Alternative Loop Rings with Solvable Unit Loops

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Let $L$ be an RA loop, that is, a loop whose loop ring in any characteristic is an alternative, but not associative, ring. We find necessary and sufficient conditions for the (Moufang) unit loop of $RL$ to be solvable when $R$ is the ring of rational integers or an arbitrary field.

1. INTRODUCTION

It is of interest to determine conditions under which the unit loop of a loop ring has certain properties, such as nilpotency, finite conjugacy, or solvability. Work on solvability was started in 1971 by Bateman [Bat71], who considered the group algebra of a finite group over a field. It has been continued for various other groups and rings of coefficients by Motose et al. [MT71, MN72], Sehgal [Sch75], Passman [Pas77], Gonçalves [Gon86],

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Bovdi [Bov92], and others. The survey article by Polcino Milices [Mil82] also contains a section on the solvability of unit groups.

Let $R$ denote a commutative and associative ring with identity and let $L$ be an RA loop, that is, a loop whose loop ring in arbitrary characteristic is an alternative, but not associative, ring. Then $L$ is a Moufang loop, as is the set $\mathcal{U}(RL)$ of units in the alternative ring $RL$. In previous papers, the authors have investigated the possibility that $\mathcal{U}(RL)$ is FC each, $\mathcal{U}(RL)$ has just finitely many conjugates of the form $x^{-1}\mu x$ [GM96b] or nilpotent [GM97]. In this paper, we investigate solvability. Unlike the associative case, where certain situations cause difficulty, we obtain complete answers when $R$ is the ring $\mathbb{Z}$ of rational integers or an arbitrary field.

An alternative ring is a ring which satisfies the identities $(yx)x = yx^2$ and $(xy)y = xy^2$ and a Moufang loop is a loop which satisfies the identity

$$ (xy \cdot z)y = x(y \cdot yz). $$

Any associative ring is alternative and any group is a Moufang loop so, when we want to refer to an alternative ring or a Moufang loop which is not associative, we say so explicitly. Alternative rings and Moufang loops are nearly associative in the sense that they are diassociative: the subring (or subloop) generated by any pair of elements is associative. In fact, if three elements in an alternative ring (or Moufang loop) associate, then they generate an associative ring (or a group).

The set of units (invertible elements) of an alternative ring with 1 is a Moufang loop. Of special interest in this paper is the alternative matrix algebra $\mathfrak{M}(F)$ over a field $F$ of Max Zorn and its loop of units. The elements of $\mathfrak{M}(F)$ are $2 \times 2$ matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in F$, $x, y \in F^3$. Such matrices are added in the obvious way, but multiplied according to the following variation of the usual rule,

$$ \begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ y_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + x_1 \cdot y_2 & a_1x_2 + b_2x_1 - y_1 \times y_2 \\ a_2y_1 + b_1y_2 + x_1 \times x_2 & b_1b_2 + y_1 \cdot x_2 \end{bmatrix}, $$

where $\cdot$ and $\times$ denote the dot and cross products, respectively, in $F^3$.

Let $A$ denote a quaternion algebra over a field $F$ of characteristic different from 2. Let $\alpha$ be a nonzero element of $F$ and $u$ an element not in $A$. Let $\mathfrak{A} = A + Au$ with obvious addition, but multiplication defined by

$$ (a + bu)(c + du) = (ad + \alpha d^* b) + (da + bc^*)u $$

for $a, b, c, d \in A$, $q \mapsto q^*$ denoting any involution in $A$. The algebra $\mathfrak{A}$ is called a Cayley–Dickson algebra. Such an algebra is alternative, but not
associative; moreover, it is known that every such algebra is either a division algebra or isomorphic to Zorn’s vector matrix algebra over $F$ [GJM96, Corollary I.4.17; ZSSS82, Theorem 2.4.7].

Zorn’s algebra comes with a determinant function, $[a \, b] \mapsto ab - x \cdot y$, and the units of $\mathcal{Z}(F)$ are precisely those matrices whose determinant is nonzero. These units form a loop denoted $\text{GL}(2, F)$ and called the general linear loop. Though the name and notation were first used in JLM94 and continued in GJM96, this loop was first explored by Paige [Pai56] who showed that its centre is the set of scalar matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and that, modulo this centre, $\text{GL}(2, F)$ is simple (and not associative). Consequently, and of particular relevance in this paper, $\text{GL}(2, F)$ is never solvable.

An RA loop is a loop whose loop ring over a coefficient ring of any characteristic is an alternative, not associative, ring. Such a loop is determined by a nonabelian group $G$, an involution on $G$ and a central element $g_0$; hence the notation $\text{M}(G, *, g_0)$ has become standard. We refer the reader to GJM96 for more information about RA loops. The following lemma plays a key role in this paper.

**Lemma 1.1.** Let $L = \text{M}(G, *, g_0)$ be an RA loop, $R$ a commutative and associative ring with $1$, and $A$ a quaternion algebra contained in $RG$. If $A^* \subseteq A$ and $v \in (RG)u$ is an element of order $2$, then $B = A + Av$ is a Cayley–Dickson subalgebra of $RL$. If $A = M_2(F)$, then $B \cong \mathcal{Z}(F)$.

**Proof.** We have only to show that multiplication in $B$ is given by (1.2). Write $v = \alpha u$ for some $\alpha \in RG$. For $a, b, c, d \in A \subseteq RG$, we have

\[
[a(\alpha u)] = a[(\alpha u)u] = (\alpha a)u = (da)(\alpha u) = (da)v,
\]

\[
(bv)c = [b(\alpha u)]c = [(ab)u]c = (abc^*)(\alpha u) = (bc^*)v,
\]

and

\[
(\alpha (\alpha u)] = g_0(\alpha d)^*(\alpha b)
\]

\[
= g_0 d^* b = g_0^* b = d^* b,
\]

using in the last two steps that $\alpha^* \alpha$ is central and $1 = v^2 = (\alpha u)(\alpha u) = g_0^2 \alpha^2$. Thus $B$ is indeed a Cayley–Dickson algebra. Finally, if $A = M_2(F)$, then $B$ is not a division algebra, so it is $\mathcal{Z}(F)$. 

