Two Generator Subgroups of $SL(2, \mathbb{C})$ and the Hypergeometric, Riemann, and Lamé Equations

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For the purposes of constructing explicit solutions to second-order linear homogeneous differential equations on the Riemann sphere the Kovacic algorithm partitions the subgroups of $SL(2, \mathbb{C})$ into four classes and initially determines which class contains the differential Galois group of the input equation. We prove in the case of the hypergeometric and Riemann equations that the relevant class can be determined directly from the coefficients by elementary calculation. We also treat the (non-algebraic form of the) Lamé equation, to which the Kovacic algorithm is not directly applicable. In that instance we combine the Kovacic results with ours to produce an algorithm for determining the class of the associated group.

From the group-theoretic viewpoint the problem solved herein is the following: given arbitrary $S, T \in SL(2, \mathbb{C})$, determine which class contains the group $\langle S, T \rangle$ generated by $S$ and $T$.

1. Introduction

A subgroup $G \subset SL(2, \mathbb{C})$ has precisely one of the following four properties: (a) the projective representation fixes a line; (b) the projective representation permutes two lines, fixing neither; (c) the projective group is isomorphic to the alternating group $A_4$, the symmetric group $S_4$, or the alternating group $A_5$; or (d) the Zariski closure of $G$ is $SL(2, \mathbb{C})$. At the group-theoretic level we are concerned with the following question: given elements $S, T \in SL(2, \mathbb{C})$, can one easily determine which property holds for the subgroup $G = \langle S, T \rangle$ generated by $S$ and $T$? The answer is yes: we prove that the nature of $G$ can be determined by inspection from the three traces $t_S, t_T$ and $t_{ST}$ and the quantity $t_S^2 + t_T^2 + t_{ST}^2 - t_ST t_{TS}$ (which is adapted from the Fricke–Klein formula (9.2c) for the trace of the commutator $(S, T) = STS^{-1}T^{-1}$). For example, with our methods (i.e. using Theorem 6.1) it is trivial to verify that the group generated by

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \sqrt{5} - 4 & \frac{\sqrt{5}}{2} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \frac{\sqrt{5}}{2} & -\frac{1}{4} (4 + \sqrt{5})(2 - i\sqrt{7}) \\ 2 + i\sqrt{7} & -\frac{1}{2} \end{pmatrix}$$

is projectively isomorphic to $A_5$, i.e. is projectively icosahedral.

This partitioning of the subgroups of $SL(2, \mathbb{C})$ is fundamental to the Kovacic (1986) algorithm for producing explicit solutions to second-order linear homogeneous differential equations.

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\[ y'' + c_1(x)y' + c_2(x)y = 0 \]  \hspace{1cm} (1.1)

on the Riemann sphere \( \mathbb{P}^1 \) having rational function coefficients. The relationship can be summarized as follows.

The initial step in the algorithm is to replace the “standard form” (1.1) with the “normal form”

\[ y'' + c(x)y = 0, \]  \hspace{1cm} (1.2)

where

\[ c(x) := c_2(x) - (c_1(x)/2)^2 - c_1'(x)/2. \]  \hspace{1cm} (1.3)

This (classical) replacement involves no loss of generality: any solution \( y = y(x) \) of (1.1) has the form \( y = y_1(x)y_2(x) \), where \( y_1 = y_1(x) \) satisfies (1.2) and \( y_2 = y_2(x) \) satisfies

\[ y' + \frac{1}{2}c_1(x)y = 0, \]  \hspace{1cm} (1.4)

and this last equation is trivial to integrate; solutions of (1.1) and (1.2) are in this sense interchangeable.

But the normal form has two major advantages: the differential Galois group \( G \) is a subgroup of \( SL(2, \mathbb{C}) \) (this is standard, e.g. see Magnus, 1976, p. 702, or, for an algebraic proof, Kaplansky, 1976, p. 77); and the nature of this group (as described in the opening paragraph) determines the structure of solutions (see Kovacic, 1986, pp. 5 and 7). The algorithm proceeds by testing (in order) which of the four cases holds, and then constructs solutions (except in the last case, which excludes “elementary” solutions) by utilizing the resulting knowledge of their structure.

The algorithm has been implemented on MAPLE and other symbolic manipulation packages, and continues to be refined and extended, e.g. see Duval and Loday-Richaud (1992), Singer and Ulmer (1993a,b), Ulmer and Weil (1996) and references therein.

The indicated partitioning of the subgroups of \( SL(2, \mathbb{C}) \) has other applications, including applications within ordinary differential equations not directly related to the construction of explicit solutions. For example, the structure of the differential Galois group of (1.2) was recently shown to reflect the degree of integrability of related Hamiltonian systems. In fact the Kovacic algorithm has been adapted to that context (e.g. see Churchill and Rod, 1991; Rod and Sleeman, 1995; Baider et al., 1996, Molares-Ruiz and Ramis, 1997) precisely so as to determine that structure, but in practice the detail can be quite tedious, even in the hypergeometric case. (The root of the problem is that one must investigate each case “in order”.) Very recent work suggests that these methods might prove useful for establishing the existence of chaos, as in Morales-Ruiz and Peris (1996).

In this paper we use an elementary group-theoretic result to prove that in important special cases all the information about the differential Galois group of (1.2) provided by the Kovacic algorithm can be obtained with far less work (generally with a few trivial hand calculations). In particular, we show this is the case for the hypergeometric and Riemann equations, and illustrate our methods by verifying Schwarz’s classical list of those hypergeometric equations admitting only algebraic solutions. We also treat the (non-algebraic form of the) Lamé equation, to which the Kovacic algorithm is not directly applicable. In that instance we are able to combine the Kovacic results with ours to produce an algorithm for determining the class of the associated group.
It is understood that practically every example we present can be handled by alternate (and well-established) means. Indeed, those alternate techniques often provide more information, but at considerable cost in terms of the work involved.

2. Initial Statements of Results

The most general Fuchsian equation on the Riemann sphere \( P^1 \) has the form

\[
y'' + \left( \sum_{j=1}^{m} \frac{A_j}{x-a_j} \right) y' + \left( \sum_{j=1}^{m} \frac{B_j}{(x-a_j)^2} + \sum_{j=1}^{m} \frac{C_j}{x-a_j} \right) y = 0,
\]

where the complex numbers \( a_1, \ldots, a_m \) are distinct and the only restriction on the complex constants \( A_j, B_j, C_j \) is \( \sum_{j=1}^{m} B_j = 0 \) (e.g. see Poole, 1960, Chapter V, or Birkhoff and Rota, 1989, Chapter 9). The normal form (2.2) is

\[
y'' + \left( \sum_{j=1}^{m} \frac{\hat{B}_j}{(x-a_j)^2} + \sum_{j=1}^{m} \frac{\hat{C}_j}{x-a_j} \right) y = 0,
\]

where \( \hat{B}_j = \frac{1}{4} \left( 1 + 4B_j - (1 - A_j)^2 \right) \) and \( \hat{C}_j = C_j - \frac{1}{2} A_j \left[ \sum_{i \neq j} \frac{A_i}{a_i-a_j} \right] \), which we note is again Fuchsian. When it is understood that (2.2) is derived from (2.1) we refer to the latter as the standard form of (2.2).

We are primarily concerned with the calculation of the monodromy and differential Galois groups of (2.2) in the case \( m = 2 \), but at this point we can be more general. We recall the realationship between the groups in the Fuchsian case.

**Proposition 2.3.** The differential Galois groups of the Fuchsian equations (2.1) and (2.2) are the Zariski closures of the respective monodromy groups.

**Proof.** See Proposition III of Tretkoff and Tretkoff (1979). \( \square \)

Denote \( \infty \in P^1 \) by \( a_{m+1} \) and set

\[
A_{m+1} := A_{\infty} := 2 - \sum_{j=1}^{m} A_j, \quad B_{m+1} := B_{\infty} := \sum_{j=1}^{m} (B_j + C_j a_j).
\]

Then the trace \( t_j \) of the monodromy generator (“circuit matrix”) of (2.1) at the singularity \( a_j \) is

\[
t_j := 2e^{-\pi i A_j} \cos \pi \sqrt{(A_j - 1)^2 - 4B_j}, \quad j = 1, \ldots, m + 1,
\]

and that of the local monodromy of (2.2) at \( a_j \) is

\[
t_j := -2 \cos \pi \sqrt{(A_j - 1)^2 - 4B_j}, \quad j = 1, \ldots, m + 1.
\]

Observe that \( \sqrt{(A_j - 1)^2 - 4B_j} \) is the “exponent difference”, i.e. the difference of the “characteristic exponents” \( \rho_j^\pm := \frac{1}{2} \left( (1 - A_j) \pm \sqrt{(1 - A_j)^2 - 4B_j} \right) \) of (2.1) at \( a_j \).
Remark 2.7. (a) We compute monodromy groups using analytic continuation along inverses of loops based at some distinguished point \( x_0 \) of \( X = \mathbb{P}^1 \setminus \{ a_1, \ldots, a_{n+1} \} \). As opposed to the case with direct continuation, this convention results in a representation of \( \pi_1(X, x_0) \) in \( GL(2, \mathbb{C}) \) for (2.1) and (2.2). But it does necessitate minor adjustments to trace formulae when the circuit matrices are computed by direct continuation, as in Poole (1960) and Birkhoff and Rota (1989).

(b) The monodromy representation of (2.2) has its image in \( SL(2, \mathbb{C}) \).

(c) For \( j = 1, \ldots, n + 1 \) let \( [\gamma_j] \) be a homotopy generator corresponding to \( a_j \), i.e. let \( \gamma_j \) be a loop in \( X \) based at \( x_0 \) encircling only \( a_j \) in a positive sense. Then by standard homotopy theory (and proper labelling) we have \( \prod_{j=1}^m [\gamma_j] = [\gamma_{m+1}]^{-1} \). In particular, when \( m = 2 \) the equality \( [\gamma_3]^{-1} = [\gamma_1][\gamma_2] \) holds, and for a representation in \( SL(2, \mathbb{C}) \) it follows that \( t_3 = t_{[\gamma_1][\gamma_2]} \) for the corresponding traces. This is a key point for all that follows. Indeed, when \( m > 2 \) the calculation of \( t_{[\gamma_1][\gamma_2]} \) can be intractable. For recent work on the general problem see Katz (1996).

To formulate and motivate our main result we need several group-theoretic preliminaries.

We view \( GL(2, \mathbb{C}) \) and \( SL(2, \mathbb{C}) \) as groups of linear operators on \( \mathbb{C}^2 \) (as opposed to groups of matrices). We write the identity operator as \( I \), and we let \( \hat{H} \) and \( H \) denote the normal subgroups \( \{ \mathbb{C}\{0\} \cdot I \subset GL(2, \mathbb{C}) \) and \( \{ I, -I \} = \hat{H} \cap SL(2, \mathbb{C}) \subset SL(2, \mathbb{C}) \) respectively. For any subgroup \( G \subset GL(2, \mathbb{C}) \) we let \( H_G := G \cap \hat{H} \), and we refer to \( G/H_G \) as the projective group of \( G \). By a matrix representation of a subgroup \( G \subset SL(2, \mathbb{C}) \) we mean a group of unimodular (i.e. determinant 1) \( 2 \times 2 \) matrices isomorphic to \( G \).

The commutator \( GST^{-1}T^{-1} \) of elements \( S, T \in GL(2, \mathbb{C}) \) is denoted \( [S, T] \).

\( D \) denotes the group of diagonal unimodular matrices, and \( P \) (for “permutation”) denotes the collection of unimodular matrices having the form \( \begin{pmatrix} 0 & \eta \\ -\eta^{-1} & 0 \end{pmatrix} \), where \( \eta \in \mathbb{C}\{0\} \). The quaternion group is the eight element subgroup of \( D \cup P \) generated by the matrices \( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

The symmetric and alternating groups on \( n \) letters are denoted \( S_n \) and \( A_n \), respectively.

