 10 | – | – | 16 | 17 | – | 20 |
| 7 | – | – | – | – | – | – | – | – | – | 7 | 7 | – | – | 11 | – | 10 | 10 | – | – | 17 | 14 | – | 21 |
| 8 | – | – | – | – | – | – | – | – | – | – | – | 8 | 8 | 12 | 12 | – | – | 15 | 13 | – | 14 | 18 | 18 |
| 9 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 9 | – | – | – | 16 | – | – | 20 |
| 10 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 10 | 10 | – | – | 17 | 17 | – | 21 |
| 11 | – | – | – | – | – | – | – | – | – | – | – | – | – | 11 | – | – | 11 | – | – | 17 | 14 | – | 21 |
| 12 | – | – | – | – | – | – | – | – | – | – | – | – | – | 12 | 12 | – | – | 15 | 15 | – | 14 | 18 | 18 |
| 13 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 13 | – | – | 15 | 13 | – | 18 | 19 | 22 |
| 14 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 14 | – | 21 |
| 15 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 15 | 15 | – | 18 | 18 | 18 |
| 16 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 16 | – | – | 20 |
| 17 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 17 | 17 | – | 21 |
| 18 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 18 | 18 | 18 |
| 19 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 19 | 22 |
| 20 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 20 |
| 21 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 21 |
| 22 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | 22 |
| 23 | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |

Also, $P_{23}(1, 18) = 15$, $P_{23}(1, 15) = 12$, and $P_{23}(1, 12) = 5$. Since $P_{23}(1, 5) = 1$, then we **Stop**. Therefore, the shortest path is: $1 \to 5 \to 12 \to 15 \to 18 \to 23$.

Using the least squares model, we regress the estimated points and obtain the following equation as membership function for the shortest path. By addition of various fuzzy numbers on the corresponding path, the membership function is obtained

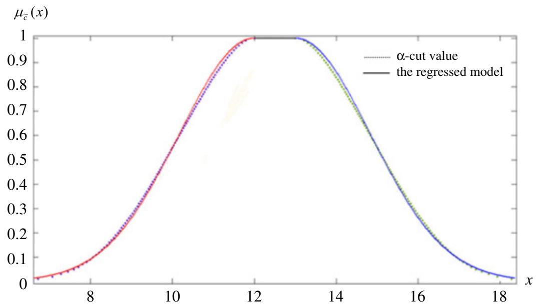


**Fig. 7.** The obtained membership function using the least squares model.

**Table 15**
The results of parameter variation.

| $n$ | Path | $\beta'$ | $\beta$ | $\lambda'$ | $\lambda$ |
|---|---|---|---|---|---|
| 10 | 1–5–12–15–18–23 | 8.954 | 9.613 | 59.439 | 64.622 |
| 20 | 1–5–12–15–18–23 | 8.905 | 9.556 | 59.452 | 64.612 |
| 25 | 1–5–12–15–18–23 | 8.881 | 9.528 | 59.442 | 64.621 |
| 40 | 1–5–12–15–18–23 | 8.828 | 9.467 | 59.411 | 64.647 |
| 50 | 1–5–12–15–18–23 | 8.805 | 9.439 | 59.395 | 64.660 |
| 80 | 1–5–12–15–18–23 | 8.761 | 9.388 | 59.363 | 64.688 |
| 100 | 1–5–12–15–18–23 | 8.743 | 9.367 | 59.349 | 64.700 |
| 200 | 1–5–12–15–18–23 | 8.7 | 9.317 | 59.31 | 64.73 |
| 250 | 1–5–12–15–18–23 | 8.689 | 9.304 | 59.305 | 64.738 |
| 500 | 1–5–12–15–18–23 | 8.66 | 9.276 | 59.284 | 64.75 |

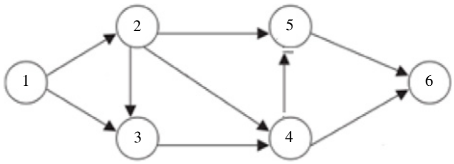


**Fig. 8.** The network.

**Table 16**
The arc lengths of network (both crisp and fuzzy).

| Arc | Fuzzy lengths | Crisp lengths | Arc | Fuzzy lengths | Crisp lengths | Arc | Fuzzy lengths | Crisp lengths |
|---|---|---|---|---|---|---|---|---|
| (1, 2) | (1, 2, 7) | 2 | (1, 3) | (2, 3, 4) | 3 | (2, 3) | (3, 5, 8) | 5 |
| (2, 4) | (1, 2, 6) | 2 | (2, 5) | (2, 5, 6) | 5 | (3, 4) | (1, 3, 4) | 3 |
| (4, 5) | (1, 4, 5) | 4 | (4, 6) | (1, 2, 5) | 2 | (5, 6) | (4, 5, 8) | 5 |

by (11)–(14) as follows:

$$\mu_{\tilde{c}}(x) = \begin{cases} e^{-\left(\frac{x-59}{9.154}\right)^2} & x < 59 \\ 1 & 59 < x < 65 \\ e^{-\left(\frac{x-65}{8.559}\right)^2} & x > 65. \end{cases}$$

The regressed membership function is presented in Fig. 7.

To analyze the variations of $\beta$, $\beta'$, $\lambda$, and $\lambda'$ with respect to $n$, we use different sizes of $n$ and obtain the left and right membership function parameters. The results are reported in Table 15. Note that the path does not change for different values of $n$ bigger than 100, and the variations of membership function parameters are negligible.

## 6. Discussion

Our approach can easily be used when the distances are crisp values. This can serve as a tool for comparing crisp versus fuzzy arc length for a particular case. Consider the example shown by Fig. 8.

Both the fuzzy and crisp distances of arcs are given in Table 16. As an example, we considered the crisp value of each arc to be the corresponding maximum membership value. The obtained results for the shortest paths corresponding to the two cases show the difference of the optimal paths. For the fuzzy case, the optimal path is 1–3–4–6, while for the crisp case, the optimal path is 1–2–4–6. The distance value of optimal path in the fuzzy network, using our proposed algorithm is 5.93, while for the crisp situation is 6 (for the path 1–2–4–6). Moreover, for the optimal path 1–3–4–6 in the fuzzy network, we will obtain the value of 8 for the crisp case, showing the difference of $8 - 5.93 = 2.07$ in favor of the fuzzy case. This example shows the possible effectiveness of considering fuzzy arc lengths in a network.

## 7. Conclusions

A novel practical approach was proposed for computing a shortest path in a fuzzy network having mixed fuzzy arc lengths. In doing this, an $\alpha$-cut method was presented to compute the addition of various fuzzy numbers as arc lengths. To obtain an approximation of the corresponding membership function for the addition, we proposed a linear least squares model. Finally, using a recently proposed distance function, we showed how to decide distances for comparison of fuzzy arc lengths to be used in our proposed dynamic programming algorithm for finding an optimal (shortest) path. The effectiveness of our approach was shown by working out illustrating examples. The proposed model, while being practically simple, has the flexibility to consider a mixture of various types of fuzzy arc lengths in a general network. We also gave a comparative case of fuzzy and crisp fuzzy cases to point out a possible effectiveness of a fuzzy network.

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