If $a, b,$ and $c$ are elements of a loop, we denote the commutator of $a$ and $b$ and the associator of $a, b,$ and $c$ by $(a, b)$ and $(a, b, c)$, respectively. Thus, by definition,

\[
ab = (ba)(a, b) \quad \text{and} \quad (ab)c = [a(bc)](a, b, c).
\]
The subloop of a loop $L$ generated by all commutators and associators is denoted $L'$. In an RA loop, there exists an element $s \neq 1$, necessarily central and of order 2, with the property that every commutator and every associator is either 1 or $s$. Thus, if $L = M(G, \ast, g_n)$ is an RA loop, $L' = G' = \{1, s\}$. In an RA loop $L$, two elements commute if and only if they associate with every third element. This is a consequence of the LC property described in [GJM96, Sect. IV.2; CG86, Sect. 2]: $g, h \in L$ commute if and only if $g$ or $h$ or $gh$ is central. Unlike the general situation with Moufang loops, the test for normality of a subloop of an RA loop is precisely what it is for groups: a subloop $N$ of an RA loop $L$ is normal if and only if $NT(x) \subseteq N$ for all $x \in L$, where $T : L \to L$ is the inner map defined by $T(x) = R(x)L(x)^{-1}$ [GJM96, Corollary IV.1.11; CG90, Corollaries 2.4 and 2.11].

Solvability for Moufang loops is defined just as it is for groups [Gla68]. A Moufang loop $L$ is solvable if there is a finite subnormal series of subloops extending from 1 to $L$ with each quotient loop a commutative group. Subloops and homomorphic images of solvable loops are solvable. Moreover, if $M$ is a normal subloop of a Moufang loop $L$, then $L$ is solvable if and only if $M$ and $L/M$ are solvable. If $L$ is RA, the LC property implies that $L^2 \subseteq Z(L)$ for any $L \in L$. Thus $L$ is an extension of its centre by an elementary abelian 2-group (in fact, an extension of its centre by $C_2 \times C_2 \times C_2$), and hence solvable. (Here, $C_2$ denotes the cyclic group of order 2.)

It turns out that many conditions for the solvability of a unit loop $\#(RL)$ involve the set of torsion elements of $L$. In contrast to the general situation, the torsion elements of an RA loop always form a subloop [GJM96, Lemma VIII.4.1; GM95, Lemma 2.1].

2. INTEGRAL LOOP RINGS

In this section, we find necessary and sufficient conditions for the unit loop of an integral alternative loop ring to be solvable. It is typical of situations where strong hypotheses are applied to the unit loop that the answer involves hamiltonian loops. For example, the elements of $\pm L$ are the only units of $\mathbb{Z}L$ if and only if $L$ is a hamiltonian Moufang 2-loop [GJM96, Theorem VIII.3.2; GP86].

A Moufang loop is hamiltonian if it is not commutative and every subloop is normal. A group is hamiltonian if and only if it is the direct product $Q_8 \times E \times A$, where $Q_8$ is the quaternion group of order 8, $E$ is elementary abelian of exponent 2, and $A$ is a group all of whose elements are of odd order [Hal59, Theorem 12.5.4]. A Moufang loop which is not associative is hamiltonian if and only if it is the direct product $C \times E \times A$
where $C = M_{16}(Q_8)$ is the Cayley loop [GJM96, Sect. 4.1] and $E$ and $A$ are as before [GJM96, Theorem II.4.8; Nor52].

The following theorem is the analogue for alternative loop rings which are not associative of a theorem of Sehgal for group rings [Seh78, Theorem VI.4.8]. The proof requires the concept of “support.” The support of an element $\alpha = \sum_{x} \alpha_x x$, $x \in R$, in a loop ring $RL$ is the set

$$\text{supp}(\alpha) = \{x \in R | \alpha_x \neq 0\}.$$ 

**Theorem 2.1.** Let $L$ be an RA loop with torsion subloop $T$. Then $\mathcal{U}(ZL)$ is solvable if and only if

$$T \text{ is either an abelian group or a hamiltonian Moufang 2-loop and every subloop of } T \text{ is normal in } L.$$  

**(2.1)**

**Proof.** The proof that solvability implies condition (2.1) is virtually identical to the group case, because RA loops are diassociative. We refer the reader to Seh78, Theorem VI.4.8 for details.

It is for the converse that new arguments are required in the nonassociative context. Suppose then that $T$ is either an abelian group or a hamiltonian Moufang 2-loop with every subloop of $T$ normal in $L$. As shown in Lemma 2.3 of [GM95] (see also [GJM96, Proposition XII.1.3]), we have $\mathcal{U}(ZL) = [\mathcal{U}(ZT)]L$. If $T$ is a hamiltonian 2-loop, then $\mathcal{U}(ZT) = T$ by [GJM96, Theorem VIII.3.1; GP86, Theorem 7] and so $\mathcal{U}(ZL) = T$ is solvable. Suppose $T$ is an abelian group.