A subgroup \( G \subset GL(2, \mathbb{C}) \) is: diagonalizable if it admits a matrix representation contained in \( D \); reducible if it admits a matrix representation contained within the group \( \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}\{0\} \right\} \); \( DP \) (or imprimitive) if it admits a matrix representation contained in \( D \cup P \) (e.g. the quaternion group); quaternionic if \( G \) is isomorphic to the quaternion group; projectively dihedral if \( G/H_G \) is isomorphic to the 2n element dihedral group \( D_{2n} \) for some \( n > 2 \) (some authors do not require the strict inequality); projectively tetrahedral if \( G/H_G \) is isomorphic to \( A_4 \); projectively icosahedral if \( G/H_G \) is isomorphic to \( S_5 \).

Note that a diagonalizable subgroup of \( SL(2, \mathbb{C}) \) is both reducible and \( DP \).

The following partition of the subgroups of \( SL(2, \mathbb{C}) \) is crucial to the Kovacic algorithm for computing solutions of (2.1) and (2.2). The significance for second-order linear differential equations on the Riemann sphere, e.g. the existence or non-existence of Liouvillian solutions, is explained on pp. 5 and 7 of Kovacic (1986).

**Proposition 2.8.** A subgroup \( G \subset SL(2, \mathbb{C}) \) is either: (a) reducible, which includes the diagonalizable and Abelian cases; or (b) \( DP \) but not reducible, which includes the
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Quaternionic and projectively dihedral cases; or (c) projectively tetrahedral, projectively octahedral, or projectively icosahedral, in which case none of the previous possibilities occurs; or (d) the Zariski closure of $G$ is $SL(2, \mathbb{C})$, in which case none of the previous possibilities occurs.

**Proof.** When $G$ is diagonalizable, reducible, DP, and/or finite the Zariski closure will have the same property, and we may therefore assume $G$ is algebraic. In that case the result can be found on pp. 7 and 27 of Kovacic (1986).

An element $S \in SL(2, \mathbb{C})$ is resonant if the eigenvalues $\{\lambda, \lambda^{-1}\}$ are roots of unity; otherwise $S$ is non-resonant. Note that either property can be determined directly from the trace $t_S = \lambda + \lambda^{-1}$ of $S$: the characteristic polynomial is $z^2 - t_S z + 1$ and the eigenvalues are therefore $\frac{1}{2}(t_S \pm \sqrt{t_S^2 - 4})$. The trace $t_S$ is (non-)resonant when $S$ is (non-)resonant.

We can now state our main result.

**Theorem 2.9.** Suppose the Fuchsian equation (2.1) has exactly three singular points, say $a_1, a_2$ and $a_3 = a_\infty = \infty$, and that $G \subset SL(2, \mathbb{C})$ is the differential Galois or monodromy group of the normal form (2.2). For $j = 1, 2, 3$ set

$$t_j := -2 \cos \pi \sqrt{(A_j - 1)^2 - 4B_j},$$

where $A_3 := A_\infty$ and $B_3 := B_\infty$ are as in (2.4), i.e. let $t_j$ denote the trace of the local monodromy generator at $a_j$. Then the following statements hold:

(a) $G$ is reducible iff either of the following two equivalent conditions holds:

(i) $t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 = 4$;

(ii) at least one of the four possible determinations of the expression

$$((A_1 - 1)^2 - 4B_1)^{1/2} + ((A_2 - 1)^2 - 4B_2)^{1/2} + ((A_\infty - 1)^2 - 4B_\infty)^{1/2}$$

is an odd integer.

(b) Suppose $G$ is not reducible. Then $G$ is DP iff either of the following two equivalent conditions holds:

(i) At least two of $t_1, t_2$ and $t_3$ vanish;

(ii) At least two of $2((A_1 - 1)^2 - 4B_1)^{1/2}, 2((A_2 - 1)^2 - 4B_2)^{1/2}$ and $2((A_\infty - 1)^2 - 4B_\infty)^{1/2}$ are odd integers.

Moreover, in this irreducible DP case the group $G$ is finite iff:

(i) all three $t_j$ vanish in (i), or, equivalently, all three of the numbers in (ii) are odd integers, in which case $G$ is quaternionic; or

(ii) the non-vanishing trace is resonant, in which case $G$ is projectively dihedral.

(c) $G$ is finite but not reducible and not DP iff $G$ is projectively tetrahedral, projectively octahedral or projectively icosahedral. Moreover:

(i) $G$ is projectively tetrahedral iff $t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 = 2$ and $t_1, t_2, t_3 \in \{0, \pm 1\}$. 
(ii) \( G \) is projectively octahedral iff \( t_1^2 + t_2^2 + t_3^2 - t_1 t_2 t_3 = 3 \) and \( t_1, t_2, t_3 \in \{0, \pm 1, \pm \sqrt{2}\} \).

(iii) \( G \) is projectively icosahedral iff \( t_1^2 + t_2^2 + t_3^2 - t_1 t_2 t_3 \in \{2 - \mu_2, 3, 2 + \mu_1\} = \{1 + \mu_2^2, 3, 1 + \mu_2^2\} \) and \( t_1, t_2, t_3 \in \{0, \pm \mu_2, \pm 1, \pm \mu_1\} \), where \( \mu_1 := \frac{1}{2}(1 + \sqrt{5}) \) and \( \mu_2 := \mu_1^{-1} = -\frac{1}{2}(1 - \sqrt{5}) \).

(d) When none of (a)–(c) hold the Zariski closure of \( G \) is \( \text{SL}(2, \mathbb{C}) \) when \( G \) is the monodromy group, whereas \( G = \text{SL}(2, \mathbb{C}) \) when \( G \) is the differential Galois group.

Assertion (a) is well-known, e.g. see the proof of Theorem 2.24 of Baider and Churchill (1990). For the equivalence of the two conditions in (b) use (2.6). What remains reduces, by means of Remark 2.7(c) and Proposition 2.3, to an analogous result on two-generator subgroups of \( \text{SL}(2, \mathbb{C}) \) which is stated in Section 6.

We note that when \( G \) is not reducible a matrix representation of \( G \) can be determined from the three quantities \( t_1, t_2 \) and \( t_\infty \) (see Lemma 2, p. 703, of Magnus, 1976).

Remark 2.10. In applying Theorem 2.9 the following observation concerning resonant elements can be useful: when \( \cos \pi r \neq 0 \) and \( 0 \leq \ell \leq n \) we have \( -2 \cos \pi r = 2 \cos(\pi \ell/n) \) \( \iff \) \( r = (\ell/n + 1) + 2k \) for some integer \( k \), and as a result we see that

\[-2 \cos \pi r = 2 \cos(\pi \ell/n) \iff \text{ at least one of } \ r \pm \frac{\lambda}{2} \text{ is an odd integer,}\]

where “odd integer” includes “negative odd integer”. In particular: \( t_j = 2 \) iff the exponent difference \( \sqrt{(A_j - 1)^2 - 4B_j} \) is an odd integer; \( t_j = 1 \) iff one of the two numbers \( \sqrt{(A_j - 1)^2 - 4B_j} \pm \frac{1}{2} \) is an odd integer; \( t_j = 0 \) iff \( 2 \sqrt{(A_j - 1)^2 - 4B_j} \) is an odd integer or, equivalently, if \( \sqrt{(A_j - 1)^2 - 4B_j} - \frac{1}{2} \) is an integer; \( t_j = \mu_1 := \frac{1}{2}(1 + \sqrt{5}) = 2 \cos(\pi/5) \) iff one of the two numbers \( \sqrt{(A_j - 1)^2 - 4B_j} \pm \frac{1}{2} \) is an odd integer; etc.

Example 2.11. The Legendre equation is

\[ y'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}y = 0, \]

where \( \lambda \) is a parameter which we assume is real. It can also be written

\[ y'' + \left\{ \frac{1}{x-1} + \frac{1}{x+1} \right\}y' + \left\{ \frac{-\lambda/2}{x-1} + \frac{\lambda/2}{x+1} \right\}y = 0, \]

and is therefore Fuchsian; the normal form is

\[ y'' + \frac{1}{4} \left\{ \frac{1}{(x-1)^2} + \frac{1}{(x+1)^2} - \frac{2\lambda + 1}{x-1} + \frac{2\lambda + 1}{x+1} \right\}y = 0. \]

The exponents and exponent differences (of the standard form) at \( a_1 := 1 \) and \( a_2 := -1 \) are both zero, the exponents at \( \infty \) are \( \frac{1}{2}(1 \pm \sqrt{1+4\lambda}) \), and the corresponding exponent difference is thus \( \sqrt{1+4\lambda} \). In particular,

\[ t_1 = t_2 = -2, \ t_3 = -2 \cos \pi \sqrt{1+4\lambda}, \]

\[ t_1^2 + t_2^2 + t_3^2 - t_1 t_2 t_3 = 4(\cos(\pi \sqrt{1+4\lambda}) + 1)^2 + 4. \]
and Theorem 2.9 gives the following information about the differential Galois or monodromy group $G$ of the normal form of the equation:

- $G$ is reducible iff $\cos(\pi \sqrt{1 + 4\lambda}) = -1$, which is easily seen to hold iff $\lambda = k(k + 1)$, where $k$ is an integer;
- otherwise the Zariski closure of $G$ is $SL(2, \mathbb{C})$ in the monodromy case, whereas $G = SL(2, \mathbb{C})$ in the differential Galois case.

We remark that the values $\lambda = k(k + 1)$ described in the reducible case are precisely the parameter values for which Legendre’s equation (in standard form) admits polynomial solutions (see, e.g. pp. 110 and 114 of Poole, 1960).

A subgroup $G \subset SL(2, \mathbb{Z})$ has the Ziglin property, or is Ziglin, if the standard action of $G$ on $\mathbb{C}^2$ (i.e. by evaluation) preserves some non constant rational function. The monodromy and differential Galois groups of certain linearized equations associated with integrable Hamiltonian systems have this property, and the determination of such groups is therefore important for applications (e.g. see Morales-Ruiz and Simó, 1994, 1996; Rod and Sleeman, 1995; Baider et al., 1996).

**Theorem 2.12.** A subgroup $G \subset SL(2, \mathbb{C})$ is Ziglin iff the Zariski closure is Ziglin. Moreover, the following assertions hold for any such $G$:

(a) If $G$ is diagonalizable then $G$ is Ziglin (e.g. the action of $G$ by evaluation preserves the product of the coordinate functions of any diagonalizing basis). Alternatively, if $G$ is reducible and finitely generated but not diagonalizable then $G$ is Ziglin iff all generators are resonant.

(b) If $G$ is DP then $G$ is Ziglin.

(c) If $G$ is projectively tetrahedral, octahedral or icosahedral then $G$ is Ziglin. Indeed, any finite subgroup of $SL(2, \mathbb{C})$ is Ziglin.

(d) If the Zariski closure of $G$ is $SL(2, \mathbb{C})$ then $G$ is not Ziglin.

Finally, when $G$ contains a nonresonant element it is Ziglin iff it is a DP-group.

**Proof.** For the initial assertion see Proposition 2.9 of Baider et al. (1996), and for (a)–(d) see Corollary 3.4 of Churchill and Rod (1991). For the final assertion see Theorem 3.5 of Baider and Churchill (1990). □

**Example 2.13.** From Theorem 2.12 and Example 2.11 we see that the monodromy and differential Galois groups of the normal form of the Legendre equation are Ziglin iff $\lambda = k(k + 1)$ for some integer $k$.

