Stabilizing Heegaard splittings of toroidal 3-manifolds

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Abstract

Let T be a separating incompressible torus in a 3-manifold M. Assuming that a genus g Heegaard splitting $V \cup_S W$ can be positioned nicely with respect to T (e.g., $V \cup_S W$ is strongly irreducible), we obtain an upper bound on the number of stabilizations required for $V \cup_S W$ to become isotopic to a Heegaard splitting which is an amalgamation along T. In particular, if T is a canonical torus in the JSJ decomposition of M, then the number of necessary stabilizations is at most $4g - 4$. As a corollary, this establishes an upper bound on the number of stabilizations required for $V \cup_S W$ and any Heegaard splitting obtained by a Dehn twist of $V \cup_S W$ along T to become isotopic.

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1. Introduction

Recent study of Heegaard splittings indicates that generically, Heegaard splittings of Haken manifolds are amalgamations along incompressible surfaces; that is, they can be decomposed into Heegaard splittings of the manifolds obtained by cutting along those surfaces (see, e.g., [3] for the genus 1 case, and [2,8,9,17] and for the genus $\geqslant 2$ case). There are many examples, however, of Heegaard splittings (such as strongly irreducible splittings) that are not of this nature. In this paper we investigate the question of how many stabilizations are needed to make a Heegaard splitting isotopic to an amalgamation along an incompressible torus.

The peculiarity of Heegaard splittings of 3-manifolds containing incompressible tori can be seen in the recent establishment of the generalized Waldhausen conjecture [7,10], which states that a 3-manifold has only finitely many Heegaard splittings of a given genus up to isotopy, assuming the 3-manifold contains no incompressible tori. If, however, a 3-manifold contains an incompressible torus T, then taking a given Heegaard splitting and Dehn twisting along T can yield infinitely many Heegaard splittings of the same genus (see, e.g., [1]). Hence, a question one may ask is how many stabilizations are needed for these Heegaard splittings to become isotopic.

Upon consideration of these questions, we have the following results (see below for relevant definitions):

Theorem 1.1. *Let M be a closed, orientable 3-manifold, and let T be a separating incompressible torus in M. If $V \cup_S W$ is a Heegaard splitting of M that can be isotoped so that $V \cap T$ consists of k annuli, then after at most k stabilizations, $V \cup_S W$ is isotopic an amalgamation along T.*
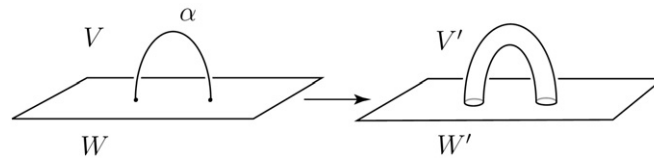
E-mail address: rdt@aucegypt.edu.

Fig. 1. A stabilization of $V \cup_S W$.

In particular, if $V \cup_S W$ is strongly irreducible then (after possible isotopy) the hypotheses of the theorem are satisfied as S can always be isotoped to intersect T in essential simple closed curves (see, e.g., [15]). We can further refine the above bound by restricting our choice of T.

Corollary 1.2. *Let M be a closed, orientable, irreducible 3-manifold, and let T be a separating canonical torus in the JSJ decomposition of M. If $V \cup_S W$ is a genus g Heegaard splitting of M that can be isotoped so that $V \cap T$ consists of annuli, then after at most $4g - 4$ stabilizations $V \cup_S W$ is isotopic an amalgamation along T.*

The assumption that M is irreducible is added here as it is required by the definition of the JSJ decomposition of M (see Section 6). As an immediate consequence we obtain:

Corollary 1.3. *Let M be a closed, orientable (irreducible) 3-manifold, and let T be a separating incompressible torus (canonical torus in the JSJ decomposition) in M. Suppose $V \cup_S W$ and $P \cup_\Sigma Q$ are genus g Heegaard splittings of M such that $P \cup_\Sigma Q$ is obtained from $V \cup_S W$ via any power of a Dehn twist along T. Moreover, assume that $V \cup_S W$ can be isotoped so that $V \cap T$ consists of k annuli. Then $V \cup_S W$ and $P \cup_\Sigma Q$ are isotopic after at most k stabilizations ($4g - 4$ stabilizations).*

2. Definitions

Let M be a closed, orientable 3-manifold. Definitions of standard terms regarding 3-manifolds can be found, for example, in [6] and [12].