We claim first that $\mathcal{U}(ZT)$ is a normal subloop of $\mathcal{U}(ZL)$. For this, let $\mu \in \mathcal{U}(ZT)$ and $v \in \mathcal{U}(ZL)$ and write $v = v_{1} x, v_{1} \in \mathcal{U}(ZT)$. Let $B$ be the subloop of $L$ generated by the supports of $\mu$ and $v_{1}$ and let $G$ be the subloop of $L$ generated by $B$ and $x$. Since $B$ is commutative, $G$ is a group [GJM96, Corollary IV.2.4; GM96d, Lemma 3.1] and so $v^{-1} \mu v = x^{-1} x \mu x \in \mathcal{U}(ZT)$ since $x^{-1} x \in T$ for any $x \in T$. It remains to show that $\mu R(v_{1}, v_{2}) \in \mathcal{U}(ZT)$ for any $v_{1}, v_{2} \in \mathcal{U}(ZL)$. For this, we may assume that $\mu = t \in T$. Recall that $tR(v_{1}, v_{2}) = (tv_{1} : v_{2}) (v_{1} v_{2})^{-1}$ so, by diassociativity of $\mathcal{U}(ZL)$, it is sufficient to show that $tv_{1} v_{2} = t' v_{1} v_{2}$ for some $t' \in T$. Remembering that $v_{1}, v_{2} \in [\mathcal{U}(ZT)]L$, to establish this last fact, it is enough to show that for any $t_{1}, t_{2} \in T$ and $x_{1}, x_{2} \in L$,

$$(t \cdot t_{1} x_{1})(t_{2} x_{2}) = t' [(t_{1} x_{1})(t_{2} x_{2})]$$

**(2.2)**

for some element $t' \in T$ which is independent of $t_{1}$ and $t_{2}$.

In a Moufang loop, an associator $(x, y, z)$ is 1 if and only if $x, y,$ and $z$ generate a group; hence, in an RA loop, where there is just one nonidentity associator, an associator $(x, y, z)$ is independent of the order of $x, y, z$. 

Furthermore, an RA loop satisfies the identity

\[(xy, z, w) = (x, z, w)(y, z, w)\]

[GJM96, Theorem IV.1.14; CG86, Theorem 3]. Therefore, the associator \((t, t_1', t_2', t_3')\) is the product of four associators, those involving \(t\) and either \(t_1\) or \(t_2\) being 1 because \(T\) is commutative. (Remember that two elements of an RA loop commute if and only if they associate with every third element.) It follows that \((t, t_1', t_2', t_3') = (t, t_1', t_2')\) is 1 or \(s\). This gives (2.2) with \(t' = t\) or \(t' = st\); in either case, \(t' \in T\). As asserted, \(\mathcal{U}(T)\) is indeed normal in \(\mathcal{U}(L)\).

Finally, let \(A = [\mathcal{U}(T)]\mathcal{Z}\) be the product of \(\mathcal{U}(T)\) and the centre \(\mathcal{Z}\) of \(L\). Then \(A\) is a commutative group and a normal subloop of \(\mathcal{U}(L)\). Moreover, for any \(\mu \in \mathcal{U}(L)\), writing \(\mu = \mu_1\ell, \mu_1 \in \mathcal{U}(T)\), we notice that

\[\mu^2 = \mu_1 \ell \mu_1 \ell = \mu_1 (\ell \mu_1 \ell^{-1}) \ell^2 \in [\mathcal{U}(T)]\mathcal{Z}\]

since \(\text{supp}(\mu_1) \subseteq T\) and \(\ell^2 \in \mathcal{Z}(L)\). Thus \(L/A\) is a diassociative loop of exponent 2, hence a commutative group, and the normal series \(\mathcal{U}(L) \supseteq A \supseteq 1\) witnesses solvability of \(\mathcal{U}(L)\).

**Corollary 2.2.** If \(L\) is a finite RA loop, then \(\mathcal{U}(L)\) is solvable if and only if \(L\) is a Hamiltonian Moufang 2-loop.

In Chapter XII of [GJM96] (specifically, Corollary XII.2.14) (see also [GM95, Theorem 3.3]), it is shown that if \(L\) is an RA loop with torsion subloop \(T\), then the integral unit loop \(\mathcal{U}(L)\) is nilpotent if and only if \(T\) is an abelian group or a Hamiltonian Moufang 2-loop and

\[T\text{ is abelian or }L\text{ is abelian and }x \in L\text{ is an element that does not centralize }T, \text{ then }x^{-1}tx = t^{-1}\text{ for all }t \in T.\]

In the case that \(T\) is Hamiltonian, conditions (2.3) and (2.1) (which are labelled (10) and (10') respectively in [GJM96]) are equivalent because it is easily seen that \(x^{-1}tx = t^{\pm 1}\) for any \(x \in L\) and \(t \in T\). Similarly, (2.3) and (2.1) are equivalent for finite \(L\) since in that case \(T = L\) is not abelian. Thus solvability and nilpotence of \(\mathcal{U}(L)\) are equivalent for finite RA loops and for RA loops whose torsion subloops are Hamiltonian 2-loops. They are different in general as the following example shows.

**Example 2.3.** Let \(A\) be the direct product of a cyclic group \(\langle a \rangle\) of order 16 and an infinite cyclic group \(\langle b \rangle\) and let \(G\) be the group generated by \(A\) and two additional elements \(x\) and \(y\), subject to the relations

\[ax = xa, ay = ya \text{ for all }a \in A, \quad x^2 = b, \ y^2 = a^2, \ (x, y) = a^8.\]
It is easy to see that $\mathcal{Z}(G) = A$, that $G/\mathcal{Z}(G) \cong C_2 \times C_2$, where $s = a^8$ is the only nonidentity commutator in $G$, and that the set $T$ of torsion elements in $G$ is precisely the subgroup $\langle y, a \rangle$ generated by $y$ and $a$. In particular, $T$ is an abelian group. Also, since $x^{-1}yx = y^9 \neq y^{\pm 1}$, $T$ satisfies (2.1) but not (2.3).

Now let $L = G \cup Gu$, where $u^2 = b$, and extend the multiplication from $G$ to $L$ as in [GJM96, Sect. II.5.2]. Then $L = M(G, \ast, b)$ is an RA loop (see [GJM96, Sec. II.5.2, Corollary III.3.4 and Corollary III.3.6]) and the torsion subloop of $L$ is the abelian group $T$. Our results show that $\mathcal{Z}(FL)$ is solvable, but not nilpotent.

3. LOOP ALGEBRAS OVER FIELDS

Since all torsion indecomposable RA loops are 2-loops [GJM96, Corollary V.1.3; CG86, Theorem 6], it is not surprising that characteristic 2 generally represents a very special case in theorems concerning RA loops.