Before turning to the proof we apply Theorem 2.9 to the hypergeometric, Riemann, and Lamé equations.
3. The Hypergeometric Equation

The classical hypergeometric equation is the Fuchsian equation

\[ y'' + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)}y' - \frac{\alpha\beta}{x(1-x)}y = 0 \quad (3.1) \]

on \( P^1 \), where \( \alpha, \beta \) and \( \gamma \) are arbitrary complex constants. The normal form (2.2) is

\[ y'' + \frac{1}{4} \left\{ \frac{1 - \lambda^2}{x^2} + \frac{1 - \nu^2}{(x-1)^2} - \frac{\lambda^2 - \mu^2 + \nu^2 - 1}{x} + \frac{\lambda^2 - \mu^2 + \nu^2 - 1}{x-1} \right\} y = 0, \quad (3.2) \]

with parameters \( \lambda, \mu \) and \( \nu \) defined by

\[
\begin{align*}
\lambda & := 1 - \gamma, \\
\nu & := \gamma - (\alpha + \beta), \\
\mu & := \pm(\alpha - \beta). 
\end{align*}
\]  

(3.3)

The exponents at the singularity \( a_1 = 0 \) are 0 and \( 1 - \gamma \), those at the singularity \( a_2 = 1 \) are 0 and \( \gamma - (\alpha + \beta) \), and those at \( a_{\infty} \) are \( \alpha \) and \( \beta \). In particular, the parameters \( \lambda, \mu \) and \( \nu \) are the exponent differences at the three points. The traces (2.6) are given by

\[
\begin{align*}
t_1 & = -2 \cos \pi (\gamma - 1) = -2 \cos \pi \lambda \\
t_2 & = -2 \cos \pi (\gamma - (\alpha + \beta)) = -2 \cos \pi \nu \\
t_{\infty} & = -2 \cos \pi (\alpha - \beta) = -2 \cos \pi \mu. 
\end{align*}
\]

(3.4)

**Proposition 3.5.** The monodromy group \( G \) of the hypergeometric equation (3.2) is:

(a) reducible iff at least one of \( \alpha, \beta, \gamma - \alpha \) and \( \gamma - \beta \) is an integer or, equivalently, iff at least one determination of \( \pm \lambda \pm \nu \pm \mu \) is an odd integer; and

(b) DP but not reducible iff at least two of \( \lambda - \frac{1}{2}, \nu - \frac{1}{2}, \) and \( \mu - \frac{1}{2} \) are integers and the third is not of the form \( n - \frac{1}{2} \) for some integer \( n \). Moreover, \( G \) is quaternionic iff all three of \( \lambda - \frac{1}{2}, \nu - \frac{1}{2}, \) and \( \mu - \frac{1}{2} \) are integers.

**Proof.** (a) is immediate from Theorem 2.9(a); see, e.g. Proposition 2.22 and Theorem 2.24 of Baider and Churchill (1990). (Alternatively, see p. 90 of Poole, 1960, or Proposition 8, p. 231, of Duval and Loday-Richaud, 1992.) (b) is immediate from Theorem 2.9(b), (3.4), and Remark 2.10. □

**Remarks 3.6.** (a) Necessary and sufficient conditions for \( G \) to be (projectively) tetrahedral, octahedral or icosahedral are also easily determined from (3.4), the conditions given in Theorem 2.9(c), and Remark 2.10, but are somewhat cumbersome to write down. For example, \( G \) is projectively tetrahedral if \( \lambda - \frac{1}{2} \) is an integer and each of \( \nu \pm \frac{1}{3} \) and \( \mu \pm \frac{1}{3} \) includes an odd integer, or if \( \nu - \frac{1}{4} \) is an integer and each of \( \lambda \pm \frac{1}{4} \) and \( \mu \pm \frac{1}{4} \) includes an odd integer, etc. For such reasons it is preferable to tabulate up to permutations after normalizing the exponent differences (see Example 3.7(b) and, for more detail on the normalization, Corollary 2.6 of Beukers and Heckman, 1989).

(b) Recall from the remarks following the statement of Theorem 2.9 that a matrix
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representation of $G$ can be determined from the three quantities $t_1, t_2$ and $t_\infty$ in any irreducible case. There we cited Magnus (1976), but for the hypergeometric equation one can also use the $n = 2$ case of a theorem of A.H.M. Levelt (see Theorem 3.5 of Beukers and Heckman, 1989). Indeed, Levelt’s result amounts to an algorithm for computing generators of the monodromy group of any higher hypergeometric equation when that group is irreducible (see Proposition 3.3 of Beukers and Heckman, 1989.)

Examples 3.7. (a) In Kovacic (1986) the differential equation

$$y'' + \left\{ \frac{3}{16x^2} + \frac{2}{9(x-1)^2} - \frac{3}{16x(x-1)} \right\} y = 0$$

is considered in connection with an algorithm for solving second-order linear equations. The author works through the algorithm, first eliminating the reducible case, then the DP case, and subsequently proves (implicitly) that the monodromy group is projectively tetrahedral. With our methods this particular aspect of the example can be established as follows. The equation is seen to be the hypergeometric equation in normal form with $\lambda = \frac{1}{2}$ and $\mu = \nu = \frac{1}{2}$; from (3.4) we have $t_1 = 0$ and $t_2 = t_3 = 1$; by (ci) of Theorem 2.9 the monodromy group is projectively tetrahedral.

In fact this example is but one case of a classic study of the hypergeometric equation which we now recall.

(b) In a celebrated paper Schwarz (1873) enumerated those values of the parameters $\lambda, \nu, \mu$ for the hypergeometric equation, up to permutation and appropriately normalized, for which the monodromy group is projectively finite. The irreducible cases are listed in Table 1, which is numbered so as to agree with the list presented in Poole (1960, p. 128). The remaining columns are computed using Remark 2.10, straightforward calculation, and Theorem 2.9(c). (Note that $\mu$ are $\nu$ are associated with $t_3$ and $t_2$, respectively, and as a result the ordering of the second and third entries in column three corresponds to a transposition of the second and third entries in column two.)

Beukers and Heckman (1989) published a sweeping generalization of the Schwarz results: a method for determining the nature (e.g. being projectively finite) of the monodromy groups of all higher-order hypergeometric equations. (Our methods apply only to the classical second-order case (3.1).)

(c) In Yoshida (1986), the author is interested in the Ziglin property for the hypergeometric equation in the case $\lambda = \frac{1}{2m}$, $\nu = \frac{1}{2}$, $\mu = \frac{1}{2m} \sqrt{(m-1)^2 + 4Lm}$, where $m > 1$ is a positive integer and $L$ (there called $\lambda_m$) is a parameter. In essence he proves, by computing relevant circuit matrices explicitly, that when the quantity $t_3 = -2 \cos \frac{\pi}{2m} \sqrt{(m-1)^2 + 4Lm}$ is non-resonant the monodromy group cannot be Ziglin. We capture this result as follows: by (a) and (b) of Theorem 2.9 the group is not DP; by Theorem 2.12 it is therefore not Ziglin.

Similar arguments can be used to establish results found, e.g. in Baider et al. (1996), Churchill and Rod (1991), Irigoyen (1996), Yoshida (1987), and Yoshida (1988).

See Duval and Loday-Richaud (1992) for additional applications of the Kovacic algorithm to the hypergeometric equation.
Table 1. The Schwarz table.

<table>
<thead>
<tr>
<th>Number</th>
<th>$\lambda, \mu, \nu$</th>
<th>$t_1, t_2, t_3$</th>
<th>$f_2^0 + f_2^1 + f_2^2 - t_1 t_2 t_3$</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$0, -2 \cos \pi/n, 0$</td>
<td>$4 \cos^2 \pi/n$</td>
<td>Quaternionic when $n = 2$; Projectively Dihedral when $n &gt; 2$</td>
</tr>
<tr>
<td>II</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$1, 1, -1$</td>
<td>2</td>
<td>Projectively Tetrahedral</td>
</tr>
<tr>
<td>III</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$0, -\sqrt{2}, 0$</td>
<td>3</td>
<td>Projectively Octahedral</td>
</tr>
<tr>
<td>IV</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$-\sqrt{2}, -\sqrt{2}, 0$</td>
<td>3</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>V</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$1, -1, 0$</td>
<td>2</td>
<td>Projectively Dihedral when $n &gt; 2$</td>
</tr>
<tr>
<td>VI</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$0, -1, 0$</td>
<td>1</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>VII</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$-\mu_2, -1, -1$</td>
<td>3</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>VIII</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$1, -\mu_1, -\mu_3$</td>
<td>1</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>IX</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$0, -1, -\mu_2$</td>
<td>3</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>X</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$\mu_2, -1, -1$</td>
<td>3</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>XI</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$-\mu_2, -\mu_2, -\mu_2$</td>
<td>1</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>XII</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$1, -\mu_1, -1$</td>
<td>3</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>XIII</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$\mu_1, -\mu_1, -\mu_1$</td>
<td>1</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>XIV</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$0, -1, -\mu_2$</td>
<td>1</td>
<td>Projectively Icosahedral</td>
</tr>
<tr>
<td>XV</td>
<td>$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$</td>
<td>$\mu_2, -1, -\mu_2$</td>
<td>1</td>
<td>Projectively Icosahedral</td>
</tr>
</tbody>
</table>

4. Riemann’s Equation

This is the second-order Fuchsian equation $y'' + c_1(x)y' + c_2(x)y = 0$, where

\[
c_1(x) = \frac{1 - \eta_1 - \mu_1}{x} + \frac{1 - \eta_2 - \mu_2}{x - 1},
\]

\[
c_2(x) = \frac{\eta_1 \mu_1}{x^2} + \frac{\eta_2 \mu_2}{(x - 1)^2} + \frac{\eta_3 \eta_4 - \eta_1 \mu_1 - \eta_2 \mu_2}{x(x - 1)},
\]

and the complex constants $\eta_j, \mu_j$ are subject to the single constraint $\sum(\eta_j + \mu_j) = 1$.

The normal form is

\[
y'' + \frac{1}{4} \left( \frac{1 - (\eta_1 - \mu_1)^2}{x^2} + \frac{1 - (\eta_2 - \mu_2)^2}{(x - 1)^2} + \frac{\nu}{x} - \frac{\nu}{x - 1} \right) y = 0,
\]

where

\[
\nu := 1 - (\eta_1 - \mu_1)^2 - (\eta_2 - \mu_2)^2 + (\eta_3 - \mu_3)^2.
\]

The exponents at the singularity $a_1 = 0$ are $\eta_1$ and $\mu_1$, those at the singularity $a_2 = 1$ are $\eta_2$ and $\mu_2$, and those at $a_\infty$ are $\eta_3$ and $\mu_3$. The traces (2.6) are therefore

\[
t_j = -2 \cos \pi(\eta_j - \mu_j), \quad j = 1, 2, \infty,
\]

where $\eta_\infty := \eta_3$ and $\mu_\infty := \mu_3$.

Proposition 4.4. The monodromy group $G$ of the Riemann equation (4.1) is:
(a) reducible iff at least one of $\eta_1 + \eta_2 + \eta_3, \eta_1 + \mu_2 + \eta_3$ and $\mu_1 + \eta_2 + \eta_3$ is an integer; and
(b) DP but not reducible iff at least two of the three expressions $2(\eta_j - \mu_j)$ are odd integers and the third is not an even integer. Moreover, $G$ is quaternionic iff all three expressions are odd integers.

Assertion (a) is standard, e.g. see Corollary 2.28 of Baider and Churchill (1990).

**Proof.** Use Theorems 2.9(a) and (b) and the appropriate items of Remark 2.10. \(\Box\)

Remark 3.6(a) is also relevant here.

