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Galois coverings of pointed coalgebras

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ABSTRACT

We introduce the concept of a Galois covering of a pointed coalgebra. The theory developed shows that Galois coverings of pointed coalgebras can be concretely expressed by smash coproducts using the coaction of the automorphism group of the covering. Thus the theory of Galois coverings is seen to be equivalent to group gradings of coalgebras. An advantageous feature of the coalgebra theory is that neither the grading group nor the quiver is assumed finite in order to obtain a smash product coalgebra.

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1. Introduction

Every coalgebra is equivalent to a basic coalgebra, whose simple comodules appear with multiplicity one. Over an algebraically closed field a basic coalgebra is pointed and embeds in the path coalgebra of its quiver. This motivates the study of pointed coalgebras as subcoalgebras of path coalgebras. The theory of coverings of coalgebras extends the theory of coverings of finite-dimensional algebras via quivers with relations.

The theory of Galois coverings of quivers with relations is a standard tool in the representation theory of finite-dimensional algebras see e.g. [1,10,20]. In this paper we introduce the concept of coverings of pointed coalgebras, based on Galois coverings of quivers. The theory developed shows that coverings of coalgebras can be concretely expressed by smash coproducts using the coaction of the automorphism group of coverings. Thus the theory of Galois coalgebra coverings is seen to be equivalent to group gradings of coalgebras. One advantageous feature of the coalgebra theory is that neither the grading group nor the quiver is assumed finite in order to obtain a smash product coalgebra. In the case that the grading group is infinite we do not need to pass to coverings of \mathbb{k} -categories, as done in [7].

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We begin with a review of path coalgebras and embeddings of pointed coalgebras into the path coalgebra of their quivers. Next we turn to the theory of Galois coverings of quivers in Section 3. In this context, coverings and smash coproducts are known in graph theory as derived graphs [12]. A covering of a quiver is specified by an arrow weighting (also known as a voltage assignment), where group elements are assigned to edges. Coverings are obtained from a smash coproduct construction (i.e. the derived graph with respect to a voltage assignment) on the base quiver. For the associated path coalgebras, the covering path coalgebra becomes a smash coproduct coalgebra with a canonical homomorphism onto the base path coalgebra (Theorem 4.2). We wish to restrict to subcoalgebras of the path coalgebras and we provide conditions on the covering that characterize when this is possible in Theorem 4.5, using a designated normal subgroup of the fundamental group of the base quiver. The conditions guarantee that the covering coalgebra is a smash coproduct over the homogeneous base coalgebra, which is graded by the automorphism group of the covering. Thus the covering quiver has a subcoalgebra whose comodule category is equivalent to the category of graded comodules over the base coalgebra. Moreover, the methods here provide a construction of a universal covering coalgebra which provides a way of constructing all connected gradings of a pointed subcoalgebra of a path coalgebra. As a converse to these results, in Theorem 5.1 we show that any connected grading of a pointed coalgebra gives rise to a smash coproduct that serves as a covering coalgebra.

A grading of a pointed coalgebra is specified by an arrow weighting that comes from a vertex lifting of a covering of the quiver. Different liftings yield possibly different gradings and isomorphic smash coproducts. We show explicitly in Section 6 how liftings and gradings are related via vertex weightings, and how they are equivalent via smash coproduct coalgebra isomorphisms.

We close with examples including single and double loop quivers, the Kronecker quiver, Example 8.3, and the basic coalgebra of quantum $SL(2)$ at root of unity, Example 8.4. In these examples we examine coverings the gradability of arbitrary finite-dimensional comodules.

We partially follow the approach of E.L. Green in [11] where coverings of quivers with relations were studied for locally finite quivers with relations. Green's paper contains a definition of a Galois covering for quivers with relations, and establishes a universal object in the category of such coverings. He also showed that the representation theory of finitely generated group-graded algebras is essentially equivalent to the theory of the representation theory of Galois coverings of finite quivers with relations. More precisely, given a finitely generated algebra presented by a finite quiver with relations, the category of representations of a covering with Galois group G is equivalent to the category of G -graded modules over the algebra. Other related work concerning coverings of quivers with relations and coverings of \mathbb{k} -categories includes [16,7,15]. In this paper we work directly with coverings via smash coproduct coalgebras, thereby obtaining results on graded comodules. Thus we do not work directly with representations of quivers as is done in [11].

In a manner dual to the phenomenon for quivers with relations and finite-dimensional algebras, a pointed coalgebra B can be embedded in $\mathbb{k}Q$ in more than one way (see Section 2). However, for many presentations of algebras by quivers with relations, there is a canonical presentation which results in universal fundamental group [15]. It would be interesting to extend this sort of result to coalgebras.

The reader may refer to [17,13] for basic information on covering spaces and [12] for combinatorial versions of covering graphs and smash coproduct quivers, known there as derived graphs of voltage graphs. Basic coalgebra theory, including the theory of pointed coalgebras and path coalgebras, is covered in e.g. [3]. The articles [6,5,2,21,19,22,14,23] contain results concerning path coalgebras.

2. Path coalgebras

The vertex set of a quiver Q is denoted by Q_0 and the arrow set is denoted by Q_1 . For each arrow a the start (or source) of a is denoted by $s(a)$ and the terminal (or target) of a is written $t(a)$ where $s, t: Q_1 \rightarrow Q_0$ are functions. Paths in Q are, by definition, finite concatenations of arrows, and are always directed. Paths are written from right to left and are of the form $a_n a_{n-1} \cdots a_1$ with $a_i \in Q$ and $s(a_i) = t(a_{i-1})$ for $i = 2, \dots, n$. Using formal inverses of arrows, we also consider possibly nondirected paths $a_n^\pm a_{n-1}^\pm \cdots a_1^\pm$ with $s(a^-) = t(a)$ and $t(a^-) = s(a)$, referred to as walks in Q . Here we are slightly abusing terminology since paths and walks are in general not elements of Q . The

set of paths of length n is denoted by Q_n and their span by $\mathbb{k}Q_n$. For each pair $x, y \in Q_0$ we write $Q(x, y)$ (resp. $Q^\pm(x, y)$) for the set of paths (resp. walks) starting at x and ending at y . Here we have $B(x, y) = \mathbb{k}Q(x, y) \cap B$ and $B = \bigoplus_{x, y \in Q_0} B(x, y)$.

J.A. Green (see [3]) showed that the structure theory for finite-dimensional algebras carries over well to coalgebras, with injective indecomposable comodules replacing projective indecomposables. Define the (Gabriel- or Ext-) quiver of a coalgebra C to be the directed graph $Q(C)$ with vertices \mathcal{G} corresponding to isomorphism classes of simple comodules and $\dim_{\mathbb{k}} \text{Ext}^1(h, g)$ arrows from h to g , for all $h, g \in \mathcal{G}$. The blocks of C are (the vertices of) components of the undirected version of the graph $Q(C)$. In other words, the blocks are the equivalence classes of the equivalence relation on \mathcal{G} generated by arrows. The indecomposable or “block” coalgebras are the direct sums of injective indecomposables having socles from a given block. When C is a pointed coalgebra we may identify \mathcal{G} with the set of group-like elements $G(C)$ of C , and we may take the arrows to be a basis of a nonuniquely chosen \mathbb{k} -space I_1 spanned by nontrivial skew primitive elements. Here $C_0 \oplus I_1 = C_1$.