**Theorem 3.1.** Let $L$ be an RA loop and $F$ a field. If $\text{char} \; F = 2$, then $[\mathcal{Z}(FL)]'$ is an abelian group, so $\mathcal{Z}(FL)$ is solvable.

**Proof.** As always, let $L' = \{1, s\}$. For any $g, h \in L$, either $gh = hg$ or $gh = shg$, so $\mu\nu - \nu\mu \in FL(1 - s)$ for any $\mu, \nu \in FL$. Suppose $\mu$ and $\nu$ are units of $FL$. Noting that $1 - s$ is central in $FL$ we have $(\nu\mu)^{-1}\mu\nu - 1 = (\nu\mu)^{-1}(\mu\nu - \nu\mu) = \alpha(1 - s)$ for some $\alpha \in FL$, so $\mu^{-1}\nu^{-1}\mu\nu = \langle\mu, \nu\rangle = 1 + \alpha(1 - s)$.

For any $g, h, k \in L$, either $(gh)k = g(hk)$ or $(gh)k = sg(hk)$. Thus, for any $\mu, \nu, \omega \in FL$, we have $(\mu\nu)\omega - \mu(\nu\omega) \in FL(1 - s)$. Consequently, if $\mu, \nu, \omega$ are units, then $(\mu\nu\omega)^{-1}(\mu\nu\omega) - 1 = \alpha(1 - s)$ for some $\alpha \in FL$ and so $(\mu\nu\omega)^{-1}(\mu\nu\omega) = (\mu, \nu, \omega) = 1 + \alpha(1 - s)$.

Now suppose that each of $\mu, \nu$, and $\omega$ is either a commutator or an associator. Writing $\mu = 1 + \alpha(1 - s)$, $\nu = 1 + \beta(1 - s)$ and $\omega = 1 + \gamma(1 - s)$, $\alpha, \beta, \gamma \in FL$, direct calculation shows that $\mu\nu - \nu\mu = 0$ and $(\mu\nu)\omega - \mu(\nu\omega) = 0$ because $(1 - s)^2 = 0$ in characteristic 2. Thus commutators and associators in $\mathcal{Z}(FL)$ commute and associate. The result follows. 

**Corollary 3.2.** Let $L$ be a torsion RA loop and $F$ a field. Then $\mathcal{Z}(FL)$ is solvable if and only if $\text{char} \; F = 2$.

**Proof.** Sufficiency follows from Theorem 3.1. For necessity, assume that $\mathcal{Z}(FL)$ is solvable for some torsion RA loop $L$. Any two noncommuting elements of $L$ generate a finite nonabelian group $G$ such that $\mathcal{Z}(FG)$ is solvable. If $\text{char} \; F = 0$, we would have $\mathcal{Z}(QG)$ solvable, which is a contradiction [Seh78, Corollary VI.4.14]. Thus $\text{char} \; F = p > 0$. Since
$\mathcal{Z}(FG)$ is solvable for any nonabelian group $G$ contained in $L$, results of Bateman and Passman [Bat71, Pas77] show that $G'$ is a $p$-group, unless $p = 2$ or $p = 3$. Since $G'$ is, in fact, a 2-group, indeed $p = 2$ or $p = 3$. We complete the proof by showing that the assumption $p = 3$ leads to a contradiction.

Suppose then that $p = 3$. Since $L$ is a torsion loop, we can write $L = L_2 \times L_{3^r}$ as the direct product of a 2-loop $L_2$ and an abelian group $L_{3^r}$ [GJM96, Proposition V.1.1; GM96d, Lemma 3.2]. Since $L$ is locally finite [GJM96, Lemma VIII.4.1; GM95, Lemma 2.1], replacing $L$ by the subloop generated by any three elements which do not associate, we may assume that $L_2$ is finite. Letting $F_3$ denote the field of three elements, the loop algebra $F_3 L_2$ is the direct sum of fields and Cayley–Dickson algebras [GJM96, Corollary VI.4.8; GM96c, Theorem 2.8]. Since a finite Cayley–Dickson algebra cannot be a division algebra, it is necessarily a Zorn’s algebra [GJM96, Proposition I.4.15 and Corollary I.4.17] the unit loop of which, $\mathbb{GL}(F_3, 2)$, is not solvable as noted earlier. The result follows.

Since a hamiltonian Moufang loop is a torsion loop, the following corollary is immediate.

**Corollary 3.3.** Let $F$ be a field and let $L$ be a hamiltonian Moufang loop which is not associative. Then $\mathcal{Z}(FL)$ is solvable if and only if char $F = 2$.

This result is especially interesting in the light of the analogous result of Passman for finite hamiltonian groups [Pas77].

**Proposition 3.4.** Let $F$ be a field and let $G$ be a hamiltonian group. If $\mathcal{Z}(FG)$ is solvable, then $F = F_3$ is the field of 3 elements and $G$ is a 2-group. The converse holds if $G$ is finite.

We now wish to study solvability of the unit loop of a loop algebra $FL$ over an arbitrary field without the assumption that $L$ is torsion. The corresponding problem for group rings remains unsettled in general, although A. Bovdi has made progress for nilpotent nontorsion groups [Bov92]. In positive characteristic $p$, whether or not $L$ contains $p$-elements turns out to be critical, so our main results are described by two theorems, Theorems 3.8 and 3.10 below.

We begin with several lemmas, the first two giving conditions under which an alternative loop algebra $FL$ contains $\mathcal{Z}(F)$, Zorn’s vector matrix algebra over $F$.

**Lemma 3.5.** Let $F$ be a field of characteristic different from 2. Let $L$ be an RA loop with torsion subloop $T$. Suppose there exists $t \in T$ such that the subloop $\langle t \rangle$ generated by $t$ is not normal in $L$. Then $FL$ contains Zorn’s vector matrix algebra $\mathcal{Z}(F)$; hence $\mathcal{Z}(FL)$ is not solvable.
Proof. Since the unit loop of Zorn’s vector matrix algebra is not solvable, we have only to show that FL contains 3(F).