**Example 4.5.** The Fuchsian equation

$$y'' - \left(\frac{2/3}{x} + \frac{4/3}{x-1}\right)y' + \left(\frac{2/3}{x} + \frac{4/3}{(x-1)^2} - \frac{5/16}{x(x-1)}\right)y = 0$$

is Riemann’s equation with parameter values $\eta_1 = \eta_2 = 1, \mu_1 = \frac{2}{3}, \mu_2 = \frac{4}{3}, \eta_3 = -\frac{3}{4}$, and $\mu_3 = \frac{3}{4}$, and the normal form is

$$y'' + \left(\frac{2/9}{x^2} + \frac{2/9}{(x-1)^2} + \frac{109/144}{x} - \frac{109/144}{x-1}\right)y = 0.$$ 

From (4.3) the relevant traces at $a_1 = 0, a_2 = 1$ and $a_3 = \infty$ are seen to be $t_1 = t_2 = -1$ and $t_3 = 0$. By (i) of Theorem 2.9(c) the monodromy group of the normal form is projectively tetrahedral.

Remarks 4.6. (a) It is well-known that the Riemann equation can be reduced to the hypergeometric equation by a change of the dependent variable (e.g. see p. 297 of Birkhoff and Rota, 1989), and in this sense the results of the present section are redundant. But in specific applications one or the other equation may arise “naturally” (e.g. for the case of the Riemann equation see Churchill et al., 1996), and indicating the results for both equations eliminates the need for additional calculations.

(b) As already suggested in Remark 2.7(c), when a Fuchsian equation on the Riemann sphere has more than three singular points one needs more than the corresponding traces to completely determine the nature of the monodromy group. However, in special cases methods analogous to those developed here can be used to achieve a reasonable understanding of the monodromy group with very little work. Example: suppose $s_1, \ldots, s_{m+1} \in SL(2, \mathbb{C})$ satisfy the following conditions: $\Pi s_j = I$; $t_{s_{m+1}}$ is transcendental over $\mathbb{Q}(t_1, \ldots, t_m)$; and $t_{s_j} \neq 0$ for $j = 1, \ldots, m$. Then the subgroup of $S(L, \mathbb{C})$ generated by $s_1, \ldots, s_{m+1}$ is Zariski dense. (This is a rephrasing of Theorem 3.7 of Baider and Churchill (1990).)

5. The Lamé Equation

Let $\mathcal{L} = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} \subset \mathbb{C}$ be a lattice, with $\omega_1$ and $\omega_2$ independent over $\mathbb{R}$, and let $\wp: \mathbb{C} \to \mathbb{P}^1$ be the corresponding Weierstrass $\wp$-function. The associated Lamé equation
is

$$\frac{d^2 y}{dz^2} + [\lambda - n(n+1)\wp(z)] y = 0, \quad (5.1)$$

where $\lambda, n \in \mathbb{R}$ are parameters. (In particular, $n$ is not restricted to integer values.) We view (5.1) as a Fuchsian equation on the complex torus $T := \mathbb{C}/\mathcal{L}$ and accordingly regard all points of $\mathbb{C}$ as being expressed mod $\mathcal{L}$. The exponents at 0 are $-n$ and $n + 1$.

In $T$ set

$$p_1 := \frac{1}{2} \omega_1, \quad p_2 := \frac{1}{2} (\omega_1 + \omega_2), \quad p_3 := \frac{1}{2} \omega_3, \quad p_4 := 0,$$

and let

$$q_j := \wp(p_j) \in \mathbb{P}^1, \quad j = 1, \ldots, 4.$$

Note that $q_4 = \infty$. The $q_j$ are distinct, and the $p_j$ are precisely the points where $\wp$ has branching order one (e.g. see p. 156 of Knapp, 1992). The Fuchsian equation

$$\frac{d^2 y}{dx^2} + \frac{1}{2} \left( \frac{1}{x - q_1} + \frac{1}{x - q_2} + \frac{1}{x - q_3} \right) + \frac{\lambda - n(n+1) x}{4(x - q_1)(x - q_2)(x - q_3)} y = 0 \quad (5.2)$$

on the Riemann sphere is known as the algebraic form of the Lamé equation (e.g. see Erdélyi, 1955, pp. 56 and 57). The exponents of (5.2) at the $q_j$ are 0 and $\frac{1}{2}$, and those at $\infty$ are $-n/2$ and $\frac{1}{2}(n + 1)$.

In classical formulations the relationship between equations (5.1) and (5.2) is the “variable substitution” $x = \wp(z)$. Alternatively, in terms of operators one can regard (5.1) as the pullback of (5.2) by the $\wp$-function. A third approach is to view the equations as describing the horizontal sections of holomorphic connections on holomorphic rank-two vector bundles over the respective Riemann surfaces, with (5.1) being the pullback of (5.2) by $\wp$ (e.g. see Churchill et al., 1997).

**Proposition 5.3.** Let $G$ and $G_{AL}$ denote the monodromy groups, or differential Galois groups, of (5.1) and (5.2), respectively. Then the following statements hold:

(a) $G$ is reducible iff $n$ is an integer, which is also the case iff $G$ is Abelian.

(b) In the irreducible case $G$ is $DP$ iff $G_{AL}$ is finite and $n + \frac{1}{2}$ is an integer, which is also the case iff $G$ is quaternionic. In particular, $G$ cannot be projectively dihedral.

(c) The finite possibilities for $G$ not covered by (a) and (b) are:

(i) $G$ is projectively tetrahedral iff $G_{AL}$ is finite and one of $n \pm \frac{1}{2}$ is an integer, which is also the case iff $G_{AL}$ is projectively octahedral;

(ii) $G$ is never projectively octahedral; and

(iii) $G$ is projectively icosahedral iff $G_{AL}$ is finite and one of $n \pm \frac{1}{10}, n \pm \frac{3}{10}$ is an integer, which is also the case iff $G_{AL}$ is projectively icosahedral.

(d) In all other cases $G$ has Zariski closure $SL(2, \mathbb{C})$.

An algorithm for determining if $G_{AL}$ is finite, and for determining if $G_{AL}$ is projectively octahedral or projectively icosahedral when this is the case, is found in Baldassarri (1981). Restrictions on $\lambda$ are involved, e.g. when $n + \frac{1}{2}$ is an integer there are at most $|n + \frac{1}{2}|$ values of $\lambda$ for which (5.2) has finite monodromy and differential Galois groups (see Theorem 2.6 of Baldassarri, 1981). Alternatively, see Corollary 5.4.

In connection with (b) and (ciii) it is interesting to note that $G_{AL}$ cannot be projectively
We claim that if $G$ is a subgroup of $G$, then $\pi$ is the standard biholomorphic mapping $\pi$ to $P$. Denote the standard biholomorphic mapping $\pi$ of $K$ and $\gamma$ to $\pi$ of $K$ and $\gamma$, respectively, and we let $\rho_{AL} = \pi_{1}(Y, y_0) \to GL(2, \mathbb{C})$ denote the respective monodromy representations of (5.1) and (5.2). Moreover, we let $\rho : \pi_{1}(X, x_0) \to SL(2, \mathbb{C})$ and $\rho_{AL} : \pi_{1}(Y, y_0) \to GL(2, \mathbb{C})$ denote the respective monodromy representations of (5.1) and (5.2), where $x_0 \in X$ is arbitrary and $y_0 := \varphi(x_0)$. Finally, we let $\rho : \pi_{1}(X, x_0) \to SL(2, \mathbb{C})$ denote the monodromy representation of the restriction of (5.1) to $X$, and we note that the corresponding monodromy group $G := \rho(\pi_{1}(X, x_0))$ is identical with $G$. The claim follows.

In the remaining discussion no distinction is made (notationally) between points and curves in $C$ and their images in $T, X,$ and $X$. Moreover, the restrictions of $\varphi$ to $X$ and $X$ are also denoted $\varphi$.

We choose the standard generators for $\pi_{1}(Y, y_0)$, i.e. the homotopy classes $[\gamma_j]$ of positively oriented loops $\gamma_j$ surrounding $q_j$, $j = 1, \ldots, 4$. For simplicity we assume the standard relation $\prod_{j=1}^{4} [\gamma_j] = e$ (the identity element), even though this may actually require a permutation of the subscripts of $q_1, q_2,$ and $q_3$. For $j = 1, \ldots, 4$ we let $m_j := \rho_{AL}([\gamma_j]) \in GL(2, \mathbb{C})$. Using the stated exponents for (5.2) and the relation just recalled we see that $\det(m_j) = -1$ for $j = 1, \ldots, 4$ and that $m_j^2 = I$ for $j = 1, 2, 3$.

Now let $\Sigma$ be the projective curve associated with $y^2 = 4 \prod_{j=1}^{4} (x - q_j)$, let $\varphi : T \to \Sigma$ denote the standard birational mapping $z \mapsto (\varphi(z), \varphi'(z), 1)$ (e.g. see pp. 160 and 161 of Knapp, 1992), and set $\sigma_0 := \varphi(x_0)$. Moreover, let $\Sigma_X := \varphi(X)$. We regard $\Sigma$ as being constructed from $P^1$ and a copy thereof by gluing along slits from $q_1$ to $q_2$ and $q_3$ to $q_4$ respectively, and we let $\sigma : \Sigma \to P^1$ denote the standard projection $(x, y) \mapsto x$. This gives $\pi_{1}(\Sigma, \sigma_0) = x_0$, and one sees easily that the homotopy classes $[\alpha'1], [\beta^3]'$ of lifts of $\gamma_1 \cdot \gamma_2$ and $\gamma_1 \cdot \gamma_2$ will generate both $\pi_{1}(\Sigma, \sigma_0)$ and $\pi_{1}(\Sigma_X, \sigma_0)$ (e.g. imagine $q_j$ as the point $(-i)^{j-1}e^{\pi i/4}, j = 1, \ldots, 4,$ and imagine $y_0$ as the point $(-1, 0)$).

The mapping $\varphi : \Sigma \to Y$ induces an embedding of $G = G_{AL}$ into $G_{AL}$ (e.g. see Proposition 2.13 of Churchill et al., 1997), and as a result we may henceforth regard $G$ as a subgroup of $G_{AL}$. Since $\varphi = \pi \circ \varphi$, it follows from the work of the previous paragraph that $G = \{m_3 \cdot m_2, m_1 \cdot m_2\}$.

(I) We claim that $G = G_{AL} \cap SL(2, \mathbb{C})$, i.e. that $G$ consists of those elements of $G_{AL}$ having determinant one. Indeed, first note from $G = \langle m_3 \cdot m_2, m_1 \cdot m_2 \rangle$ and $m_j^2 = I$ for $j = 1, 2, 3$, that $G$ contains all two-letter words in $G_{AL} = \langle m_3, m_2, m_1 \rangle$, and as a result all even-letter words. But odd-letter words have negative determinant (since $\det(m_j) = -1$ for $j = 1, 2, 3$, hence cannot be in $G \subset SL(2, \mathbb{C})$. The claim follows.

(II) We claim that $G \leq G_{AL}$, i.e. that $G$ is a normal subgroup of $G_{AL}$. This is clear.

(III) We claim that $G$ if finite iff $G_{AL}$ is finite. Simply note from (I) that the sequence $\{I \} \to G \to G_{AL} \xrightarrow{\det} \{1, -1\} \simeq \mathbb{Z}_2 \to \{1\}$ is exact, hence $G_{AL}/G \simeq \mathbb{Z}_2$, and (III) follows.