The **path coalgebra** $\mathbb{k}Q$ of a quiver Q is defined to be the span of all paths in Q with coalgebra structure

$$\Delta(p) = \sum_{p=p_2p_1} p_2 \otimes p_1 + t(p) \otimes p + p \otimes s(p),$$

$$\varepsilon(p) = \delta_{|p|,0}$$

where p_2p_1 is the concatenation $a_t a_{t-1} \cdots a_{s+1} a_s \cdots a_1$ of the subpaths $p_2 = a_t a_{t-1} \cdots a_{s+1}$ and $p_1 = a_s \cdots a_1$ ($a_i \in Q_0$). Here $|p| = t$ denotes the length of p and the starting vertex of a_{i+1} is required to be the end of a_i . Thus vertices are group-like elements, and if a is an arrow $g \leftarrow h$, with $g, h \in Q_0$, then a is a (g, h) -skew primitive, i.e., $\Delta a = g \otimes a + a \otimes h$. It follows that kQ is pointed with coradical $(kQ)_0 = kQ_0$ and the degree one term of the coradical filtration is $(kQ)_1 = kQ_0 \oplus kQ_1$. Moreover have the coradical grading $\mathbb{k}Q = \bigoplus_{n \geq 0} \mathbb{k}Q_n$ by path length. The path coalgebra may be identified with the cotensor coalgebra $\bigoplus_{n \geq 0} (\mathbb{k}Q_1)^{\square n}$ of the $\mathbb{k}Q_0$ -bicomodule $\mathbb{k}Q_1$, cf. [18].

Let B be a pointed coalgebra over a field \mathbb{k} . Then B embeds nonuniquely as an admissible subcoalgebra of the path coalgebra $\mathbb{k}Q$ of the Gabriel quiver associated to B . We review this embedding here (see [6,24], or [22]; cf. [18]). Write $B = kQ_0 \oplus I$ for a (nonunique) coideal I , with projection $\pi_0 : B \rightarrow B_0$ along I , and write $\mathbb{k}Q = \mathbb{k}Q_0 \oplus J$ where $J = \mathbb{k}Q_1 \oplus \mathbb{k}Q_1 \square \mathbb{k}Q_1 \oplus \cdots$ and $\mathbb{k}Q = \mathbb{k}Q_0 \oplus J$ is the cotensor coalgebra over $\mathbb{k}Q_0$. We identify B_0 with $\mathbb{k}Q_0$, and let $I_1 = I \cap B_1$, which we identify with $\mathbb{k}Q_1$. Define the embedding

$$\theta : B \rightarrow kQ = kQ_0 \oplus J$$

by $\theta(d) = \pi_0(d) + \sum_{n \geq 1} \pi_1^{\otimes n} \Delta_{n-1}(d)$ for all $d \in B$, where $\pi_1 : \mathbb{k}Q \rightarrow I_1$ is a B_0 -bicomodule projection onto $I_1 = \mathbb{k}Q_1$.

It is very well known that an algebra might be presented by quivers with relations in essentially different ways. Of course this happens coalgebraically as well. For example, let Q be the quiver

$$\begin{array}{c} a \\ z \xleftarrow{a} y \xleftarrow{c} z \\ b \end{array}$$

and consider the subcoalgebra of $\mathbb{k}Q$ spanned by the arrows and vertices together with the degree two element ac . If we replace ac by $ac + bc$ we obtain a different, but isomorphic, subcoalgebra. In the embedding above, the nonuniqueness can be seen to be a result of the choice of the skew primitive space I_1 . It will be apparent in Section 4 that these subcoalgebras are associated to different subgroups of fundamental group of the quiver.

Henceforth we shall assume a fixed embedding of the pointed coalgebra B into its path coalgebra so that $B \subseteq \mathbb{k}Q$ is an admissible subcoalgebra, i.e., B contains the vertices and arrows of Q . We shall always assume that B is indecomposable and, equivalently, that Q is connected as a graph.

3. Quivers and coverings

The quiver Q may be realized as a topological space, momentarily dropping the orientation of arrows, as in e.g. [17,13]. Let $F: \tilde{Q} \rightarrow Q$ be a topological Galois covering with base points $\tilde{x}_0, x_0 \in Q_0$, $F(\tilde{x}_0) = x_0$. All coverings are assumed to be connected. The topological covering space \tilde{Q} can be realized as a quiver first by giving it a graph structure and then by assigning an orientation consistent with the orientation on Q . The vertices of \tilde{Q} are the union of the fibers of the vertices of Q , and the arrows are the liftings of the arrows of Q . We can then consider F to be a morphism of quivers, as done in [11]. Coverings can be viewed as quiver maps purely combinatorially, cf. [16,15]. One can discretely construct a universal cover as the quiver whose vertices are in bijection with equivalence (homotopy) classes of walks in Q . Then any covering is isomorphic to an orbit quiver under the free action of a subgroup G of the fundamental group $\pi_1(Q, x_0)$ of Q .

For the purposes of this paper we discretely define a covering of a quiver Q to be a surjective morphism of connected quivers $F: \tilde{Q} \rightarrow Q$ such that for all $\tilde{x} \in \tilde{Q}_0$, F restricts to bijections between the set of arrows starting at \tilde{x} and the set of arrows starting at $F(\tilde{x})$, and also between the set of arrows ending at \tilde{x} and the arrows ending at $F(\tilde{x})$.

Let $\tilde{Q} \xrightarrow{F} Q \xleftarrow{F'} Q'$ be two coverings of Q . A morphism $\theta: F \rightarrow F'$ in the category of coverings of Q is a quiver map $\theta: \tilde{Q} \rightarrow Q'$ such that $F'\theta = F$. We shall refer to such maps θ as a *covering morphisms of quivers*.

The fundamental group functor provides an injective group homomorphism $F_*: \pi_1(\tilde{Q}, \tilde{x}) \rightarrow \pi_1(Q, x)$ for $\tilde{x} \in \tilde{Q}_0$ lifting $x \in Q_0$ for any covering $F: \tilde{Q} \rightarrow Q$. The covering is said to be *Galois* (also known as *regular* or *normal*) if $F_*(\pi_1(\tilde{Q}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(Q, x_0)$. In this case $\pi_1(Q, x_0)/F_*(\pi_1(\tilde{Q}, \tilde{x}_0))$ is the automorphism group of F , viewed either combinatorially as a group of quiver automorphisms, or topologically as a group of covering space automorphisms.

A fundamental property of coverings is that each walk in Q can be lifted uniquely to a walk in \tilde{Q} once the starting point in \tilde{Q} is specified. More specifically if w is a walk starting at $x \in Q_0$ and $F(\tilde{x}) = x$, then there is a unique walk \tilde{w} starting at \tilde{x} and lifting w . If w is path, then the lifting \tilde{w} is a path. We say that a function $L: Q_0 \rightarrow \tilde{Q}_0$ is a *lifting* of F if $FL = \text{id}_{Q_0}$. For a walk w in Q , we shall write $L(w)$ to denote the unique lifting of w starting at $L(s(w))$.

For a group G , we say that a function $\delta: Q_1 \rightarrow G$ is an *arrow weighting*. For later use we also define a *vertex weighting* to be a function $\gamma: Q_0 \rightarrow G$. The map δ may be extended to all walks by setting

$$\delta(a_n^{e_n} \cdots a_2^{e_2} a_1^{e_1}) = \delta(a_n)^{e_n} \cdots \delta(a_2)^{e_2} \delta(a_1)^{e_1},$$

$e_i = \pm$. We shall henceforth identify arrow weightings and their extension to walks or paths in this manner.