Definition 2.1. A *Heegaard splitting* for M is a decomposition of M into two handlebodies V and W of the same genus such that M is obtained as the identification space of V and W identified along their boundaries via some homeomorphism from ∂V to ∂W.

The closed orientable surface $S = \partial V = \partial W$ is called the *splitting surface*, and we write this Heegaard splitting as $V \cup_S W$. For convenience, we do not distinguish between V, W and S, and their respective embeddings in M. The *genus* of $V \cup_S W$ is defined to be the genus of S. Two Heegaard splittings $V \cup_S W$ and $P \cup_\Sigma Q$ are said to be *isotopic* if there exists an isotopy of M taking V to P.

Given a Heegaard splitting $V \cup_S W$ of M, one can generate new Heegaard splittings of M. Let α be a properly embedded, boundary parallel arc in one of the handlebodies, say V. Create a new handlebody W' in M by attaching a 1-handle X to W along the boundary such that α is the core of X. As α is boundary parallel in V, $V' = \overline{V - X}$ is also a handlebody.

Definition 2.2. The Heegaard splitting $V' \cup_{S'} W'$ resulting from the above process is called a *stabilization* of $V \cup_S W$. A Heegaard splitting which is obtained by a stabilization of another Heegaard splitting is called *stabilized*.

Note that $V' \cup_{S'} W'$ has genus one larger than $V \cup_S W$. Repeating the above process k times yields a *k-times stabilization* of $V \cup_S W$, or $V \cup_S W$ *stabilized k times*. It is a nice exercise to show that a Heegaard splitting being stabilized is equivalent to the property that there exist properly embedded essential disks in each of the handlebodies that intersect in a single point. Moreover, two Heegaard splittings obtained by stabilization of the same splitting are isotopic. A classical theorem of Reidemeister and Singer [11,16] states that any two Heegaard splittings of the same manifold can each be stabilized some indefinite number of times to become isotopic.
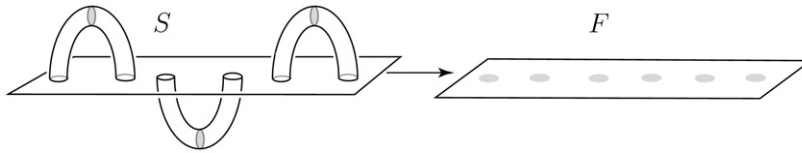
Fig. 2. An amalgamation along F.

Definition 2.3. A Heegaard splitting $V \cup_S W$ of M is said to be *reducible* if there exist essential disks $D \subset V$ and $E \subset W$ such that $\partial D = \partial E$. Otherwise $V \cup_S W$ is called *irreducible*.

Definition 2.4. A Heegaard splitting $V \cup_S W$ is said to be *weakly reducible* if there are essential disks $D \subset V$ and $E \subset W$ such that $\partial D \cap \partial E = \emptyset$. Otherwise it is called *strongly irreducible*.

A reducible Heegaard splitting is easily seen to be weakly reducible. As referred to in the introduction, a useful property of a strongly irreducible Heegaard splitting is that it can be isotoped to intersect an incompressible surface in simple closed curves essential on both surfaces. A result of Haken [5] implies that irreducible Heegaard splittings arise only in irreducible manifolds.

Definition 2.5. Let F be a separating incompressible surface in M. A Heegaard splitting $V \cup_S W$ of M is called an *amalgamation along F* if (after isotopy) F is obtained from simultaneous compressions on both sides of S.

This definition of amalgamation only makes sense in the context that M is closed. The more general definition can be found, e.g., in [14].

The condition that F is obtained from simultaneous compressions on both sides of S is equivalent to saying that S can be obtained from F by a series of ambient 1-surgeries on pairwise disjoint arcs properly embedded in M cut along F. Also note that amalgamation is not unique; $V \cup_S W$ can be an amalgamation along several different surfaces in M. By [4], an irreducible Heegaard splitting is either strongly irreducible or an amalgamation along some incompressible surface.

3. Amalgamations

The purpose of this section is to determine when a Heegaard splitting is an amalgamation along a given incompressible torus. Let A be an annulus properly embedded in a handlebody V. We say A is *essential* in V if it is incompressible and not boundary parallel in V. A *spanning arc* of A is an essential arc in A, i.e. a properly embedded arc in A that cuts A into a disk. A *spanning disk* for A in V is a boundary compressing disk for A in V, i.e. a disk D such that $\partial D = a \cup b$ where $a = D \cap A$ is a spanning arc of A and $b = D \cap \partial V$. Note that this implies D is essential in V cut along A.