We next specify generators for $\pi_{1}(X, x_0)$, and to this end first note that the four points...
0, ω1, ω2 and ω2 of \( \mathcal{L} \) determine a parallelogram \( \mathcal{P}_\mathcal{L} \subset \mathbb{C} \). We shift \( \mathcal{P}_\mathcal{L} \setminus \{p_2/2\} \) by \(-p_2/2\) so as to obtain a parallelogram \( \mathcal{P}' \) punctured at the origin, and we regard \( X \) as the result of identifying opposite edges of \( \mathcal{P}' \). To simplify the exposition imagine \( \mathcal{P}' \) as having corners \((-2, -1), (2, -1), (2, 1) \) and \((-2, 1) \), and let \( \alpha \) and \( \beta \) denote the edges connecting \((-2, -1) \) to \((2, -1) \) and \((2, -1) \) to \((2, 1) \), respectively. Moreover, choose \( x_0 := (-2, -1) \) in \( X \) as the basepoint for homotopy calculations. Then \( \pi_1(X, x_0) \) is free on the generators \([\alpha]\) and \([\beta]\), and the commutator \(([\alpha], [\beta]) \) is represented by a tiny positively oriented loop surrounding 0. (In this statement we can view 0 either as a point of \( X \) or as the missing point of \( \mathcal{P}' \).)

Now let \( S = \rho([\alpha]) \) and \( T := \rho([\beta]) \), so that \( G = \langle S, T \rangle \). Using the exponents \(-n, n + 1\) of (5.1) at 0 and the Fricke–Klein formula \( t_{(S,T)} = t_S^2 + t_T^2 + t_{ST}^2 - t_STt_{ST} - 2 \) one computes that

\[
(IV) \quad t_S^2 + t_T^2 + t_{ST}^2 - t_STt_{ST} = 2 \cos(2\pi n) + 2.
\]

(a) From (IV) and Theorem 6.1(a) we see that \( G \) is reducible iff \( n \) is an integer. For the Abelian equivalence see, e.g. Corollary 6.9 of Churchill et al. (1997).

(b) We first note from the analytic argument on pp. 160 and 161 of Morales-Ruiz and Simó (1994) that \( n + \frac{1}{2} \in \mathbb{Z} \) must hold if (a) fails and (5.1) admits DP solutions, in which case the right-hand side of (IV) vanishes. If \( G \) is DP, i.e. if at least two of the three traces vanish, it follows that the third must also vanish, and \( G \) is then quaternionic by Theorem 6.1(b). The remaining assertions are also clear from that result together with (IV).

(c) We need the fact that when \( G_{AL} \) is finite the only possibilities are: quaternionic or projectively dihedral (in which case \( G \subset G_{AL} \) is clearly DP); projectively octahedral; and projectively icosahedral (see Section 2 of Baldassarri and Dwork, 1979, particularly pp. 55 and 56, together with Propositions 1.1 and 3.1 of Baldassarri, 1981).

(cii) The forward implication of the initial statement is immediate from (III), Theorem 6.1(c), and (IV). Conversely, if \( G_{AL} \) is finite the same holds for \( G \subset G_{AL} \), and if one of \( n + \frac{1}{2} \) is an integer \( G \) cannot be reducible or DP by (a), (b), and (IV). We conclude from Theorem 6.1(c) and (IV) that \( G \) is projectively tetrahedral.

For the final equivalence see pp. 54 and 55 of Baldassarri (1981).

(ciii) When \( G \) is projectively octahedral or projectively icosahedral we see from Theorem 6.1(c) and (IV) that one of \( n \pm \frac{1}{3} \) \( n \pm \frac{2}{3} \) and \( n \pm \frac{1}{3} \) must be an integer, which from (III) and pp. 56 and 57 of Baldassarri (1981) is equivalent to \( G_{AL} \) being projectively icosahedral.

Conversely, suppose \( G_{AL} \) is projectively icosahedral, i.e. projectively \( A_5 \). Then \( G_{AL} \) is projectively simple, and from (II) and the inclusion \( G \subset G_{AL} \) we conclude that \( G \) is projectively icosahedral (otherwise it is projectively trivial, contradicting (a)).

(d) By Theorem 6.1(d).

---

**Corollary 5.4.** Suppose a value for \( \lambda \) in (5.1) has been specified. Then:

1. \( G \) is quaternionic iff \( n + \frac{1}{2} \) is an integer and the Kovacic algorithm, applied to (5.2), returns the general solution;
2. \( G \) cannot be projectively dihedral;
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(3) $G$ is projectively tetrahedral iff one of $n \pm \frac{1}{4}$ is an integer and the Kovacic algorithm, applied to (5.2), returns the general solution;
(4) $G$ cannot be projectively octahedral; and
(5) $G$ is projectively icosahedral iff one of $n \pm \frac{1}{6}, n \pm \frac{1}{10}$ and $n \pm \frac{3}{10}$ is an integer and the Kovacic algorithm, applied to (5.2), returns the general solution.

We note that “returning the general solution” is equivalent to the equation having only Liouvillian solutions (see Kovacic, 1986).

PROOF. The Kovacic algorithm returns the general solution to a second-order equation on $\mathbb{P}^1$ precisely in cases (a)–(c) of Theorem 2.9; otherwise it returns the input equation, and the Zariski closure of $G$ is $SL(2, \mathbb{C})$.

COROLLARY 5.5. Suppose none of $n, n + \frac{1}{2}, n \pm \frac{1}{4}, n \pm \frac{1}{6}, n \pm \frac{1}{10}$ and $n \pm \frac{3}{10}$ is an integer. Then the monodromy and differential Galois groups of (5.1) are not Ziglin. More precisely, with the possible exception of this discrete set of values for $n$ the differential Galois group of (5.1) is $SL(2, \mathbb{C})$ and the monodromy group has Zariski closure $SL(2, \mathbb{C})$.

This corollary is of interest in connection with work of Morales-Ruiz and Simó (1994) and Morales-Ruiz and Simó (1996).

The remainder of the paper is devoted to the proof of Theorem 2.9.

6. Reduction to Group Theory

The comments following the statement of Theorem 2.9 reduce the proof to that of the following purely group-theoretic result.

THEOREM 6.1. Suppose $S, T \in SL(2, \mathbb{C})$ and $G := \langle S, T \rangle$. Then the following statements hold:

(a) $G$ is reducible iff $t_S^2 + t_T^2 + t_{ST}^2 - t_ST t_{ST} = 4$.
(b) Suppose $G$ is not reducible. Then $G$ is DP iff at least two of the three traces $t_S, t_T$ and $t_{ST}$ vanish. Moreover, in this irreducible DP case the group $G$ is finite iff:
   (i) all three traces vanish, in which case $G$ is quaternionic; or
   (ii) the non-vanishing trace is resonant, in which case $G$ is projectively dihedral.
(c) $G$ is finite but not reducible and not DP iff $G$ is projectively tetrahedral, projectively octahedral, or projectively icosahedral. Moreover:
   (i) $G$ is projectively tetrahedral iff $t_S^2 + t_T^2 + t_{ST}^2 - t_ST t_{ST} = 2$ and $t_S, t_T, t_{ST} \in \{-1, 0, 1\}$;
   (ii) $G$ is projectively octahedral iff $t_S^2 + t_T^2 + t_{ST}^2 - t_ST t_{ST} = 3$ and $t_S, t_T, t_{ST} \in \{-\sqrt{2}, -1, 0, 1, \sqrt{2}\}$; and
   (iii) $G$ is projectively icosahedral iff $t_S^2 + t_T^2 + t_{ST}^2 - t_ST t_{ST} \in \{2 - \mu_2, 3, 2 + \mu_1\}$ and $t_S, t_T, t_{ST} \in \{-\mu_1, 1, -\mu_2, 0, \mu_2, 1, \mu_1\}$, where $\mu_1 := \frac{1}{2}(1 + \sqrt{5})$ and $\mu_2 := (\mu_1)^{-1} = \frac{1}{2}(1 - \sqrt{5})$. 


Assertion (d) of Theorem 6.1 is immediate from Proposition 2.8. As for the other assertions, (a) is classical, and is a special case of the following result: a subgroup \( G \subset SL(2, \mathbb{C}) \) is reducible iff \( \text{tr}|(G, G) = 2 \), where \( (G, G) \subset G \) denotes the commutator subgroup and \( \text{tr} : SL(2, \mathbb{C}) \to \mathbb{C} \) is the trace function. For a proof see Proposition 1.2 of Baider and Churchill (1990). The proof of (b) is quite elementary and will be given in the following section. That of (c) is far more involved: it is given in Section 9, and ultimately rests on an elementary group-theoretic result treated in Section 8.

7. Proof of Theorem 6.1(b)

⇒ When \( G \) is \( DP \) we can regard \( G \) as a matrix subgroup of \( D \cup P \), and since \( G \) is not reducible we must have at least one of \( S \) and \( T \) in \( P \). Now simply note that elements of \( P \) have trace zero and check the possibilities.

⇐ Since \( \langle S, T \rangle = \langle T^{-1}, ST \rangle = \langle S, ST \rangle \), etc., and since \( t_S = t_{S^{-1}} \), we may assume, by relabelling if necessary, that \( t_T = t_{ST} = 0 \). Moreover, since \( t_S \neq \pm 2 \) the operator \( S \) can be diagonalized, and in terms of such a basis we can then write \( S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \) and \( T = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \). This immediately leads to \( t_{ST} = \lambda^{-1}(\lambda^2 - 1)x \), and since \( \lambda \neq \pm 1 \) we conclude that \( x = 0 \), hence that \( T \in P \). \( G \) is therefore a \( D \cup P \) group.

If all three traces vanish then in the notation of the previous paragraph we have \( S = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix} \), and we can rescale the second basis element so as to achieve \( T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) without altering the form of \( S \). \( G \) is thus quaternionic.

To see the converse first note that for any \( M \in SL(2, \mathbb{C}) \) we have \( t_{M^2} = t_M^2 - 2 \) (\( := (t_M)^2 - 2 \)), and therefore \( t_{M^4} = t_M^4 - 4t_M^2 + 2 \). But \( M^4 = I \) when \( M \) belongs to a quaternionic group, and so in that case we see that either \( t_M = 0 \) or \( t_M = \pm 2 \). Moreover, in the second instance we must have \( M = \pm I \), since otherwise the Jordan form would be \( \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \), forcing \( M \) to have infinite order.

Now suppose \( \langle S, T \rangle \subset SL(2, \mathbb{C}) \) is quaternionic. Then from the previous paragraph we see that \( t_S \neq 0 \neq t_T \) would give the contradiction \( \langle S, T \rangle \subset \{ I, -I \} \), and so w.l.o.g. we may assume \( t_S = 0 \), thereby allowing us to identify \( S \) with the matrix \( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \) by means of an appropriate basis. It follows immediately that \( \langle S \rangle \) has order 4 and contains \(-I\). Since \( t_T \neq 0 \) implies \( T = \pm I \), the group \( \langle S, T \rangle \) cannot have order 8 unless \( t_T = 0 \), and we can therefore identify \( T \), using the same basis, with a matrix of the form \( \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \).

Applying completely analogous arguments to \( ST = \begin{pmatrix} ia & ib \\ -ic & ia \end{pmatrix} \) we conclude that \( a = 0 \), and therefore \( t_S = t_T = t_{ST} = 0 \). \( \square \)

8. Two Generator Subgroups of the Permutation Groups \( A_n \), \( S_n \), and \( A_5 \)

The proof of Theorem 6.1(c) rests on the following elementary group-theoretic result. Recall that \( A_n \) and \( S_n \) denote the alternating and symmetric groups on \( n \geq 1 \) letters; we
let $e$ denote the identity in all three cases. For any non-empty subset $S = \{s_1, \ldots, s_k\}$ of $S_n$ we let $\langle s_1, \ldots, s_k \rangle \subset S_n$ denote the subgroup generated by $S$.

**Proposition 8.1.** (a) Suppose $g, h \in A_4$ satisfy $g \notin \langle h \rangle$ and $h \notin \langle g \rangle$. Then $A_4 = \langle g, h \rangle$ iff at least one of $g$ and $h$ is a 3-cycle. Moreover, when both are 3-cycles the pair \{hg, hg^{-1}\} consists of a 3-cycle and a product of disjoint transpositions.