By [GJM96, Proposition V.1.1] (see also [CG86, Theorem 6]), we may write \( t = t_1 a \) where \( t_1 \) has order a power of 2 and \( a \) is central in \( L \). Thus, for suitable odd \( n \), we have \( t^n = t_1^n \). Write \( 1 = in + jo(t_1) \) for integers \( i \) and \( j \). Then \( t_1 = t_1^{in+jo(t_1)} = t^{in} \), so \( \langle t_1 \rangle \subseteq \langle t \rangle \). Since \( t \) is not central, neither is \( t_1 \). A noncentral subloop \( H \) of \( L \) is normal if and only if \( s \in H \) [GJM96, Corollary IV.1.9]. Since \( s \not\in \langle t \rangle \), it follows that \( s \not\in \langle t_1 \rangle \) and so \( \langle t_1 \rangle \) is not normal in \( L \). Thus we may assume that the given element \( t \) is a 2-element.

Let \( \tilde{t} = 1 + t + t^2 + \cdots + t^{o(t) - 1} \). Note that \( t^i \tilde{t} = \tilde{t} \) for any \( i \). Since \( o(t) \) is invertible in \( F \), the element \( e = \frac{1}{o(t)} \tilde{t} \) is an idempotent and, since \( \langle t \rangle \) is not normal, there exists \( x \in L \) such that \( ex \neq xe \). For any \( H \subseteq FL \), let \( H^x = x^{-1}Hx \) and write \( \alpha^x \) for \( \{ \alpha \}^x \). Observe that the support of \( ee^x \) is contained in \( \langle t \rangle \langle t \rangle^x \subseteq \langle t \rangle = \text{supp}(e) \). Thus \( ee^x \neq e \) and \( f = e(1 - e^x) \neq 0 \). Clearly \( xt \neq tx \). Recalling that \( f^2 \in Z(L) \) for all \( t \in L \) and that \( L = \{ 1, s \} \), we see that \( t^i x = xt^i \) if \( i \) is even and \( t^i x = x t^{i - 1} \) if \( i \) is odd. Thus

\[
e^x = 1 + t + st + t^2 + st^3 + \cdots + st^{o(t) - 1} = e^x, \quad (3.1)
\]

\( \alpha \mapsto \alpha^x \) denoting the standard involution on FL. In particular, \( e \) and \( e^x \) commute; so \( f \) is also an idempotent. Moreover, \( f^x = f^x = e^x(1 - e) \).

As in [HP80], we define \( e_{11} = f, e_{22} = f^x, e_{12} = fx, \) and \( e_{21} = x^{-1}f \). It is straightforward to verify that these four elements comprise the usual basis for the ring \( A = M_2(F) \) of \( 2 \times 2 \) matrices over \( F \). In the calculations for this (and in those which follow), all elements associate since all have support in the loop generated by \( t \) and \( x \) (which is a group, by diassociativity).

We claim that \( A \) is invariant under the involution on FL. Since \( f = e(1 - e^x) = e(1 - e^x) \) and since \( e \) and \( e^x \) commute, we have \( e_{11} = f^x = e_{22} \). Thus \( e_{22}^x = e_{11} \). Next, we compute \( e_{12}^x = x^sf^x = xfx = xse_{12} = s(e_{11} + e_{22})e_{12} \). Now \( e_{11} + e_{22} = f + f^x = e(1 - e^x) + e^x(1 - e) = (e - e^x)^2 \). By (3.1), \( e - e^x = (1 - s)\alpha \) for some \( \alpha \in FL \), so \( (e - e^x)^2 = 2(1 - s)\alpha^2 \) and \( s(e - e^x)^2 = -2(1 - s)\alpha^2 = -(e - e^x)^2 = -(e_{11} + e_{22}) \). This gives

\[
e_{12}^x = s(e_{11} + e_{22})e_{12} = s(e - e^x)^2e_{12} = -(e_{11} + e_{22})e_{12} = -e_{12}.
\]

Similarly, \( e_{21}^x = -e_{21} \) and, indeed, \( A^x \subseteq A \) as claimed.

Now since \( t \) and \( x \) do not commute, the subloop of \( L \) generated by \( t, x \), and \( Z(L) \) is a group \( G \) and \( L = G \cup Gu \) for some \( u \not\in G \). We seek an element \( v \in (FG)u \) of order 2 and to do this, find a \( 2 \times 2 \) matrix ring and take \( v \) to be the element \([0 \ 1] \). Since \( L \) has the LC property and \( Z(L) \cap
Gu = ∅, the elements t and xu do not commute. Thus, as shown previously for e^t, we have e^{it} = e^a. In particular, e^{it} = e^a ≠ e. We construct a copy of \(M_2(F)\) as before and note that, in this copy, the matrices \(g_{11}\) and \(g_{22}\) are, respectively, the matrices \(e_{11}\) and \(e_{22}\) determined earlier, since \(g_{11} = e(1 - e^{it}) = e(1 - e^a) = e_{11}\) and \(g_{22} = g_{11}^{it} = e_{11}^{it} = e_{22}^a = e_{22}^a\), but that \(g_{12} = g_{11}(xu) = e_{11}(xu)\) and \(g_{21} = (xu)^{-1}g_{11} = (xu)^{-1}e_{11}\). are in \((FG)u\). In particular, the element \(g_{12} + g_{21}\), corresponding to \(\{1, 2\}\) is in \((FG)u\) as desired.

Finally, Lemma 1.1 shows that \(A + Av\) is Zorn's vector matrix algebra over \(F\).

**Lemma 3.6.** Let \(L\) be an RA loop which contains the quaternion group Q8 and let \(F\) be a field of characteristic 3. Then \(\mathcal{U}(FL)\) contains \(\mathcal{J}(F)\) and hence is not solvable.

**Proof.** Once again, the final statement employs the fact that the loop of units of \(\mathcal{J}(F)\) is not solvable.