Lemma 3.1. *Any essential annulus A in a handlebody V has a spanning disk.*

Proof. Using Van Kampen's Theorem one can compute a presentation for $\pi_1(V)$ using the components of V cut along A. As $\pi_1(V)$ is a free group, the generator of $\pi_1(A)$ must be mapped to a generator in the fundamental group of one of the components of V cut along A. This implies that there exists an essential disk in this component which meets A in a single spanning arc. Such a disk is a spanning disk of A in V.   □

Lemma 3.2 *(The Amalgamation lemma). Suppose T is a separating incompressible torus in M so that $M = N_0 \cup_T N_1$ and let $V \cup_S W$ be a Heegaard splitting of M. Then $V \cup_S W$ is an amalgamation along T if and only if there is an isotopy of S to a surface intersecting T in essential simple closed curves such that each (annulus) component of $V \cap T$ has a spanning disk in V contained in N_ε, and each (annulus) component of $W \cap T$ has a spanning disk in W contained in $N_{\varepsilon'}$ where $\{\varepsilon, \varepsilon'\} = \{0, 1\}$.*
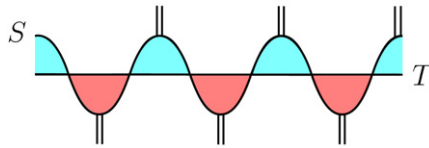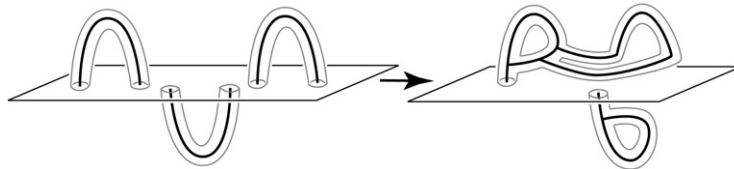
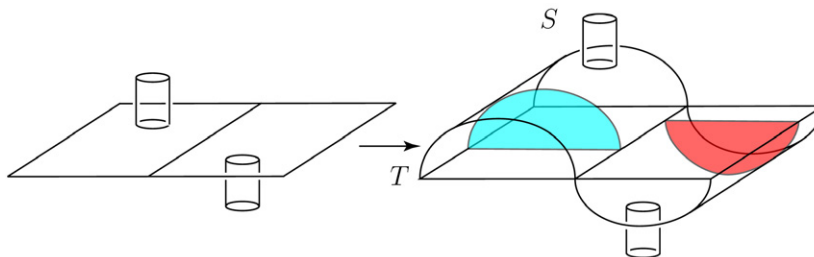Fig. 3. A schematic for the Amalgamation lemma.

Fig. 4.

Fig. 5.

Proof. Assume first that $V \cup_S W$ is an amalgamation along T. Then S can be obtained from T by ambient 1-surgery on pairwise disjoint arcs properly embedded in N_0 and N_1. These arcs can be isotoped by arc slides so that on each side of T only one arc meets T in one of its ends (see Fig. 4). Thus after isotopy, $S \cap T = \overline{T - (D_1 \cup D_2)}$, where D_1 and D_2 are disks in T. Isotope S into N_0 and N_1 as in Fig. 5 so that $S \cap T$ consists of 2 simple closed curves essential in T. They are essential in S as well by the fact that T is incompressible. The existence of spanning disks on opposite sides of T for the resulting 2 annuli is then obvious, as in Fig. 5.

For the other direction assume without loss of generality that each component of $V \cap T$ has a spanning disk in V contained in N_0, and each component of $W \cap T$ has a spanning disk in W contained in N_1. Let $A_1, \ldots, A_n$ be the components of $V \cap T$ with respective spanning disks $D_1, \ldots, D_n$, and $B_1, \ldots, B_n$ the components of $W \cap T$ with respective spanning disks $E_1, \ldots, E_n$. By the definition of spanning disk and the fact that D_i is contained in N_0, it follows that $D_i \cap T$ is a single spanning arc of A_i, for $1 \leqslant i \leqslant n$. Similarly, $E_i \cap T$ is a single spanning arc of B_i, for $1 \leqslant i \leqslant n$.