Next let $G = \pi_1(Q, x_0)/F_*(\pi_1(\tilde{Q}, \tilde{x}_0))$ be the automorphism group of the Galois covering F . There is an arrow weighting $\delta_L: Q_1 \rightarrow G$ as follows. Let $w \in Q^\pm(x, y)$. Let $L(w)$ be the lifting of w starting at $L(x)$ and ending at $L(y)^g$, $g \in G$. We define the map by letting $\delta_L(w) = g$. Clearly δ_L depends on the choice of lifting L . Restricting to arrows, we obtain an arrow weighting δ_L . Now let $w = a_n^{e_n} \cdots a_2^{e_2} a_1^{e_1} \in Q^\pm(x, z)$. It is true that $\delta_L(a_n^{e_n} \cdots a_2^{e_2} a_1^{e_1}) = \delta_L(a_n)^{e_n} \cdots \delta_L(a_2)^{e_2} \delta_L(a_1)^{e_1}$, for if $w = rq$, with $q \in Q^\pm(x, y)$ and $r \in Q^\pm(y, z)$, then $L(q) \in \tilde{Q}^\pm(L(x), L(y)^{\delta_L(q)})$ and $L(r) \in \tilde{Q}^\pm(L(y), L(z)^{\delta_L(r)})$; hence $L(r)^{\delta_L(q)} L(q) \in \tilde{Q}^\pm(L(x), L(z)^{\delta_L(r)\delta_L(q)})$ is a lifting of rq and we conclude that $\delta_L(rq) = \delta_L(r)\delta_L(q)$. Thus the weighting δ_L respects concatenation in agreement with the extension from arrows to walks at the beginning of this paragraph.

An arrow weighting $\delta: Q_1 \rightarrow G$ is said to be *connected* if for all $x, y \in Q_0$ and $g \in G$, there exists a walk w in Q from x to y such that $\delta(w) = g$. Assume δ is connected and let $Q \rtimes G$ be the quiver with underlying vertex set $(Q \rtimes G)_0 = Q_0 \times G = \{x \rtimes g \mid x \in Q_0, g \in G\}$ and arrow set $(Q \rtimes G)_1 = Q_1 \times G = \{a \rtimes g \mid a \in Q_1, g \in G\}$, declaring that $s(a \rtimes g) = s(a) \rtimes g$ and $t(a \rtimes g) = t(a) \rtimes \delta(a)g$. This construction shall be called the *smash coproduct quiver* of Q and G . Let $F: Q \rtimes G \rightarrow Q$ be defined by $F(u \rtimes g) = u$ for $u \in Q_0 \cup Q_1, g \in G$, and note that F is a surjective quiver morphism. Also, δ is connected if and only if $Q \rtimes G$ is connected as a graph. It is known that:

Proposition 3.1. *If $\delta : Q_1 \rightarrow G$ is a connected arrow weighting, the canonical map $F : Q \rtimes G \rightarrow Q$ is a Galois covering with automorphism group G .*

Proof. By the hypothesis that δ is connected, it is immediate $Q \rtimes G$ is connected as a graph. By the construction of $Q \rtimes G$, each $y \in Q_0$ has fiber $\{y \rtimes g \mid g \in G\}$ and each subgraph $z \xleftarrow{b} y \xleftarrow{a} x$ in Q with arrows a, b and vertices x, y, z lifts to a subgraph

$$z \rtimes \delta(b)g \xleftarrow{b \rtimes g} y \rtimes g \xleftarrow{a \rtimes \delta^{-1}(a)g} x \rtimes \delta^{-1}(a)g$$

for all $g \in G$. This observation makes it evident that there is a bijection between the set of arrows ending at (resp. starting at) y and the arrows ending at (resp. starting at) $y \rtimes g$. Also it is clear that these arrows sets are disjoint for different g . It follows that $F : Q \rtimes G \rightarrow Q$ is a covering map.

We note that if $[w] \in \pi_1(Q, x_0)$ is a walk class with closed walk w , then the map $\pi_1(Q, x_0) \rightarrow G$ defined by $[w] \mapsto \delta(w)$ is onto since δ is connected. Its kernel is the set of walk classes of walks that lift to closed walks in $Q \rtimes G$, i.e. the normal subgroup $F_*(\pi_1(Q \rtimes G))$. Thus the covering F is Galois and the automorphism group of the covering is $G = \pi_1(Q, x_0)/F_*(\pi_1(Q \rtimes G))$. \square

Alternatively one can observe that the right action of G on $Q \rtimes G$ is free and that the orbit quiver is isomorphic to Q .

4. Graded coalgebras from coverings

A coalgebra C is said to be graded by the group G if C is the direct sum of \mathbb{k} -subspaces $C = \bigoplus_{g \in G} C_g$ and

$$\Delta(C_g) \subseteq \sum_{\substack{ab=g \\ a, b \in G}} C_a \otimes C_b$$

for all $g \in G$ and $\varepsilon(C_g) = 0$ for all $g \neq 1$. The element $c \in C_g$ is said to be *homogeneous of degree* $\delta(c) = g$ and we shall adapt Sweedler notation and write $\Delta(c) = \sum c_1 \otimes c_2$ always assuming homogeneous terms in the sum. In this case C is a right $\mathbb{k}G$ -comodule coalgebra via the structure map $\rho : C \rightarrow C \otimes \mathbb{k}G$, $\rho(c) = c \otimes g$ for all $g \in G$ and $c \in C_g$. Conversely, every right $\mathbb{k}G$ -comodule coalgebra is a G -graded coalgebra.

Let C be a G -graded coalgebra. The smash coproduct coalgebra of C and $\mathbb{k}G$ is denoted by $C \rtimes \mathbb{k}G$ is defined as the \mathbb{k} -vector space $C \otimes \mathbb{k}G$ with $\Delta : C \rtimes \mathbb{k}G \rightarrow (C \rtimes \mathbb{k}G) \otimes (C \rtimes \mathbb{k}G)$ given by

$$\Delta(c \rtimes g) = \sum (c_1 \rtimes \delta(c_2)g) \otimes (c_2 \rtimes g)$$

for all homogeneous $c \in C$ and $g \in G$. Let $F : C \rtimes \mathbb{k}G \rightarrow C$, $c \rtimes g \mapsto c$ be the canonical colagebra map onto C . G acts canonically on the right as coalgebra automorphisms of $C \rtimes \mathbb{k}G$ via $(c \rtimes g)^h = c \rtimes gh$ for all $h \in G$. It is clear the action of G preserves F , i.e. $F = F \circ h$.

Proposition 4.1. (See [8].) *The category of graded right comodules Gr^C of C is equivalent the comodule category $\mathcal{M}^{C \rtimes \mathbb{k}G}$ over $C \rtimes \mathbb{k}G$.*

A graded comodule M acquires the structure of a $C \rtimes \mathbb{k}G$ -comodule given by $\rho'(m) = \sum m_0 \otimes (m_1 \rtimes \delta(m)^{-1})$, $m \in M$. If N is a $C \rtimes \mathbb{k}G$ -comodule, then N acquires a right $\mathbb{k}G$ -comodule structure via the coalgebra map $C \rtimes \mathbb{k}G \rightarrow \mathbb{k}G$, $c \rtimes g \mapsto \varepsilon(c)g^{-1}$; N is a right C -comodule via the coalgebra map $C \rtimes \mathbb{k}G \rightarrow \mathbb{k}G$, $c \rtimes g \mapsto c$. Thus N corresponds to an object of Gr^C . See [8] for more details.

Consider a Galois covering $F : \tilde{Q} \rightarrow Q$ with automorphism group G and lifting L . We saw in the previous section that there is an arrow weighting associated to L . The path coalgebra $\mathbb{k}Q$ is similarly

a G -graded coalgebra as follows: Let p be a path in $Q(x, y)$. Let $L(p) = \tilde{p}$ be the lifting of p starting at $L(x) = \tilde{x}$ and ending at $L(y)^g$. We define the degree map $\delta(p) = g$. Suppose p is the concatenation of paths $p = rq$. Then $L(p) = L(r)^{\delta(q)}L(q)$ so $\delta(p) = \delta(r)\delta(q)$. It follows that δ determines a coalgebra grading of $\mathbb{k}Q$ depending on the choice of lifting L . It is apparent that grading is determined by the arrow weighting obtained by restricting to Q_1 . We accordingly form the smash coproduct $\mathbb{k}Q \rtimes \mathbb{k}G = \mathbb{k}Q \rtimes_L \mathbb{k}G$, with the canonical projection $F_L: \mathbb{k}Q \rtimes_L \mathbb{k}G \rightarrow \mathbb{k}Q$, $F(p \rtimes g) = p$, using this coaction of $\mathbb{k}G$ on $\mathbb{k}Q$, sometimes leaving the lifting L implicit.