Let \(Q_8 = \langle a, b | a^4 = 1, b^2 = a^2, ba = a^3b \rangle\). Let \(G = \langle a, b, \mathcal{U}(L) \rangle\) be the group generated by \(a, b, \) and the centre of \(L\). Let \(u\) be any element which does not associate with \(a\) and \(b\). Then \(L = G \cup Gu\). It is easily checked that the element \(e_{11} = \frac{1}{2}(2 + a + b)(1 - a^2)\) is an idempotent in \(FG\). Let \(e_{22} = e_{11}^a = \frac{1}{2}(2 - a - b)(1 - a^2)\), \(e_{12} = e_{11}(ab)\), and \(e_{21} = (ab)^{-1}e_{11}\). The four elements \(e_{ij}, i, j = 1, 2\), lie in a group algebra and it is straightforward that they comprise a basis for a subalgebra \(A\) of \(FG\) isomorphic to \(M_2(F)\). Clearly, \(e_{11}^a, e_{22}^a \in A\); also \(e_{12}^a = (ab)^{e_{11}^a} = a^2(ab)e_{22} = a^2(ab)e_{22}(ab)^{-1}(ab) = a^2e_{11}(ab) = -e_{12}\) and, similarly, \(e_{21}^a = -e_{21}\). Thus \(A^* \subseteq A\) and, just as in the proof of Lemma 3.5, the element \(v = e_{11}u + u^{-1}e_{11} \in (FG)u\) has order 2, so \(A + Av \cong \mathcal{J}(F)\), by Lemma 1.1.

**Lemma 3.7.** Let \(F\) be a field of characteristic \(p \geq 0\), \(p \neq 2\), and let \(L\) be an RA loop with torsion subloop \(T\). Suppose \(\mathcal{U}(FL)\) is solvable. Then \(T\) is an abelian group and every subloop of \(T\) is normal in \(L\).

**Proof.** Lemma 3.5 implies that every subloop of \(T\) is normal in \(L\). In particular, \(T\) is either commutative (in which case it is an abelian group) or a hamiltonian Moufang loop. In the latter case, \(T\) is a group because of Corollary 3.3. Proposition 3.4 then gives \(F = F_3\), and Lemma 3.6 shows this situation cannot occur. Thus \(T\) is an abelian group.

**Theorem 3.8.** Let \(F\) be a field of characteristic \(p \geq 0\) and suppose \(L\) is an RA loop which contains no elements of order \(p\) (in the case \(p > 0\)). Then \(\mathcal{U}(FL)\) is solvable if and only if (i) \(p = 2\) or (ii) \(T\) is an abelian group and every idempotent of \(FT\) is central in \(FL\).
Proof. Suppose \( \mathcal{A}(FL) \) is solvable and \( p \neq 2 \). Lemma 3.7 says that \( T \) is an abelian group and that every subgroup of \( T \) is normal in \( L \). Let \( e \in FT \) be an idempotent. Since \( T \) is commutative, \( e \) commutes with every element of \( T \). Let \( x \in L \setminus T \). Since \( T \) is locally finite [GJM96, Lemma VIII.4.1; GM96a, Lemma 1.4], \( A = \langle \text{supp}(e) \rangle \) is a finite abelian group and it contains no elements of order \( p \). Thus \( FA = F_1 \oplus F_2 \oplus \cdots \oplus F_m \) is the direct sum of fields. Let \( e_1, e_2, \ldots, e_m \) be the corresponding primitive idempotents; that is, \( F_i = (FA)e_i \). It is sufficient to show that each \( e_i \) commutes with \( x \). Let \( G \) be any nonabelian group containing \( \text{supp}(e) \) and \( \mathcal{A}(L) \). Thus \( L = G \cup Gu \) for some \( u \). Suppose \( x \) does not commute with \( e_1 \). Then \( e_1^x \) is a primitive idempotent of \( FA \), so \( e_1^x \in \{e_2, e_3, \ldots, e_m\} \).

Without loss of generality, \( e_1^x = e_2 \). This implies \( e_1e_2 = 0 \). Notice also that \( e_1 = e_1^2 = e_1^x \) by centrality of \( x^x \). Let \( e_{11} = e_1, \ e_{22} = e_2, \ e_{12} = e_{11}x, \) and \( e_{31} = x^{-1}e_{11} \). An argument which should by now be familiar leads to a contradiction and completes the proof of necessity.

Conversely, if \( p = 2 \), then \( \mathcal{A}(FL) \) is solvable by Theorem 3.1, so assume \( p \neq 2 \). Thus \( T \) is an abelian group and every idempotent of \( FT \) is central in \( FL \). To show that \( \mathcal{A} = \mathcal{A}(FL) \) is solvable, it is sufficient to show that the commutator-associator subloop \( \mathcal{A}' \) is an abelian group. For this, we may assume that \( L \) is finitely generated (think in terms of the support of elements), hence that \( T \) is finite. We now make substantial use of the proof of Lemma XII.1.1 in [GJM96] and its corollary. (See also Lemma 2.3 in [GM95].)

Since \( T \) is finite, abelian, and of order relatively prime to \( p \), we can write \( FT = F_1 \oplus F_2 \oplus \cdots \oplus F_n \) as the direct sum of fields \( F_i = (FT)e_i \), determined by primitive idempotents \( e_1, \ldots, e_n \). If \( A \) is the direct sum of any subset of \( F_1, \ldots, F_n \), then centrality of the \( e_i \) in \( FL \) and normality of \( T \) in \( L \) imply that \( A \) is normal in \( FL \) in the sense that

\[
A\alpha = \alpha A, \quad (A\alpha)\beta = A(\alpha\beta), \quad (\alpha A)\beta = \alpha (A\beta), \quad \text{and} \quad \alpha (\beta A) = (\alpha \beta) A
\]

for any \( \alpha, \beta \in FL \). We simplify the discussion which follows with a local definition. Suppose \( \mu = \sum d_i \) and \( \nu = \sum d'_i \) are sums of nonzero elements from certain of the fields \( F_i \). Let us agree to write \( \mu \sim \nu \) if the fields involved with \( \mu \) are precisely those involved with \( \nu \).