(b) Suppose $g, h \in S_4$ satisfy $g \notin \langle h \rangle$ and $h \notin \langle g \rangle$. Then $S_4 = \langle g, h \rangle$ iff at least one of $g$ and $h$ and $hg$ is a 4-cycle and at most one is of order two.

(c) Suppose $g, h \in A_5$ satisfy $g \notin \langle h \rangle$ and $h \notin \langle g \rangle$. Then $A_5 = \langle g, h \rangle$ iff at least one of $g$, $h$ and $hg$ is a 5-cycle and at most one is of order two.

Note that the condition $g \notin \langle h \rangle$ and $h \notin \langle g \rangle$ common to all three cases is trivially necessary, and that by switching generators and/or replacing generators by inverses one can assume that the orders $|g|, |h|$ and $|gh|$ of the elements $g, h$ and $gh$ satisfy $|g| \geq |h| \geq |gh|$. In the arguments establishing the three cases bounds for the orders $|G|$ of various $G := \langle g, h \rangle$ are often stated explicitly. These bounds can be verified by direct calculation (Todd–Coxeter algorithm), by consulting standard tables (e.g. p. 134 of Coxeter and Moser, 1980), or through the use of computer packages (e.g. MAGNUS of the New York Group Theory Cooperative).

**Proof of Proposition 8.1(a).** $\Rightarrow$ $A_4 \subset S_4$ consists of the identity $e$, all products of disjoint transpositions, and all 3-cycles, and when $g$ has order two we have $|G| \leq 4 \neq 12 = |A_4|$.

$\Leftarrow$ If both $g$ and $h$ are 3-cycles then w.l.o.g. $h = (123)$ and $\{g, g^{-1}\} = \{(234), (243)\}$. This gives $\{hg, hg^{-1}\} = \{(12)(34), (124)\}$, and the final assertion follows. We are thus reduced to the case $|g| = |h| = 3$ and $|gh| = 2$, hence $|G| = 6$ or 12, and the first alternative is impossible since $A_4$ has no subgroup of order six (e.g. see Theorem 3.11, p. 37 of Rotman, 1973).

**Proof of Proposition 8.1(b).** $\Rightarrow$ If $|g| = |h| = 3$ then $g, h \in A_4$ and therefore $G \subset A_4 \neq S_4$; if $|g| = 3$ and $|h| = |gh| = 2$ then $|G| = 12 \neq 24 = |S_4|$; and if $|g| = |h| = |gh| = 2$ then $|G| = 4 \neq |S_4|$.

$\Leftarrow$ Here $|g| = 4$. If $|h| = |gh| = 2$ then $|G| \leq 8 \neq |S_4|$. If $|h| = 3$ then $G$ contains elements of orders 3 and 4, and therefore has order 12 or 24. But $A_4$ is the unique subgroup of $S_4$ of order 12 (e.g. see Exercise 1, p. 37 of Rotman, 1973), and since $A_4$ contains no 4-cycle we conclude that $G = S_4$. If $|h| = 4$ then w.l.o.g. $g = (1234)$ and $h \in \{(1243), (1324), (1423), (1432)\}$. But in all four cases $gh$ is a 3-cycle, hence $G$ contains elements of orders 4 and 3, and as just seen we then have $G = S_4$. □

Two preliminaries are useful for the proof of Proposition 8.1(c).

**Lemma 8.2.** Suppose $g, h \in A_5$ are 5-cycles satisfying $g \notin \langle h \rangle$ and $h \notin \langle g \rangle$. Then at least one of $hg$ and $hg^{-1}$ is not a 5-cycle.

**Proof.** Assume to the contrary, w.l.o.g. let $h = (12345)$, and read all indices mod 5 in the remainder of the proof. Since $hg$ has no fixed point, for no $k$ do we have $g : k \mapsto k + 4$; since $hg^{-1}$ has no fixed point, for no $k$ do we have $g : k \mapsto k + 1$; since
has no fixed point, for no $k$ do we have $g : k \mapsto k$. Therefore, for all $k$ we must have
$g : k \mapsto \{k + 2, k + 3\}$, i.e. $g$ “advances” points 2 or 3 units (mod 5). However, since
$g \notin \langle h \rangle$, $g$ cannot advance all points by the same amount, and by cyclically permuting
labels (if necessary) we may assume that $g : 1 \mapsto 3$ and $g : 2 \mapsto 5$. But $g : 3 \mapsto \{5, 1\}$,
hence $g : 3 \mapsto 1$, and this is impossible since $g$ is a 5-cycle. □

By replacing $g$ by $g^{-1}$, if necessary, one can assume under the hypotheses of Lemma 8.2
that $hg$ is not a 5-cycle. This explains the alternative hypotheses for $hg$ in the next lemma.

**Lemma 8.3.** When $g, h \in A_5$ are 5-cycles satisfying $g \notin \langle h \rangle$ and $h \notin \langle g \rangle$ the following
statements hold. (a) If $hg$ is a 3-cycle then for some $1 \leq j \leq 4$ the element $h^jg$ is a
product of disjoint transpositions. (b) If $hg$ is a product of disjoint transpositions the
element $h^{-1}g$ is a 3-cycle.

**Proof.** (a) W.l.o.g. $h = (12345)$ and $hg$ fixes 1; hence $g : 1 \mapsto 5$.
If $hg$ also fixes 2 then $g : 2 \mapsto 1$, and therefore $g = (1432)$ (the alternative for $g$ is
$(15432) = h^{-1} \in \langle h \rangle$). Here $h^4g = h^{-1}g = (14)(25)$ is a product of disjoint transpositions.

Similar arguments apply if $hg$ fixes 3, 4 or 5.

(b) W.l.o.g. we may assume $h = (12345)$ and that $hg$ fixes 1, hence that $hg \in
\{(23)(45), (24)(35), (25)(34)\}$.
If $hg = (23)(45)$ then $g = h^{-1} \cdot hg = (15432)(23)(45) = (153)$, contradicting the
assumption that $g$ is a 5-cycle. If $hg = (24)(35)$ then $h^{-1}g = h^4g = h^3 \cdot hg = (14253)(24)$
$(35) = (145)$, as desired. If $hg = (25)(34)$ then we have $g = h^{-1} \cdot hg = (15432)(25)(34) =
(15)(24)$, contradicting the assumption that $g$ is a 5-cycle. □

**Proof of Proposition 8.1(c).** ⇒ A non-identity element of $A_5$ is either a 5-cycle, a
3-cycle, or a product of disjoint transpositions, and therefore $|g|, |h|, |gh| \in \{5, 3, 2\}$. We
assume the usual normalization $|g| \geq |h| \geq |gh|$. Our first task is to prove that $|g| = 5$.

If $|g| = |h| = |gh| = 2$ then $|G| = 4 \neq 60 = |A_5|$, and therefore $G \neq A_5$. If $|g| = 3, |h| =
|gh| = 2$ then $|G| \leq 6 \neq |A_5|$. If $|g| = |h| = 3, |gh| = 2$ then $|G| \leq 12 \neq |A_5|$.

The case $|g| = |h| = |gh| = 3$ reduces to the last considered as follows. Since $g \notin \langle h \rangle$ we
can assume there is a point moved by $g$ but not by $h$, and since each of $g$ and $h$ is of order
three there must be a point moved by both. It follows easily that w.l.o.g. $g = (123)$ and
that $h$ is contained in the set \{234), (243), (235), (254), (245)\}. Hence we have the
following possibilities for $hg$: $\langle 234 \rangle(123) = (13)(24); \langle 243 \rangle(123) = (143); \langle 235 \rangle(123) =
(13)(25); \langle 253 \rangle(123) = (153); \langle 245 \rangle(123) = (14523); and (254)(123) = (15423)$. Since $hg$
assumed a 3-cycle we must be dealing with the second or fourth entries in this list, and
in both instances we see by explicit calculation that $h^{-1}g$ is a product of disjoint
transpositions, hence order two.

We conclude that $|g| = 5$ when $G = A_5$. But in this case $|h| = |gh| = 2$ implies $|G| \leq
10 < |A_5|$, and therefore $|h| \geq 3$ as claimed (assuming the normalization $|g| \geq |h| \geq |gh|$).
⇐ We make use of the fact that $A_5$ has no proper subgroups of order $> 12$ (e.g. see
Exercise 15, p. 69 of Herstein, 1964). Since we are assuming $g$ is a 5-cycle, it follows
immediately that $G = A_5$ if $|h| = 3$. Moreover, if $|h| = |gh| = 2$ then $|G| \leq 10$, and
therefore $G \neq A_5$. If $|h| = 5$ then Lemmas 8.2 and 8.3 in combination guarantee elements
in $G$ of orders three and two, and $G = A_5$ follows. □
9. Proof of Theorem 6.1(c)

The spectrum of a linear operator $S$ is denoted $\sigma(S)$, and the commutator $STS^{-1}T^{-1}$ of operators $S, T$ is written $(S, T)$. When we write $S^n = \pm I$ we mean, in addition to what is written explicitly, that $S^n \neq \pm I$ for all $1 \leq j < n$. Recall that $H$ denotes the two element normal subgroup $\{I, -I\} \subset SL(2, \mathbb{C})$.

The order of a group $G$ is written $|G|$, and when a bound for this order is stated explicitly this bound is computed as in the previous section. The order of an element $g \in G$ is denoted $|g|$.

**Remarks 9.1.** For ease of reference it proves convenient to list the following elementary properties of an operator $S \in SL(2, \mathbb{C})$. (a) The Jordan form of $S$ is either diagonal or one of \[
\begin{pmatrix}
\pm 1 & 1 \\
0 & \pm 1
\end{pmatrix}
\] In particular: the Jordan form is diagonal iff $t_S \neq \pm 2$ or $S = \pm I$; the Jordan form of any element of finite order is diagonal. (b) $S^2 = I$ iff $t_S = \pm 2$ and $S$ has finite order. (c) $S^2 = -I$ iff $t_S = 0$ iff $\sigma(S) = \{i, -i\}$. (d) $S^3 = I$ iff $t_S = -1$ iff $\sigma(S) = \{e^{2\pi i/3}, e^{-2\pi i/3}\}$. (e) $S^3 = -I$ iff $t_S = 1$ iff $\sigma(S) = \{e^{\pi i/3}, e^{-\pi i/3}\}$. (f) $S^4 = I$ iff $S^2 = -I$, in which case (c) applies. (g) $S^4 = -I$ iff $t_S = \sqrt{2}$ or $t_S = -\sqrt{2}$, and the alternatives hold respectively iff $\sigma(S) = \{e^{\pi i/4}, e^{-\pi i/4}\}$ or $\sigma(S) = \{e^{3\pi i/4}, e^{-3\pi i/4}\}$. (h) $S^5 = I$ iff $t_S = -\frac{1}{2}(1 - \sqrt{5})$ or $t_S = -\frac{1}{2}(1 + \sqrt{5})$, and the alternatives hold respectively iff $\sigma(S) = \{e^{2\pi i/5}, e^{-2\pi i/5}\}$ or $\sigma(S) = \{e^{4\pi i/5}, e^{-4\pi i/5}\}$, (i) $S^5 = -I$ iff $t_S = \frac{1}{2}(1 - \sqrt{5})$ or $t_S = \frac{1}{2}(1 + \sqrt{5})$, and the alternatives hold respectively iff $\sigma(S) = \{e^{\pi i/5}, e^{-\pi i/5}\}$ or $\sigma(S) = \{e^{3\pi i/5}, e^{-3\pi i/5}\}$.