Theorem 4.2. *Let $F: \tilde{Q} \rightarrow Q$ be a Galois covering and let $L: Q_0 \rightarrow \tilde{Q}_0$ be a lifting. There is a coalgebra isomorphism $\psi: \mathbb{k}\tilde{Q} \rightarrow \mathbb{k}Q \rtimes \mathbb{k}G$ with $F_L \circ \psi = F$ and a Galois covering isomorphism from $F_L: Q \rtimes G \rightarrow Q$ to $F: \tilde{Q} \rightarrow Q$.*

Proof. If \tilde{p} is a path in \tilde{Q} , then \tilde{p} is a lifting of $F(\tilde{p})$. Hence there exists $\sigma(\tilde{p}) \in G$ such that $LF(\tilde{p}) = \tilde{p}^{\sigma(\tilde{p})}$. Since $\sigma(\tilde{p}) = \sigma(s(\tilde{p}))$, $\sigma = \sigma_L: \tilde{Q}_0 \rightarrow G$ is a vertex weighting on \tilde{Q} that extends to a function on all paths in \tilde{Q} , as specified. Let $g \in G$ and let $p \in Q(x, y)$. Define maps $\phi: \mathbb{k}Q \rtimes \mathbb{k}G \rightarrow \mathbb{k}\tilde{Q}$ and $\psi: \mathbb{k}\tilde{Q} \rightarrow \mathbb{k}Q \rtimes \mathbb{k}G$ by $\phi(p \rtimes g) = L(p)^g$ and $\psi(\tilde{p}) = F(\tilde{p}) \rtimes \sigma(\tilde{p})$ for paths p, \tilde{p} . Then

$$\Delta(p \rtimes g) = \sum_{p=rq} (r \rtimes \delta(q)g) \otimes (q \rtimes g)$$

and

$$\Delta(L(p)^g) = \sum_{p=rq} L(r)^{\delta(q)g} \otimes L(q)^g$$

using the fact that $L(q)$ is the lifting starting at $L(x)$ and ending at $L(y)^{\delta(q)} = s(L(r)^{\delta(q)})$. It follows immediately that ϕ is coalgebra map. Next observe that $\psi\phi(p \rtimes g) = \psi(L(p)^g) = F(L(p)^g) \rtimes \sigma(L(p)^g) = p \rtimes g$. Similarly, $\phi\psi = id_{\mathbb{k}\tilde{Q}}$. Thus we have shown that ϕ and ψ are mutually inverse coalgebra isomorphisms. It is immediate that $F_L\psi = F$. The isomorphism restricts to an isomorphism on vertices and arrows, so there is a covering isomorphism $\tilde{Q} \cong Q \rtimes G$ as well. \square

The result shows that the isomorphism type of the smash coproduct coalgebra $\mathbb{k}Q \rtimes \mathbb{k}G$ does not depend on the choice of lifting L , which is determined by the grading δ_L . In addition we have

Proposition 4.3. *The smash coproduct coalgebra $\mathbb{k}Q \rtimes \mathbb{k}G$ is isomorphic to the path coalgebra $\mathbb{k}(Q \rtimes G)$ of the smash coproduct quiver.*

Proof. Define a map $E: \mathbb{k}Q \rtimes \mathbb{k}G \rightarrow \mathbb{k}(Q \rtimes G)$ on elements $p \rtimes g$ where $p = a_n \cdots a_2 a_1$ ($a_i \in Q_1$) is a path in Q and $g \in G$ by letting $E(p \rtimes g)$ be the concatenation

$$\begin{aligned} &(a_n \rtimes \delta(a_{n-1}) \cdots \delta(a_1)g) \cdots (a_2 \rtimes \delta(a_1)g)(a_1 \rtimes g) \\ &= (a_n \rtimes \delta(a_{n-1} \cdots a_1)g) \cdots (a_2 \rtimes \delta(a_1)g)(a_1 \rtimes g) \end{aligned}$$

noting that E identifies the group-likes $x \rtimes g$ and the vertices (denoted by the same symbol). Similarly E identifies the skew-primitives $a \rtimes g$ and the arrows. It is straightforward to check that E is a coalgebra isomorphism. \square

We provide some terminology and notation:

- Let $B \subseteq \mathbb{k}Q$ be an admissible subcoalgebra. Let $b = \sum_{i \in I} \lambda_i p_i \in B(x, y)$ with $x, y \in Q_0$ and distinct paths p_i . We say that b is a *minimal element* of B if $\sum_{i \in I'} \lambda_i p_i \notin B(x, y)$ for every nonempty

proper subset $I' \subset I$, and $|I| \geq 2$. Clearly every element of B is a linear combination of paths and minimal elements.

- Fix a base vertex $x_0 \in Q_0$. We define a symmetric relation \sim on paths by declaring $p \sim q$ if there is a minimal element $b = \sum_{i \in I} \lambda_i p_i \in B(x, y)$ where the p_i are distinct paths, $\lambda_i \in \mathbb{k}$, $x, y \in Q_0$ and $p = p_1$, $q = p_2$. We define $N(B, x_0)$ to be the normal subgroup of $\pi_1(B, x_0)$ generated by equivalence (homotopy) classes of walks $w^{-1}p^{-1}qw$ where p, q are paths in $Q(x, y)$ with $p \sim q$ and w is a walk from x_0 to x .
- Consider a Galois covering $F: \tilde{Q} \rightarrow Q$ with automorphism group G and lifting L . For each minimal element $b = \sum \lambda_i p_i \in B$ we put $L(b) = \sum \lambda_i L(p_i)$ and we let \tilde{B} denote the \mathbb{k} -span of $\{L(b) \mid L \text{ a lifting, } b \in B \text{ a minimal element or a path}\}$. We say that the restriction $F: \tilde{B} \rightarrow B$ is a Galois coalgebra covering if every minimal element of B can be lifted to \tilde{B} in the following sense: for every minimal element $b \in B(x, y)$ with $x, y \in Q_0$ and $\tilde{x} \in \tilde{Q}_0$, there exists $\tilde{y} \in \tilde{Q}_0$ and a minimal element $\tilde{b} \in \tilde{B}(\tilde{x}, \tilde{y})$ such that $F(\tilde{b}) = b$.

Proposition 4.4. *Let $F: \tilde{B} \rightarrow B$ be a Galois coalgebra covering. Then*

- (a) $F(\min(\tilde{B})) = \min(B)$,
- (b) $F_*(N(\tilde{B}, \tilde{x}_0)) = N(B, x_0)$ for all $x_0 \in Q, \tilde{x}_0 \in \tilde{Q}$ with $F(\tilde{x}_0) = x_0$.

Proof. Since each path in Q lifts to a unique path in \tilde{Q} starting at \tilde{x} , it follows that each minimal element in $b \in B$ can be lifted uniquely to an element of $\tilde{B}(\tilde{x}, \tilde{y})$ starting at \tilde{x} and ending at \tilde{y} for some $\tilde{y} \in \tilde{Q}_0$. Now letting \tilde{x} vary over $F^{-1}(x)$, we see that $F^{-1}(b)$ consists of the set of liftings, one for each \tilde{x} . It is immediate that each such lifting is minimal in \tilde{B} . Conversely, if $\tilde{b} \in \tilde{B}(\tilde{x}, \tilde{y})$ is minimal, then it is the unique lifting of $F(\tilde{b})$ starting at \tilde{x} ; it follows that $F(\tilde{b})$ is minimal. The conclusions follow. \square

The fundamental example is given as follows. Let $B \subseteq \mathbb{k}Q$ be a homogeneous admissible subcoalgebra with respect to the grading given by an arrow weighting $\delta: Q_1 \rightarrow G$. The grading is said to be connected if the arrow weighting is connected. If $b = \sum_{i \in I} \lambda_i p_i \in B(x, y)$ is a minimal element, then it is necessarily homogeneous. Consider the canonical map $F: \mathbb{k}Q \rtimes \mathbb{k}G \rightarrow \mathbb{k}Q$ defined by $F(p \rtimes g) = p$ and consider the restriction to $B \rtimes \mathbb{k}G \rightarrow B$. Then under the identification of $\mathbb{k}Q \rtimes \mathbb{k}G$ with $\mathbb{k}\tilde{Q}$ we easily see that $\tilde{B} = B \rtimes \mathbb{k}G$. The liftings of the minimal element $b \in B$ are given by $b \rtimes g$ with $g \in G$.