Writing \( L \) as the disjoint union of cosets of \( T \), any unit (in fact, any element) \( \mu \) in \( FL \) can be written \( \mu = \sum_{q \in Q} \mu_q q \), the \( \mu_q \) nonzero sums of elements from the \( F_i \). Moreover, \( \mu^{-1} \) can be written \( \mu^{-1} = \sum_{q \in Q} q^{-1}\mu_q' \), summing over the same set \( Q \), and \( \mu_q' \sim \mu_q \) for each \( q \in Q \).

If \( q_1 \neq q_2 \), then \( \mu_{q_1} \) and \( \mu_{q_2} \) are the sums of elements from disjoint collections of fields, so

\[
\mu_1 \sim \mu_{q_1} \quad \text{and} \quad \mu_2 \sim \mu_{q_2} \quad \text{implies} \quad \mu_1 \mu_2 = 0.
\]
Now let $\nu = \sum r \in R \nu r$ be another unit in $FL$, with $\nu^{-1} = \sum_{r \in R} r^{-1} \nu r'$ and reflect upon the form of the commutator $\mu^{-1} \nu^{-1} \mu \nu$. This is a sum of terms of the form

$$\left( q_{i_1}^{-1} \mu_{q_{i_1}} \right) \left( r_{i_1}^{-1} \nu_{r_{i_1}} \right) \left( \mu_{q_2} q_2 \right) (\nu_{r_2}),$$

where, to improve readability, we have suppressed parentheses which would show the order in which products are taken. Of the several ways in which the terms of the commutator $(\mu, \nu)$ can be computed, let us agree to compute from left to right; thus $abcd$ means $(ab \cdot c)d$.

We first claim that any product of the form (3.3) will be 0 unless $q_1 = q_2$ and $r_1 = r_2$. The idea behind this is that normality of the sum of any collection of $F_i$ allows us to move terms and change the order of products until we find ourselves with a subproduct of the form $\mu_1 \mu_2, \mu_1 \sim \mu_{q_1}$, and $\mu_2 \sim \mu_{q_1}$, at which point we appeal to (3.2). For example,

$$\left( q_{i_1}^{-1} \mu_{q_{i_1}} \right) \left( r_{i_1}^{-1} \nu_{r_{i_1}} \right) = q_{i_1}^{-1} \left( \mu_{q_{i_1}} \cdot r_{i_1}^{-1} \nu_{r_{i_1}} \right) = q_{i_1}^{-1} \left( r_{i_1}^{-1} \nu_{r_{i_1}} \cdot \mu_{q_{i_1}} \right) = \left( q_{i_1}^{-1} \cdot r_{i_1}^{-1} \nu_{r_{i_1}} \right) \mu_{q_{i_1}}^{''},$$

with $\mu_{q_{i_1}}^{''} \sim \mu_{q_{i_1}} \sim \mu_{q_{i_1}}^{''} \sim \mu_{q_{i_1}}$, so that

$$\left[ \left( q_{i_1}^{-1} \mu_{q_{i_1}} \right) \left( r_{i_1}^{-1} \nu_{r_{i_1}} \right) \right] \left( \mu_{q_2} q_2 \right) = \left[ \left( q_{i_1}^{-1} \cdot r_{i_1}^{-1} \nu_{r_{i_1}} \right) \mu_{q_{i_1}}^{''} \cdot \mu_{q_{i_1}}^{''} \right] q_2$$

$$= \left[ \left( q_{i_1}^{-1} \cdot r_{i_1}^{-1} \nu_{r_{i_1}} \right) \mu_{q_{i_1}}^{''} \mu_{q_{i_1}}^{''} \right] q_2$$

with $\mu_{q_2} \sim \mu_{q_2} \sim \mu_{q_2}$ and so $\mu_{q_{i_1}}^{''} \mu_{q_{i_1}}^{''} = 0$, unless $q_1 = q_2$, in which case,

$$\left[ \left( q_{i_1}^{-1} \cdot r_{i_1}^{-1} \nu_{r_{i_1}} \right) \mu_{q_{i_1}}^{''} \mu_{q_{i_1}}^{''} \right] q_2 = \left[ \left( q_{i_1}^{-1} \cdot r_{i_1}^{-1} \nu_{r_{i_1}} \right) \mu_{q_{i_1}}^{''} \mu_{q_{i_1}}^{''} \right] q_1 = \alpha(q_{i_1}^{-1} r_{i_1}^{-1} q_1)$$

for some $\alpha \in FT$. It follows that the commutator

$$(\mu, \nu) = \sum_{q \in Q, r \in R} \left( q_{i_1}^{-1} \mu_{q_{i_1}} \right) \left( r_{i_1}^{-1} \nu_{r_{i_1}} \right) (\mu_{q} q) (\nu_{r})$$

is a sum of products of elements of $FT$ with commutators $(q, r) \in L$. Each such commutator is 1 or $s$, however, and so the commutator of $(\mu, \nu)$ is an element of $FT$.

By completely similar means, we can show that if $\mu, \nu, \omega \in \mathcal{U}(FL)$, then the associator $(\mu, \nu, \omega) \in FT$, which is a commutative ring. It follows that $\mathcal{U}'$, the commutator-associator subloop of $\mathcal{U}(FL)$, is an abelian group and the proof is complete.