We will also make extensive use of the Fricke–Klein formulae

\[
\begin{align*}
(a) \quad t_I &= 2, \\
(b) \quad t_{ST} &= t_{TS}, \quad \text{and} \\
(c) \quad t_{(S,T)} &= t_S^2 + t_T^2 + t_{ST}^2 - t_ST t_{ST} - 2, \quad (9.2)
\end{align*}
\]

valid for any $S, T \in SL(2, \mathbb{C})$. Easy consequences are:

\[
\begin{align*}
(a) \quad t_{TST^{-1}} &= t_S; \\
(b) \quad t_{(S^{-1}T, T^{-1})} &= t_{(S,T)}; \\
(c) \quad t_{(T, TS)} &= t_{(S,T)}; \\
(d) \quad t_{S} &= t_{S^{-1}}; \\
(e) \quad t_{ST} &= t_{ST} = 2; \\
(f) \quad (t_{S^{-1}T} - t_{ST})(t_{S^{-1}T} + t_{ST} - t_{ST}) = 0; \quad \text{and} \\
(g) \quad (t_{ST} - t_{T})(t_{ST} + t_{T} - t_{ST}) = 0. \quad (9.3)
\end{align*}
\]

To prove (a) use (9.2b) with $S$ and $T$ replaced by $TS$ and $T^{-1}$. For (b) check that $(S^{-1}T, T^{-1}) = S^{-1}T^{-1}ST$ and then use (9.2b). The identities (c)–(e) are equally trivial to establish. For (f) note from (9.2c) and (b) of (9.3) that $t_{(S,T)} = t_S^2 + t_T^2 + t_{ST}^2 - t_{S^{-1}T} t_ST - 2$, whereupon subtraction from (9.2c) and factoring gives the result. To obtain (g) simply replace $T$ in (f) by $ST$.

**Lemma 9.4.** Suppose $S, T \in SL(2, \mathbb{C})$, $t_{(S,T)} = 0$, and $G = \langle S, T \rangle$. Then $G$ is a DP-group iff two of the traces $t_S, t_T$ and $t_{ST}$ vanish and the third has absolute value $\sqrt{2}$. In particular, when $G = \langle S, T \rangle$ is a DP-group and $t_{(S,T)} = 0$ holds it must be the case that $\{t_S, t_T, t_{ST}\} \not\subset \{-1, 0, 1\}$.
LEMMA 9.5. For any \( S, T \in \text{SL}(2, \mathbb{C}) \) the condition \( t_{ST} \neq t_{S^{-1}T} \) implies \( t_{S^{-1}T} = t_{ST} - t_{ST} \). On the other hand, if \( t_{(S,T)} = 0 \) and \( t_{S} = \pm 1 \) the opposite condition \( t_{ST} = t_{S^{-1}T} \) implies \( t_{T} \in \{ \pm \frac{2}{3} \sqrt{3} \} \).

PROOF. The first assertion is immediate from (9.3f). For the second assume w.l.o.g. that \( S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \) and that \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Since \( G \) is irreducible (by Theorem 6.1(a) and (9.2c)) and not \( DP \) (by Theorem 6.1(b) and the two trace assumptions) we have \( bc \neq 0 \), and by rescaling the basis, if necessary, we may then achieve \( T = \begin{pmatrix} a & ad - 1 \\ 1 & a \end{pmatrix} \). Now check that \( t_{S^{-1}T} = t_{ST} \) iff \( (\lambda^2 - 1)(a - d) = 0 \), hence by Remarks 9.1(d) and (e) iff \( a = d \), and we are reduced to the case \( T = \begin{pmatrix} a & a^2 - 1 \\ 1 & a \end{pmatrix} \). But for \( T \) in this last form direct calculation using Remark 9.1(j) and (9.3e) gives \( t_{(S,T)} = 3a^2 - 1 = 3(t_{T}/2)^2 - 1 \), and the result now becomes a consequence of the assumption \( t_{(S,T)} = 0 \).

The following gives (ci) of Theorem 6.1.

PROPOSITION 9.6. Suppose \( S, T \in \text{SL}(2, \mathbb{C}) \) and \( G := \langle S, T \rangle \). Then \( G \) is projectively tetrahedral iff \( t_{S}^2 + t_{T}^2 + t_{ST}^2 - t_{STT}t_{ST} = 2 \) and \( t_{S}, t_{T}, t_{ST} \in \{ -1, 0, 1 \} \).

PROOF. \( \Rightarrow \) If \( H \subset G \) and \( G/H \simeq A_4 \) let \( [P] \) denote the equivalence class (i.e. coset) in \( G/H \) of \( P \in G \). Then for \( P \notin H \) we have \([P]^{3} = e \) or \([P]^{2} = e \), which we note is the case iff \( P^{1} = \pm I \) or \( P^{1} = -I \). \( (P^{2} = I) \) is impossible since \( P \in \text{SL}(2, \mathbb{C}) \) and \( P \notin H \).

In view of (b)–(d) of Remarks 9.1 these two conditions are respectively equivalent to \( t_{P} \in \{ -1, 1 \} \) and \( t_{P} = 0 \), and \( t_{S}, t_{T}, t_{ST} \in \{ -1, 0, 1 \} \) follows.

Since \( [S] \) and \( [T] \) generate \( G/H \) Proposition 8.1(a) implies that at least one of \([S]\) and \([T]\) is a 3-cycle; hence \( \sigma := t_{S}^{2} + t_{T}^{2} + t_{ST}^{2} \in \{ 1, 2, 3 \} \).

If \( \sigma = 1 \) the elements \([S],[T]\) and \([TS]\), perhaps after relabelling and/or replacing generators with inverses, satisfy the relations \( s^{2} = t^{2} = (ts)^{3} = e \). But this implies \( |G/H| \leq 6 \), and therefore \( |G| \leq 12 \). Since this contradicts \( |G| = 24 \), we conclude that \( \sigma = 1 \) is impossible.

If \( \sigma = 2 \) we are done.

If \( \sigma = 3 \) then \([ST]\) is a 3-cycle and by Proposition 8.1 the element \([TS^{-1}]\) must be a product of disjoint transpositions. In particular, \([TS^{-1}]^{2} = e \). Remark 9.1(c) then implies \( t_{STS^{-1}} = t_{S^{-1}T} = 0 \), whence \( t_{ST} = t_{ST} \) by (9.3f), and therefore \( t_{STT}t_{ST} = 1 \). But then \( t_{S}^{2} + t_{T}^{2} + t_{ST}^{2} - t_{STT}t_{ST} = \sigma - 1 = 2 \), precisely as desired.

\( \Leftarrow \) By (9.2c) the hypotheses imply \( t_{(S,T)} = 0 \), hence \((S,T)^{2} = -I \in G \) by Remark 9.1(c), and therefore \( H \subset G \). Moreover, from Theorem 6.1(a) we see that \( G \) is not reducible, and from Lemma 9.4 that \( G \) is not \( DP \). As a result of Proposition 2.8(c) we conclude that \( |G| \geq 24 \) (with \( |G| = \infty \) a possibility).

If \( P \in G \) and \( t_{P} = -1 \), then by replacing \( P \) with \(-P\) we may assume \( t_{P} = 1 \). Notice
that such replacements with $P = S, T$ do not alter the value of $t_{ST} t_{ST}$. By relabelling as needed we may then reduced to the cases $t_S = t_T = 1$ and $t_{ST} = 1$ or 0.

If $t_{ST} = 1$ then $t_{S−1} T = 0$ by (the two alternatives in) Lemma 9.5, and by replacing $S$ with $S^{−1}$ we have only the second case to consider. But from Remark 9.2(c)–(e) these conditions imply $|S|^3 = |T|^3 = |TS|^2 = 1$, which is a standard presentation of $A_4$. Since $|G/H| ≥ 12 = |A_4|$ the resulting epimorphism $φ : A_4 → G/H$ must be an isomorphism, and $G$ is therefore projectively tetrahedral. □

The following is (cii) of Theorem 6.1.

**Proposition 9.7.** Suppose $S, T ∈ SL(2, C)$ and $G := \langle S, T ⟩$. Then $G$ is projectively octahedral iff $t_S^2 + t_T^2 + t_{ST}^2 − t_{ST} t_{ST} = 3$ and $t_S, t_T, t_{ST} ∈ \{-√2, −1, 0, 1, √2\}$.

**Proof.** ⇒ As in the previous proof let $[P] ∈ G/H$ denote the equivalence class of $P ∈ G$. Assuming $G/H ∼ S_4$ and $P ∉ H$ the possibilities are now $|P|^4 = 1$, $|P|^3 = 3$, or $|P|^2 = 3$, which by the appropriate items in Remarks 9.1 force $t_P ∈ \{0, ±1, ±√2\}$, as asserted.

By relabelling, if necessary, we may assume $|t_S| ≥ |t_T| ≥ |t_{ST}|$, whereupon from Proposition 8.1(b) and Remarks 9.1 we see that $|t_S| = √2$ and that $σ := t_S^2 + t_T^2 + t_{ST}^2 ∈ \{3, 4, 5, 6\}$.

If $σ = 3$ the desired result is immediate.

If $σ = 4$ there are two possibilities: $|t_T| = √2$ and $t_{ST} = 0$; and $|t_T| = |t_{ST}| = 1$. In the first instance (9.2c) gives $t_{(S,T)} = 2$, $G$ is therefore reducible, and $G/H ∼ S_4$ is then impossible by Proposition 2.8(c). In the second instance we have $t_{(S,T)} = t_S^2 + t_T^2 + t_{ST}^2 − t_{ST} t_{ST} = 2 = 4 ± √2$. which is also impossible.

If $σ = 5$ we have $|t_T| = √2$ and $|t_{ST}| = 1$, and therefore $t_{ST} t_{ST} = ±2$. But $t_{ST} t_{ST} = −2$ implies $t_{(S,T)} = t_S^2 + t_T^2 + t_{ST}^2 − t_{ST} t_{ST} = 2 = 5 ∈ \{0, ±1, ±√2\}$, which is impossible, and the other alternative gives the desired result.

If $σ = 6$ we have $t_{ST} t_{ST} = ±2 √2$, leading immediately to the contradiction $t_{(S,T)} = 4 ± 2 √2 < 0, ±1, ±√2\}.

⇐ By Theorem 6.1(a) the group $G$ is not reducible. If $G$ is $DP$ assume w.l.o.g. that $S ∈ D$ and $T ∈ P$. Then $ST ∈ P$, hence $t_T = t_{ST} = 0$, and therefore $t_S^2 + t_T^2 + t_{ST}^2 < 3$, contradicting the hypotheses.

We conclude that $G$ is neither reducible nor $DP$. Moreover, it is also not projectively tetrahedral since the conditions of Proposition 9.6 are not satisfied. In particular, by Proposition 2.8(c) the group $G$, if finite, must have order 48 or 120.

We claim $−I ∈ G$. Indeed, if $0 ∈ \{t_S, t_T, t_{ST}\}$ this is immediate from Remark 9.1(c), so assume otherwise. Then $t_S^2 + t_T^2 + t_{ST}^2 > 3$, implying w.l.o.g. $|t_S| = √2$, whereupon (9.3d) gives $t_{ST} = t_S^2 − 2 = 0$, and we can once again appeal to Remark 9.1(c).

As in the previous proof, when $t_P < 0$ we can achieve $t_P > 0$ by replacing $P$ by $−P$, and such replacements with $P = S, T$ do not alter the value of $t_{ST} t_{ST}$. By relabelling, if necessary, we may then assume $t_S ≥ t_T ≥ t_{ST}$ with (at least) $t_S ≥ t_T ≥ 0$. Using the identity $t_S^2 + t_T^2 + t_{ST}^2 − t_{ST} t_{ST} = 3$ one is then quickly reduced to considering the possibilities $(t_S, t_T, t_{ST}) = (√2, √2, 1)$ and $(t_S, t_T, t_{ST}) = (√2, 1, 0)$.