Theorem 4.5. *The following are equivalent for a subcoalgebra $B \subseteq \mathbb{k}Q$ and Galois quiver covering $F: \tilde{Q} \rightarrow Q$.*

- (a) B is a homogeneous subcoalgebra of $\mathbb{k}Q$.
- (b) $N(B, x_0) \subseteq F_*(\pi_1(\tilde{Q}, \tilde{x}_0))$ for all $x_0 \in Q, \tilde{x}_0 \in \tilde{Q}$ with $F(\tilde{x}_0) = x_0$.
- (c) $F: \tilde{B} \rightarrow B$ is a Galois coalgebra covering.
- (d) B is a homogeneous subcoalgebra of $\mathbb{k}Q$ and the grading is connected.

Proof. (a) implies (b). Suppose that B is a graded subcoalgebra of $\mathbb{k}Q$. Let $[w^{-1}q^{-1}pw]$ be a generating element of $N(B, x)$ with $p \sim q$. This means that there are distinct paths $p, q \in Q(x, y)$ with $b = p + \lambda q + \dots \in B(x, y)$ minimal. The minimality of b forces it to be homogeneous in the grading determined by L . Accordingly, we have $t(L(p)) = t(L(q)) = L(y)^{\delta(p)} = L(y)^{\delta(q)}$. Now it is easy to check that

$$L(w)^{-1}(L(q)^{\delta(w)})^{-1}L(p)^{\delta(w)}L(w)$$

is a closed path in \tilde{Q} starting at $L(x_0)$ lifting $w^{-1}q^{-1}pw$. This shows that $[w^{-1}q^{-1}pw] \in F_*(\pi_1(\tilde{Q}, x_0))$.

Assume as in (b) that $N(B, x_0) \subseteq F_*(\pi_1(\tilde{Q}, \tilde{x}_0))$ and let $b = \sum_{i \in I} \lambda_i p_i \in B(x, y)$ with $x, y \in Q_0$ and distinct paths p_i be a minimal element of B . For each $i \in I$, let \tilde{p}_i be a lifting of p_i starting at $\tilde{x} = L(x)$ and ending at, say, \tilde{y}_i . Our assumption implies that $N(B, y) \subseteq F_*(\pi_1(\tilde{Q}, \tilde{y}_1))$ by a standard

isomorphism (given by conjugation by a walk class from y to x_0). Observe that $\tilde{p}_2\tilde{p}_1^{-1}$ is a walk from \tilde{y}_1 to \tilde{y}_2 lifting the closed walk $p_2p_1^{-1}$. Therefore $[p_2p_1^{-1}] \in N(B, y) \subseteq F_*(\pi_1(\tilde{Q}, \tilde{y}_1))$, and we see that $p_2p_1^{-1}$ also has a lifting that is a closed walk starting and ending at \tilde{y}_1 . Therefore $\tilde{p}_2\tilde{p}_1^{-1}$ is a closed walk by e.g. [17, Ch. 8, Lemma 3.3] and $\tilde{y}_1 = \tilde{y}_2$. This argument shows that the \tilde{y}_i are all equal. Thus $\tilde{b} = \sum_{i \in I} \lambda_i \tilde{p}_i$ is a lifting of b in $B(\tilde{x}, \tilde{y}_1)$ and it is easily seen to be minimal. This proves (b) implies (c).

Assume (c) and let $b = \sum_{i \in I} \lambda_i p_i \in B(x, y)$ with $x, y \in Q_0$ be a minimal element of B . For each $i \in I$, let \tilde{p}_i be a lifting of p_i starting at $\tilde{x} = L(x)$ and, by definition of the grading, ending at $L(y_i)^{\delta(p_i)}$. The assumption forces the $\delta(p_i)$ to all be equal. This shows that b is homogeneous and thus that B is a graded subcoalgebra of $\mathbb{k}Q$.

We have shown that (a)–(c) are equivalent. We complete the proof by showing that any grading of B is connected. Let $x, y \in Q_0$ and let $g \in G$. Then there exists a walk $\tilde{w} \in \tilde{Q}(L(x), L(y)^g)$. So $w = F(\tilde{w})$ is a walk from x to y with lifting \tilde{w} , and evidently $\delta(w) = g$. Thus the grading is connected. \square

Let $\tilde{Q} \xrightarrow{F} Q \xleftarrow{F'} Q'$ be two coverings of Q . Recall that a morphism $F \rightarrow F'$ in the category of coverings of Q is given by a quiver morphism $\theta: \tilde{Q} \rightarrow Q'$ such that $F'\theta = F$. Consider coalgebra coverings of $B \subset \mathbb{k}Q$ arising from the quiver coverings

$$\begin{array}{ccccc} \mathbb{k}\tilde{Q} & \xrightarrow{F} & \mathbb{k}Q & \xleftarrow{F'} & \mathbb{k}Q' \\ \cup & & \cup & & \cup \\ \tilde{B} & \longrightarrow & B & \longleftarrow & B' \end{array}$$

where the vertical maps are the presumed inclusions of admissible subcolagebras. A morphism of coalgebra coverings $(F: \tilde{B} \rightarrow B) \rightarrow (F': B' \rightarrow B)$ is a morphism of quiver coverings θ as above such that the θ restricts to coalgebra map $\tilde{B} \rightarrow B'$ (again abusively denoting both the map on path coalgebras and its restriction by θ).

Corollary 4.6. *Assume any of the equivalent conditions of the theorem hold and fix a lifting $L: Q_0 \rightarrow \tilde{Q}_0$. Then the coalgebra covering $F: \tilde{B} \rightarrow B$ is isomorphic to $F_L: B \rtimes G \rightarrow B$.*

Proof. It is easy to check that $B \rtimes G$ is the span of all liftings to $Q \rtimes G$ of paths and minimal elements of B . The result follows from Theorem 4.5, which says that a lifting L gives rise to a covering isomorphism from $F_L: Q \rtimes G \rightarrow Q$ to $F: \tilde{Q} \rightarrow Q$. \square

Lemma 4.7. *Let $\theta: \tilde{B} \rightarrow B'$ be a morphism of Galois coalgebra coverings of B . Then the map $\theta: \tilde{B} \rightarrow B'$ is a Galois coalgebra covering.*

Proof. Adopt the notation of the preceding paragraph. Fix $x_0 \in Q$, $\tilde{x}_0 \in \tilde{Q}$, $x'_0 \in Q'_0$ with $F(\tilde{x}_0) = x_0 = F'(x'_0)$. The quiver morphism $\theta: \tilde{Q} \rightarrow Q'$ is a continuous map on the topological realizations. Therefore by e.g. [17, Ch. 5, Lemma 6.7] it is a topological covering. It is immediately seen to be a quiver morphism as well, as the condition $F'\theta = F$ forces θ to behave well on vertices and arrows. Next, we have $F'_*\theta_*(\pi_1(\tilde{Q}, \tilde{x}_0)) = F_*(\pi_1(Q, x_0))$ is a normal subgroup of $\pi_1(Q, x_0)$; also we have

$$F'_*\theta_*(\pi_1(\tilde{Q}, \tilde{x}_0)) \subseteq F'_*(\pi_1(Q', x'_0)) \subseteq \pi_1(Q, x_0)$$

from which we conclude that $\theta_*(\pi_1(\tilde{Q}, \tilde{x}_0))$ is normal in $\pi_1(Q', x'_0)$. Thus θ is a Galois covering of quivers. Furthermore, note that by Proposition 4.4 $F_*(N(\tilde{B}, \tilde{x}_0)) = N(B, x_0) = F'_*(N(B', x'_0))$ and $N(B, x_0) \subseteq F_*(\pi_1(\tilde{Q}, \tilde{x}_0)) = F'_*\theta_*(\pi_1(\tilde{Q}, \tilde{x}_0))$. We deduce that $N(B', x'_0) \subseteq \theta_*(\pi_1(\tilde{Q}, \tilde{x}_0))$. By Theorem 4.5 again we see that θ is a Galois coalgebra covering, noting that \tilde{B} is the span of all liftings of minimal elements and paths in B' . \square