**Remark 3.9.** Before continuing, we deem it appropriate to remark that conditions on $F$ and $L$ equivalent to the idempotent condition in Theorem 3.8 are known [GJM96, Theorems XIII.1.6 and XIII.1.10; GM96a, Theorems 2.3 and 3.3].
Our last theorem makes use of some properties of augmentation ideals, one reference for which is [GJM96, Sect. VI.1]. Let \( R \) be a commutative and associative ring with 1. Let \( N \) be a normal subloop of an RA loop (or a group) \( L \) and let \( \epsilon_{\epsilon} : RL \to R[L/N] \) denote the linear extension to \( RL \) of the natural homomorphism \( L \to L/N \). This map is a surjective ring homomorphism whose kernel is the ideal \( \Delta(L, N) = \sum_{n \in N} RL(n - 1) \). In the special case \( N = L \), we write \( \epsilon = \epsilon_L \) and refer to this as the augmentation map on \( RL \). Note that \( \epsilon(\sum_{\epsilon \in L} \alpha_{\epsilon}/\epsilon) = \sum \alpha_{\epsilon} \). The ideal \( \Delta(L) = \Delta(L, L) \) is called the augmentation ideal of \( L \). The identity \( \epsilon_{\epsilon}(\epsilon_{\epsilon} - 1) = (\epsilon_{\epsilon} - 1) - (1 - 1) \) shows that

\[
\Delta(L) = \left\{ \sum_{\epsilon \in L} \alpha_{\epsilon}(\epsilon - 1), \alpha_{\epsilon} \in R \right\}.
\]

It is useful also to note that \( \Delta(L, N) = (RL)\Delta(N) \) for any normal subloop \( N \) of \( L \).

**Theorem 3.10.** Let \( F \) be a field of characteristic \( p > 0 \). Let \( L \) be an RA loop which is not torsion and which contains elements of order \( p \). Then \( \mathcal{A}(FL) \) is solvable if and only if (i) \( p = 2 \) or (ii) the set \( P \) of \( p \)-elements in \( L \) is a finite central group and \( \mathcal{A}(F[L/P]) \) is solvable.

**Proof.** Let \( L = \mathcal{M}(G, \ast, g_0) \) be the given RA loop. Assume \( \mathcal{A}(FL) \) is solvable and \( p \neq 2 \). Since elements of odd order are central in an RA loop (and hence contained in \( G \)) and since \( p \neq 2 \), the \( p \)-elements of \( L \) lie in \( \mathcal{Z}(G) \) and form a group. Since \( \mathcal{A}(FG) \) is solvable, a result of Bovdi says that \( P \) is finite [Bov92]. We prove that \( \mathcal{A}(F[L/P]) \) is solvable.

Since \( P \) is finite and char \( F = p \), \( \Delta(P) \) is nilpotent by a result of Jennings [Jen41]. Since \( P \) is central in \( L \) and \( \Delta(L, P) = (FL)\Delta(P) \), we have \( \Delta(L, P) \subseteq FL[\Delta(P)]^n \) for all \( n \geq 1 \), so \( \Delta(L, P) \) is nilpotent.

Let \( \mu, \nu \in FL \) and suppose \( \epsilon_p(\mu) \in F[L/P] \) is a unit with inverse \( \epsilon_p(\nu) \). Then \( \epsilon_p(\mu)\epsilon_p(\nu) = \epsilon_p(\nu)\epsilon_p(\mu) = 1 + P \). In particular, \( \mu \nu - 1 = \delta \in \Delta(L, P) \). Since \( \delta \) is nilpotent, \( \mu \nu = 1 + \delta \) is a unit in \( FL \). Similarly \( \nu \mu \) is a unit in \( FL \). Since \( FL \) is alternative, this implies that \( \mu \) and \( \nu \) are units in \( FL \) [ZSSS82, Lemma 10.3.9]. It follows that the restriction of \( \epsilon_p \) to \( \mathcal{A}(FL) \) is a surjective loop homomorphism \( \mathcal{A}(FL) \to \mathcal{A}(F[L/P]) \). Thus, as a homomorphic image of the solvable loop \( \mathcal{A}(FL) \), the loop \( \mathcal{A}(F[L/P]) \) is also solvable.

Conversely, if \( p = 2 \), then \( \mathcal{A}(FL) \) is solvable by Theorem 3.1. So assume that \( p \neq 2 \), that the set \( P \) of \( p \)-elements in \( L \) is a finite central group, and that \( \mathcal{A}(F[L/P]) \) is solvable. Let \( \epsilon_0 \) denote the restriction of \( \epsilon_p \) to \( \mathcal{A}(FL) \). As shown in the previous paragraph, \( \mathcal{A}(F[L/P]) \cong \mathcal{A}(FL)/\ker \epsilon_0 \) and \( \ker \epsilon_0 = 1 + \Delta(L, P) \) so, to show that \( \mathcal{A}(FL) \) is solvable, it suffices to prove that \( V = 1 + \Delta(L, P) \) is solvable. For \( \mu, \nu \in V \), centrality of \( \Delta(P) \)
makes it easy to verify that $\mu \nu - \nu \mu \in FL[\Delta(P)]^2$ and hence that $(\mu, \nu) = \mu^{-1} \nu^{-1} \mu \nu = \mu^{-1} \nu^{-1} (\mu \nu - \nu \mu) + 1 \in 1 + FL[\Delta(P)]^2$. Similarly, one can check that the associator $(\mu, \nu, \omega)$ of three elements $\mu, \nu, \omega \in V$ is also in $1 + \Sigma FL[\Delta(P)]^2$. Let

$$V_0 = V \quad \text{and, for } k > 0, \quad V_k = 1 + [\Delta(P)]^k.$$

Since $\Delta(P)$ is nilpotent, $V_n = \{1\}$ for some positive integer $n$. Each $V_i$ is normal in $V$ (because $\Delta(P)$ is central) and, if $\mu$, $\nu$, and $\omega$ are in $V_k$, the commutator $(\mu, \nu)$ and the associator $(\mu, \nu, \omega)$ are both in $V_{k+1}$; hence, each quotient loop $V_k/V_{k+1}$ is a commutative group. It follows that

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = \{1\}$$

is a normal series establishing solvability of $V$. This completes the proof. \[\square\]

**Remark 3.11.** Since $L/P$ has no $p$-elements, conditions equivalent to condition (ii) of Theorem 3.10 are given by Theorem 3.8.

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