When $(t_S, t_T, t_{ST}) = (√2, √2, 1)$ we can choose a basis so as to identify $S$ with the matrix $\begin{pmatrix} λ & 0 \\ 0 & λ^{-1} \end{pmatrix}$, where $λ = ± √2(1 ± i)$, and then write $T = \begin{pmatrix} a & ad − 1 \\ 1 & d \end{pmatrix}$, where the
lower left entry normalization in $T$ is possible since $G$ is not reducible and not $DP$. From $t_T = \sqrt{2}$ we have $d = \sqrt{2} - a$, whereupon solving $t_{ST} = 1$ results in $a = \sqrt{2}/2$. Direct calculation now gives $t_{ST^2} = 0$, and since $(S,T) = (S,ST)$ we see that by replacing $T$ with $ST$ we are reduced to the second case.

When $(t_S, t_T, t_{ST}) = (\sqrt{2}, 1, 0)$ we have $[S]^4 = [T]^3 = [ST]^2 = e$, which is a standard presentation of $S_4$. From the resulting epimorphism $\varphi : S_4 \to G/H$ we see that $|G/H| \leq 24$, and since $|G| = 2|G/H| \geq 48$ we conclude that $\varphi$ is an isomorphism. \hfill $\Box$

In considering the icosahedral case the following lemma proves useful.

**Lemma 9.8.** Suppose $S, T \in SL(2, C)$ and $t_S \notin \{-2, 2\}$. Then: (a) $t_T = t_{ST} = t_{S^{-1}T}$ implies $t_T = 0$; (b) $t_{ST} = 1$ and $t_{ST^2} = t_S$ imply $t_T = 0$; (c) $t_S \neq 0$, $t_{ST^2} = 1$, and $t_{ST} = (t_S)^{-1}$ imply $t_T = 0$; and (d) $0 \neq t_S = (t_T)^{-1}$ and $t_{ST} = 0$ imply $t_{S^{-1}T} = 1$.

**Proof.** The hypothesis on $S$ guarantees diagonalizability, and we may therefore assume $S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) $t_{ST} = t_{S^{-1}T}$ easily implies $a = d$, whereupon $t_{ST} = t_T$ implies $a = 0$.

(b) The combination $t_{ST} = 1$ and $t_{ST^2} = t_S$ is equivalent to the system

\begin{align*}
\lambda^2 a + d &= \lambda \\
\lambda^4 a + d &= \lambda^3 + \lambda
\end{align*}

with unique solution $a = \lambda/(\lambda^2 - 1) = -d$.

(c) $t_{ST^2} = 1$ implies $\lambda^2 a + \lambda^{-2} d = 1$, whereupon $t_{ST} = (t_S)^{-1}$ generates the following sequence of implications: $\lambda a + \lambda^{-1} d = (\lambda + \lambda^{-1})^{-1} \Rightarrow 1 = (\lambda + \lambda^{-1})(\lambda a + \lambda^{-1} d) = \lambda^2 a + \lambda^{-2} d + a + d \Rightarrow a + d = t_T = 0$.

(d) $\lambda + \lambda^{-1} = (a + d)^{-1} \Rightarrow 1 = \lambda a + \lambda^{-1} d + \lambda^{-1} a + \lambda d = t_{ST} + t_{S^{-1}T} = t_{S^{-1}T}$. \hfill $\Box$

The following is the final case of Theorem 6.1.

**Proposition 9.9.** Suppose $S, T \in SL(2, C)$ and $G := (S,T)$. Set $\mu_1 := \frac{1}{2}(1 + \sqrt{5})$ and $\mu_2 := \mu_1^{-1} = \frac{1}{2}(1 - \sqrt{5})$. Then $G$ is projectively icosahedral iff $t_S^2 + t_T^2 + t_{ST}^2 - t_ST t_{ST} \in \{2 - \mu_2, 3, 2 + \mu_1\}$ and $t_S, t_T, t_{ST} \in \{-\mu_1, -\mu_2, 0, \mu_2, 1, \mu_1\}$.

**Proof.** $\Rightarrow$ Again for $P \in G$ we let $[P]$ denote the equivalence class of $P$ in $G/H$. Here for $P \notin H$ the possibilities are $[P]^5 = e$, $[P]^3 = e$, and $[P]^2 = e$, which by the appropriate entries in Remark 9.1 will hold iff $t_P \in \{-\mu_1, -\mu_2, \mu_2, \mu_1\}$, $t_P \in \{-1, 1\}$, and $t_P = 0$ respectively. The membership assertion follows easily.

When $(S,T) = A_5$ the usual relabellings and/or replacements (including by negatives) allow us to assume $t_S \geq t_T \geq |t_{ST}| \geq 0$. Moreover, Proposition 8.1(c) and Remarks 9.1(h) and (i) then guarantee $t_S \in \{\mu_1, 1, \mu_2\}$, with $\{t_T, t_{ST}\} \cap \{\pm \mu_1, \pm \mu_2\} \neq \emptyset$ if $t_S = 1$, and at most one of $t_T$ and $t_{ST}$ contained in $\{-2, 0, 2\}$. By listing all possibilities and checking each for $t_{(S,T)} = t_S^2 + t_T^2 + t_{ST}^2 - t_ST t_{ST} - 2 \in \{-\mu_1, -1, -\mu_2, 0, \mu_2, 1, \mu_1\}$ one finds that
all conditions are satisfied only in the following ten cases:

<table>
<thead>
<tr>
<th>Case</th>
<th>( (t_S, t_T, t_{ST}) )</th>
<th>( t_{(S,T)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>((\mu_1, \mu_1, \mu_1))</td>
<td>(\mu_1)</td>
</tr>
<tr>
<td>2.</td>
<td>((\mu_1, \mu_1, 1))</td>
<td>(\mu_1)</td>
</tr>
<tr>
<td>3.</td>
<td>((\mu_1, 1, 1))</td>
<td>1</td>
</tr>
<tr>
<td>4.</td>
<td>((\mu_1, 1, \mu_2))</td>
<td>1</td>
</tr>
<tr>
<td>5.</td>
<td>((\mu_1, 1, 0))</td>
<td>(\mu_1)</td>
</tr>
<tr>
<td>6.</td>
<td>((\mu_1, \mu_2, 0))</td>
<td>1</td>
</tr>
<tr>
<td>7.</td>
<td>((1, 1, -\mu_2))</td>
<td>1</td>
</tr>
<tr>
<td>8.</td>
<td>((1, \mu_2, \mu_2))</td>
<td>(-\mu_2)</td>
</tr>
<tr>
<td>9.</td>
<td>((1, \mu_2, 0))</td>
<td>(-\mu_2)</td>
</tr>
<tr>
<td>10.</td>
<td>((\mu_2, \mu_2, -\mu_2))</td>
<td>(-\mu_2)</td>
</tr>
</tbody>
</table>

It is immediate that in each instance we have \(t_S^2 + t_T^2 + t_{ST}^2 - t_{ST}t_{ST} \in \{2 - \mu_2, 3, 2 + \mu_1\}\).

\(< \) Since \(t_{(S,T)} \neq 2\) the group is not reducible. We claim it is also not \(DP\). Otherwise by Theorem 6.1(b) we have w.l.o.g. that \(t_T = t_{ST} = 0\), in which case \(t_S^2 \in \{2 - \mu_2, 3, 2 + \mu_1\}\), and this is easily seen to be impossible for \(t_S \in \{0, \pm\mu_2, \pm 1, \pm \mu_1\}\). We are thus left to examine the non-reducible, non-\(DP\) group \(G = \langle S, T \rangle\) in the ten cases listed above.

Case 1: This reduces to Case 2 by replacing \(S\) with \(S^{-1}\). Indeed, by Lemma 9.8(a) we have \(t_{S^{-1}} \neq t_{ST}\), and therefore \(t_{S^{-1}} = t_{ST} - t_{ST} = \mu_2^2 - \mu_1 = 1\) by (9.3f).

Case 2: This reduces to Case 5 by replacing \(S\) by \(S^{-1}\) and \(T\) by \(ST\). Here Lemma 9.8(b) implies \(t_{ST} \neq t_S\), and therefore \(t_{ST} = t_{ST} - t_T = \mu_1 - 1 = 0\) by (9.3g).

Cases 1 and 2 thus reduce to Case 5.

Case 3: This reduces to Case 4 by replacing \(S\) by \(S^{-1}\). In this instance Lemma 9.8(a) implies \(t_{S^{-1}} \neq t_{ST}\), whereupon (9.3f) gives \(t_{S^{-1}} = t_{ST} - t_{ST} = \mu_1 - 1 = \mu_2\).

Case 4: This reduces to (a permutation and hence relabelling of) Case 9 by replacing \(S\) by \(ST\). For here Lemma 9.8(c) shows that \(t_{ST} \neq t_T\), whereupon (9.3g) gives \(t_{ST} = t_{ST} - t_T = \mu_1 \mu_2 - 1 = 0\).

Cases 3 and 4 thus reduce to Case 9.

Case 6: This reduces to Case 4 by replacing \(T\) by \(S^{-1}T\). Indeed, by Lemma 9.8(d) we have \(t_{S^{-1}} = 1\).

Case 7: If \(S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\) and \(T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), and if \(S^2 \neq I\), then \(t_{S^{-1}} = t_{ST} \Rightarrow a = d, and therefore t_{ST} = a(\lambda + \lambda^{-1}). But here t_S = \lambda + \lambda^{-1} = 1 and t_T = 1 would imply a = \frac{1}{2}, in which case t_{ST} = \frac{1}{2} \neq -\mu_2. We conclude that t_{S^{-1}} \neq t_{ST}, hence from (9.3f) that t_{S^{-1}} = t_{ST} - t_T = 1 + \mu_2 = \mu_1. Replacing S by S^{-1} thus reduces us to (a permutation of) Case 3.

Cases 6 and 7 are thus reduced to Cases 4 and 3, which have in turn been reduced to Case 9.

Case 8: As in the argument for Case 7 we have \(t_{S^{-1}} = t_{ST}\) iff \(a = d, and here the contradiction is 2a = t_T = a(\lambda + \lambda^{-1}) = at_S = a. Thus t_{S^{-1}} = t_{ST} - t_{ST} = 0, and so replacing S by S^{-1} reduces us to Case 9.

Case 10: Once again mimicking the argument for Case 7 we see that \(t_{S^{-1}} = t_{ST}\) iff \(a = d. If this is true then \mu_2 = t_S = \lambda + \lambda^{-1} = -t_{ST} = -a(\lambda + \lambda^{-1})\), hence \(a = -1, and thus the contradiction t_T = -2 \neq \mu_2. We conclude that t_{S^{-1}} = t_{ST} - t_{ST} = \mu_2^2 + \mu_2 = \mu_2 \mu_1 = 1 (see (9.3f)), and so replacing S by S^{-1} we find ourselves (after relabelling) as in Case 8, which we have previously seen can be reduced to Case 9.
We have thus reduced all cases to ones involving w.l.o.g. $(\mu_j, 1, 0)$, where $j = 1$ or 2. By Remark 9.1 we have $-I \in G$ and w.l.o.g. $[S]^5 = [T]^3 = [TS]^2 = \epsilon$ in $G/H$, which is a standard presentation of $A_5$. From the resulting epimorphism $\varphi : A_5 \to G/H$ we see that $|G/H| \leq 60$, whereas $|G/H|$ must be divisible by 30 since it admits elements of orders two, three and five. The alternatives of Proposition 2.8(c) now force the order to be 60, and $\varphi$ is therefore an isomorphism.

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References


Two Generator Subgroups of $SL(2, \mathbb{C})$


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