5. Coverings from gradings

Proposition 5.1. *Let $B \subseteq \mathbb{k}Q$ be a pointed coalgebra and let $\delta : Q_1 \rightarrow G$ be a connected arrow weighting determining a grading of B . Then $F : Q \rtimes G \rightarrow Q$ is a Galois covering of quivers and the restriction $F : \tilde{B} \rightarrow B$ is a coalgebra covering.*

Proof. In view of Proposition 3.1 and by Theorem 4.5 we need to show that $N(B, x_0) \subseteq F_*(\pi_1(Q \rtimes G, x_0 \rtimes 1))$. Let $[w^{-1}q^{-1}pw] \in N(B, x_0)$ be a generator where $\sum \lambda_i p_i \in B$ is a homogeneous minimal element with distinct paths $p_1 = p$ and $p_2 = q$, both in $Q(x, y)$, and walk w from x_0 to x . Then, since p and q have the same weight, $u = w^{-1}q^{-1}pw$ is a closed walk in Q having weight 1_G . It follows that u lifts to a closed walk \tilde{u} in $Q \rtimes G$ starting and ending at $x_0 \rtimes 1$. We have shown that $F_*(\tilde{u}) = [u] \in F_*(\pi_1(Q \rtimes G, x_0 \rtimes 1))$ and thus $N(B, x_0) \subseteq F_*(\pi_1(Q \rtimes G, x_0 \rtimes 1))$. \square

6. Liftings, weightings and isomorphisms

Let $F : \tilde{Q} \rightarrow Q$ be a Galois covering. Let $L, L' : Q_0 \rightarrow \tilde{Q}_0$ be liftings of F . By Theorem 4.2 there are covering isomorphisms with $\mathbb{k}Q \rtimes \mathbb{k}G \cong \mathbb{k}\tilde{Q} \cong \mathbb{k}'\mathbb{k}G$ determined by the liftings ($\rtimes = \rtimes_L, \rtimes' = \rtimes_{L'}$), so we can identify \tilde{Q} with the smash coproduct quiver $Q \rtimes G$ and then write

$$L'(x) = x \rtimes \gamma(x)$$

for all $x \in Q_0$ where $\gamma : Q_0 \rightarrow G$ is a function, depending on L' , that we call a *vertex weighting*. We write δ (resp. δ') for the grading corresponding to L (resp. L').

Given a vertex weighting γ , let $\delta^\gamma : Q_1 \rightarrow G$ associate the γ -twisted grading defined by the weight function

$$\delta^\gamma(a) = \gamma(y)^{-1}\delta(a)\gamma(x)$$

for all arrows $a \in Q_1(x, y)$, $x, y \in Q_0$. Observe that this formula extends to all paths (playing the role of the arrow a), i.e., if $p = a_n a_{n-1} \cdots a_2 a_1$ is a path in Q with vertex sequence $t(a_n) = x_n, \dots, x_1 = s(a_2) = t(a_1), x_0 = s(a_0)$, then we set $g_i = \gamma(x_i)$ and define $\delta^\gamma(p) = \delta(a_n)\delta(a_{n-1}) \cdots \delta(a_2)\delta(a_1)$. We see that

$$\begin{aligned} \delta^\gamma(p) &= g_n^{-1}\delta(a_n)g_{n-1}^{-1}g_{n-1}\delta(a_{n-1}) \cdots g_2^{-1}g_2\delta(a_2)g_1^{-1}g_1\delta(a_1)g_0 \\ &= \gamma(t(a_n))^{-1}\delta(a_n)\delta(a_{n-1}) \cdots \delta(a_2)\delta(a_1)\gamma(s(a_1)) \\ &= \gamma(t(p))^{-1}\delta(p)\gamma(s(p)). \end{aligned}$$

An isomorphism $\theta : F \rightarrow F'$ of Galois coverings of Q (or of the coalgebras coverings $\mathbb{k}\tilde{Q} \xrightarrow{F} \mathbb{k}Q \xleftarrow{F'} \mathbb{k}Q'$) is said to be a G -isomorphism if it commutes with the right action of the automorphism group

$$G = \frac{\pi_1(Q, x)}{F_*(\pi_1(\tilde{Q}, \tilde{x}))} = \frac{\pi_1(Q, x)}{F'_*(\pi_1(Q', x'))}$$

where $F(\tilde{x}) = F'(x') = x$. Let $\mathbb{k}Q \rtimes' \mathbb{k}G$ be a smash coproduct coalgebra using a grading $\delta' : Q_1 \rightarrow G$. For a G -isomorphism of smash coproduct coverings $\theta : \mathbb{k}Q \rtimes \mathbb{k}G \rightarrow \mathbb{k}Q \rtimes' \mathbb{k}G$ we have $\theta(p \rtimes g) = p \rtimes' g_p g$ for some $g_p \in G$, for all paths p and $g \in G$. We say that the weighting δ' implicit in the smash coproduct $\mathbb{k}Q \rtimes' \mathbb{k}G$ is the G -grading associated to the G -isomorphism θ .

Given the associated gradings to each lifting, vertex weighting and G -isomorphism θ we have described, we have the following result.

Proposition 6.1. *Let $F : \tilde{Q} \rightarrow Q$ be a Galois covering of quivers and fix a lifting $L : Q_0 \rightarrow \tilde{Q}_0$. There are bijections between the following sets:*

- (a) *liftings $L' : Q_0 \rightarrow \tilde{Q}_0$,*
- (b) *vertex weightings $\gamma : Q_0 \rightarrow G$,*
- (c) *G -isomorphisms of coverings $\mathbb{k}Q \rtimes \mathbb{k}G \rightarrow \mathbb{k}Q \rtimes' \mathbb{k}G$.*

Moreover, these bijections preserve the associated gradings.

Proof. By Theorem 4.2 we know that $\mathbb{k}\tilde{Q} \cong \mathbb{k}Q \rtimes \mathbb{k}G$ where \rtimes is the smash coproduct with grading induced by L . Let $L' : Q_0 \rightarrow \tilde{Q}_0$ be a lifting. By the isomorphism, we may assume $\mathbb{k}\tilde{Q} = \mathbb{k}Q \rtimes \mathbb{k}G$ with $\tilde{Q} = Q \rtimes G$ and therefore $L'(x) = x \rtimes \gamma(x)$ for all $x \in Q_0$ for some $\gamma(x) \in G$. This produces a vertex lifting γ . Since each such choice of γ provides a unique lifting, we have obtained a bijection between the sets in (a) and (b). Let $\delta' : Q_1 \rightarrow G$ be the arrow weighting induced by L' . Let p be a path in $Q(x, y)$. Then $L'(p) = p \rtimes \gamma(x)$ is a path in \tilde{Q} starting at $x \rtimes \gamma(x)$ and ending at

$$\begin{aligned} & y \rtimes \delta(p)\gamma(x) \\ &= y \rtimes \gamma(y)\gamma(y)^{-1}\delta(p)\gamma(x) \\ &= L'(y)\gamma(y)^{-1}\delta(p)\gamma(x) \\ &= L'(y)^{\delta'(p)}. \end{aligned}$$

This shows that $\delta' = \delta^\gamma$, so that the associated grading is preserved, as claimed.

We move on to demonstrating a grading-preserving isomorphism between (b) and (c). For any vertex weighting $\gamma : Q_0 \rightarrow G$, define a map $\theta_\gamma : \mathbb{k}Q \rtimes \mathbb{k}G \rightarrow \mathbb{k}Q \rtimes' \mathbb{k}G$ by $\theta_\gamma(p \rtimes g) = p \rtimes' \gamma(x)^{-1}g$ for all paths p and $g \in G$. This clearly defines a linear isomorphism with $F'\theta = F$. On the other hand, given a \mathbb{k} -linear isomorphism $\theta : \mathbb{k}Q \rtimes \mathbb{k}G \rightarrow \mathbb{k}Q \rtimes' \mathbb{k}G$ such that for all $p \rtimes g \in \mathbb{k}Q \rtimes \mathbb{k}G$ with paths p in Q and $g \in G$, $\theta(p \rtimes g) = p \rtimes g_p g$ for some $g_p \in G$, we note that

$$\begin{aligned} \Delta(\theta(p \rtimes 1)) &= \Delta(p \rtimes' g_p) \\ &= \sum_{p=rq} (r \rtimes' \delta'(q)g_p) \otimes (q \rtimes' g_p) \end{aligned}$$

and on the other hand

$$\begin{aligned} (\theta \otimes \theta)\Delta(p \rtimes 1) &= (\theta \otimes \theta) \sum_{p=rq} (r \rtimes \delta(q)) \otimes (q \rtimes 1) \\ &= \sum_{p=rq} (r \rtimes' g_r \delta(q)) \otimes (q \rtimes' g_q). \end{aligned}$$

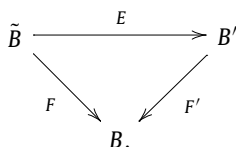
Equating the right tensor factors yields $g_p = g_q$ for all initial segments q of p . In particular, we see in this situation that g_p is determined by the starting vertex, i.e., $g_p = g_{s(p)}$ for all paths p . Let the vertex weighting γ be defined by $\gamma(x) = g_x^{-1}$ for all $x \in Q_0$. Next, equating the left tensor factors at $p = q$ results in the equation $\delta'(p) = g_{t(p)}\delta(p)g_p^{-1} = \gamma(y)^{-1}\delta(p)\gamma(x) = \delta^\gamma(p)$ for all $p \in Q(x, y)$. This shows that θ is a G -coalgebra covering isomorphism if and only if $\delta' = \delta^\gamma$ and $\theta = \theta_\gamma$. It follows that we have a bijective mapping from the set (c) to the set (b) given by $\gamma \mapsto \theta_\gamma$, which preserves the associated grading. \square

Remark 6.2. The set of vertex weightings G^{Q_0} forms group under pointwise multiplication in G . Therefore we can construe the bijections in the theorem as group isomorphisms, and the group of gradings (again pointwise) as a homomorphic image of each of the three isomorphic groups in (a)–(c). The quite arbitrary choice of the lifting L provides an identity element in the group of liftings, corresponding to the neutral vertex weighting $x \mapsto 1_G, x \in Q_0$.

7. Universality

Theorem 7.1. *Let $B \subseteq \mathbb{k}Q$ be a pointed coalgebra. Then:*

- (a) *There exists a Galois coalgebra covering $F: \tilde{B} \rightarrow B$ such that for every Galois coalgebra covering $F': B' \rightarrow B$, there exists a Galois coalgebra covering $E: \tilde{B} \rightarrow B'$ such that the following diagram commutes*



- (b) *If we fix base points $x_0 \in B_0, x'_0 \in B'_0, \tilde{x}_0 \in \tilde{B}_0$ with $F(\tilde{x}_0) = x_0 = F'(x'_0)$, then E can be uniquely chosen so that $E(\tilde{x}_0) = x'_0$.*
- (c) *The Galois covering $F: \tilde{B} \rightarrow B$ is unique up to isomorphism.*

Proof. Fix $x_0 \in Q_0$. The smash coproduct $Q \rtimes \pi_1(Q, x_0)$ is the universal covering of Q where the connected grading $\hat{\delta}: Q_1 \rightarrow \pi_1(Q, x_0)$ arises from a lifting L such that $L(x_0) = x_0 \rtimes 1_{\pi_1(Q, x_0)}$. Let $G = \pi_1(Q, x_0), N = N(B, x_0)$ and $\tilde{Q} = Q \rtimes G/N$, where the grading is induced by composing with the natural map onto G/N , i.e., δ is the composition

$$Q_1 \xrightarrow{\hat{\delta}} G \rightarrow G/N.$$

This is immediately seen to give a connected grading of $\mathbb{k}Q$. Now observe that $F_*(\pi_1(\tilde{Q}, x_0 \rtimes 1)) = N$, as the elements of N are precisely the equivalence classes of closed walks based at x_0 that lift to closed walks at $x_0 \rtimes 1$. By Theorem 4.5, we obtain a coalgebra covering $F: \tilde{B} \rightarrow B$ where $\tilde{B} = B \rtimes \mathbb{k}G$ is the span of liftings of minimal relations of B .

Let $F': B' \rightarrow B$ be the hypothetical coalgebra covering arising as the restriction of a covering $F': Q' \rightarrow Q$ with base vertex $x'_0 \in Q'_0, F'(x'_0) = x_0$. Let $N' = F'_*(N(B', x'_0))$. By Theorem 4.5 we have

$$N' \supseteq N = F_*(N(\pi_1(\tilde{Q}, \tilde{x}_0))).$$

Thus we may form the smash coproduct $Q \rtimes G/N'$ using the induced connected grading via $Q_1 \xrightarrow{\delta} G \rightarrow G/N'$. Here $F'_*(\pi_1(Q \rtimes G/N', x_0 \rtimes 1)) = N' = F'_*(N(B', x'_0))$, so by standard results e.g. [17, Ch. 5, Corollary 6.4], there is a unique isomorphism of coverings $E: Q \rtimes G/N' \xrightarrow{\sim} Q'$ sending $x_0 \rtimes 1$ to x'_0 .¹ We have the commuting diagram of quivers

¹ If we identify Q' with $Q \rtimes G/N'$ and $x'_0 = x \rtimes \bar{g}$ with $\bar{g} \in G/N$, then the isomorphism is concretely given by the right action of \bar{g} .

$$\begin{array}{ccccc}
 Q \rtimes G/N & \longrightarrow & Q \rtimes G/N' & \longrightarrow & Q' \\
 & \searrow & & \swarrow & \\
 & & F & & F' \\
 & & & & \\
 & & & & Q.
 \end{array}$$

The horizontal composed map $Q \rtimes G/N \rightarrow Q'$ provides a morphism of coalgebra coverings by Lemma 4.7. This proves the assertions (a) and (b).

The uniqueness of F follows since it is the unique Galois covering such that $F_*(\pi_1(\tilde{Q}, x_0 \rtimes 1)) = N$. \square

The covering coalgebra $Q \rtimes G/N$ in this result is the *universal Galois covering* of $B \subseteq kQ$.

8. Examples

Example 8.1. Let Q be the quiver consisting of a single loop a and single vertex x . The universal cover \tilde{Q} is a quiver of type \mathbb{A}_∞ with all arrows in the same direction. The automorphism group $\pi_1(Q, x)$ is infinite cyclic, generated by $g = [a]$. A connected grading is given by $\delta(a) = g$. Since there is a single vertex and G is abelian, Theorem 6 says that all other liftings yield the same grading. The path $i \rightarrow \dots \rightarrow i + \ell$, $i \in \mathbb{Z}$, $\ell \in \mathbb{N}$ corresponds to $a^\ell \rtimes g^i \in kQ \rtimes kG$. All other coverings of Q are given by the action of a subgroup $\langle g^n \rangle$ of G and are easily seen to be the cyclic quiver $\bar{0} \rightarrow \bar{1} \rightarrow \dots \rightarrow \overline{n-1} \rightarrow \bar{n} = \bar{0}$ of length $n \in \mathbb{N}$. The only subcoalgebras of kQ are the truncations $B = k\langle x, a, a^2 \dots a^{n-1} \rangle$. There are no minimal elements, so the universal covering \tilde{B} is isomorphic to $B \rtimes kG$ where the path $i \rightarrow \dots \rightarrow i + \ell$ corresponds to $a^\ell \rtimes g^i$, $0 \leq \ell \leq n$, $i \in \mathbb{Z}$. Each finite-dimensional comodule for $k\tilde{Q}$ corresponds to a quiver representation $\mathbb{k} \rightarrow \mathbb{k} \rightarrow \dots \rightarrow \mathbb{k}$, which pushes down to the ℓ -dimensional representation of Q corresponding to the comodule $k\langle x, a, a^2 \dots a^{\ell-1} \rangle$. Since these comodules are precisely the representatives of finite-dimensional indecomposables for kQ , it is clear that the forgetful functor $\mathcal{M}^{k\tilde{Q}} \approx \text{Gr}^{kQ} \rightarrow \mathcal{M}^{kQ}$ is dense. We note here that the indecomposable representations of Q corresponding to indecomposables over the path algebra $k[a, a^{-1}]$ with nonzero (Jordan) eigenvalue are not comodules as they are not locally nilpotent (cf. [3]).

Example 8.2. Let Q be the quiver consisting of two loops a, b and single vertex x . The fundamental group G is a free group on two generators. The quiver \tilde{Q} is the Cayley graph of the free group on a, b and the vertices of \tilde{Q} are indexed by the elements of G , and these group elements correspond to vertex weightings. Distinct vertex weightings give rise to distinct gradings, which are thus infinite in number.

Example 8.3. Let Q be the Kronecker quiver

$$x \rightrightarrows y$$

consisting of the two arrows a, b from the vertex x to the vertex y . The fundamental group is infinite cyclic. Specifying an arrow weighting $\delta: Q_0 \rightarrow G$ by $\delta(a) = 1$ and $\delta(b) = g$, we get the covering quiver $\tilde{Q} = Q \rtimes G$ of type \mathbb{A}_∞ with zig-zag orientation

$$\dots \leftarrow x_0 \rightarrow y_0 \leftarrow x_1 \rightarrow y_1 \leftarrow x_2 \rightarrow \dots$$

where $x_n = x \rtimes g^n$ and $y_n = y \rtimes g^n$. Here again there are infinitely many distinct gradings, with isomorphic smash coproducts. The finite-dimensional indecomposable comodules for $k\tilde{Q}$ are given by the representations of \tilde{Q} . By the theory of special biserial (co)algebras, see [4] (and [9]), these representations are given by the strings $\mathbb{k} - \mathbb{k} - \dots - \mathbb{k} - \mathbb{k}$ of finite length where $-$ denotes \leftarrow or \rightarrow . On the other hand, the finite-dimensional indecomposable representations of Q are well known to be given by string modules and a one-parameter family of band modules [9]. Note that the band modules

correspond to comodules in \mathcal{M}^B , as they are locally nilpotent (cf. [3]), in contrast to Example 8.1. The band comodules correspond to non-gradable comodules for $\mathbb{k}Q$. Thus the forgetful functor $\mathcal{M}^{\tilde{B}} \approx \text{Gr}^B \rightarrow \mathcal{M}^B$ is not dense.

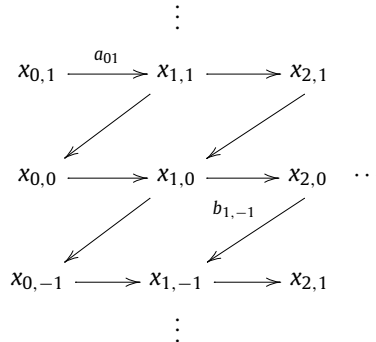
Example 8.4. Consider the coordinate Hopf algebra $\mathbb{k}_\zeta[SL(2)]$ at a root of unity ζ of odd order ℓ over a field \mathbb{k} of characteristic zero. The basic coalgebra decomposes into block coalgebras B_r , $r = 0, 1, 2, \dots, \ell - 2$. The nontrivial blocks are indexed by integers $r \leq \ell - 2$, and they are all isomorphic [4]. Each nontrivial block B is isomorphic to the subcoalgebra of path coalgebra of the quiver Q :

$$x_0 \begin{matrix} \xrightarrow{b_0} \\ \xrightarrow{a_0} \end{matrix} x_1 \begin{matrix} \xrightarrow{b_1} \\ \xrightarrow{a_1} \end{matrix} x_2 \xrightarrow{b_2} \dots$$

spanned by the by group-likes x_i corresponding to vertices, arrows $a_i, b_i, i \geq 0$, together with coradical degree two elements

$$d_0 := b_0 a_0, \\ d_{i+1} := a_i b_i + b_{i+1} a_{i+1}, \quad i \geq 0.$$

Therefore $N(B, x_0)$ is generated by homotopy classes closed walks of the form $w^{-1} b_i^{-1} a_i^{-1} b_{i+1} a_{i+1} w$ for appropriate walks w (from x_0 to x_i), and it follows that the fundamental group of $B \subset \mathbb{k}Q$ is the infinite cyclic group $G = \langle g \rangle$, generated by $g = [b_0 a_0]$ and the universal cover is a quiver of type $\mathbb{Z}\mathbb{A}_\infty$. For example letting $\delta(b) = g^{-1}$ and $\delta(a) = 1_G$ we obtain a connected grading of B and covering quiver $Q \rtimes G$



with vertices $x_{in} = x_i \rtimes g^n$; $i \in \mathbb{N}, n \in \mathbb{Z}$. The arrows are $a_{in} = a_i \rtimes g^n$ starting at x_{in} and ending at $x_{i+1,n}$, and $b_{in} = b_i \rtimes g^n$ starting at $x_{i+1,n+1} \rtimes g^{n+1}$ and ending at x_{in} . The coalgebra \tilde{B} is spanned by the vertices, arrows, paths $d_{0n} := b_{0,n-1} a_{0,n}$ and the minimal elements $d_{i+1,n} := a_{in} b_{in} + b_{i+1,n} a_{i+1,n+1}$, $i \geq 0$.

The finite-dimensional representations of B are determined in [4]. Each arrow a_i generates the length two right comodule known as a Weyl comodule, with composition series $\begin{matrix} \mathbb{k}x_i \\ \mathbb{k}x_{i+1} \end{matrix}$. The arrow b_i generates the dual Weyl comodule. Any grading of B is determined by an arrow weighting, so each Weyl and dual Weyl comodule is obviously thus graded. The coalgebra B is an example of a special biserial coalgebra and all finite-dimensional comodules are string comodules. Each of these comodules corresponds to a walk (i.e. a string) of one of the following forms

$$a_t b_{t-1}^- \cdots a_{s-1} b_{s+1}^- a_s, \\ b_t^- a_{t-1} \cdots b_{s-1} a_{s+1} b_s^-,$$

$$a_t b_{t-1}^- \cdots a_{s+1} b_s^-,$$

$$b_t^- a_{t-1} \cdots b_{s+1}^- a_s$$

where the subscripts form an interval $[s, t]$ of nonnegative integers strictly increasing from right to left in the walk. Each of these can be constructed as an iteration of pullbacks and pushouts of Weyl comodules and dual Weyl comodules (with isomorphic socles or tops), both of which are gradable. Since the simple comodules have trivial weighting, it follows that every finite-dimensional B -comodule is gradable. Thus the forgetful functor $\text{Gr}^B \rightarrow \mathcal{M}^B$ is dense.

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