By a standard innermost disk, outermost arc argument, we may assume that $D_1, \ldots, D_n, E_1, \ldots, E_n$ are all pairwise disjoint. Each spanning disk D_i defines an isotopy of S in the following way. Let $N(D_i)$ be a neighborhood of D_i such that $N(D_i) \cap S = \partial N(D_i) \cap S$ is a neighborhood of $\partial D_i \cap S$ in S, and $N(D_i) \cap T = \partial N(D_i) \cap T$ is a neighborhood of $\partial D_i \cap T$ in T. Then S can be isotoped so that $N(D_i) \cap S$ is replaced by $\partial N(D_i) - (N(D_i) \cap S)$. After further isotopy, S intersects T in a punctured annulus (see Fig. 6). After performing such an isotopy for each of the spanning disks $D_1, \ldots, D_n, E_1, \ldots, E_n$, S is such that $T - S$ is a disjoint union of open disks.

If any such open disk U is such that ∂U bounds a disk U_S in $S - T$, then by the fact that a handlebody is irreducible, $U \cup U_S$ bounds a ball. Isotope U_S through this ball so that its interior equals U. Now, for a remaining component V' of $V \cap N_0$, $V' \cap T$ is a disjoint union of disks whose interiors are a subset of the disks in $T - S$. As V' is a component of V cut along disks, V' is a handlebody and hence has a spine. After arc slides, each arc of this spine has its endpoints on T. Doing this for each component of $V \cap N_0$ and $W \cap N_1$ shows that S is obtained from T via ambient 1-surgery along these slid arcs of the spines, implying that $V \cup_S W$ is an amalgamation along T. $\quad \square$
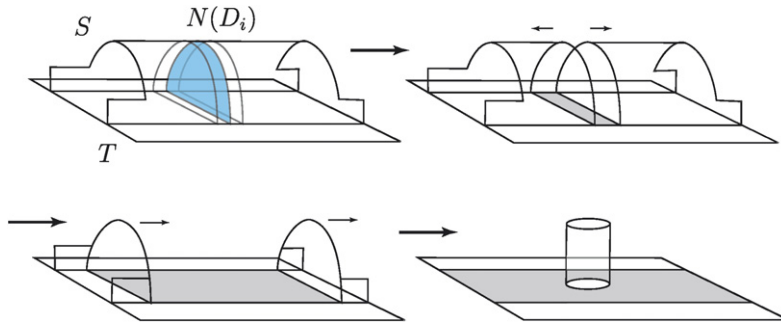
Fig. 6. Isotoping S using the disk D_i.

Remark 3.3. Suppose that a Heegaard splitting $V \cup_S W$ can be isotoped so that $V \cap T$ consists of a disjoint union of k annuli. Lemma 3.2 indicates that a $2k$-times stabilization of $V \cup_S W$ is an amalgamation along T in the following way. Add a 1-handle to either V or W such that its core is a spanning arc of a component of $W \cap T$ or $V \cap T$, respectively. The cocore of the 1-handle and a spanning disk for the annulus (which exists by Lemma 3.1) intersect in a single point, showing that the resulting splitting is a stabilization of $V \cup_S W$. By adding 1-handles disjoint from each other in this manner for every component of T cut along S (of which there are $2k$) and then pushing the handles added to V into N_1 and the handles added to W into N_0, we get the spanning disks needed for Lemma 3.2 to apply.

The purpose of the next sections is to show that the k 1-handles added to one of the handlebodies in Remark 3.3 are in fact unnecessary to obtain an amalgamation.

4. Annuli in handlebodies

This section provides the necessary technical arguments used in the proof of Theorem 1.1 to find spanning disks for annuli in a handlebody. Suppose a Heegaard splitting $V \cup_S W$ of M can be isotoped so that $V \cap T$ is a disjoint union $\mathcal{A}$ of annuli. Then by removing any boundary parallel components via additional isotopy we can assume the components of $\mathcal{A}$ are essential in V. Let Δ be a complete system of meridian disks for V.

Remark 4.1. Applying a standard cut and paste argument to Δ if necessary, we can assume that each annulus component of $\mathcal{A}$ meets Δ nontrivially, and so that $\mathcal{A} \cap \Delta$ consists of arcs properly embedded in Δ, each arc being a spanning arc of some annulus in $\mathcal{A}$.

For D a component of Δ, $D \cap \mathcal{A}$ is a collection of properly embedded arcs $\alpha_1, \ldots, \alpha_r$ in D.

Definition 4.2. Suppose that γ is an arc in $D \cap \mathcal{A}$. Then γ divides D into two disks D' and D''. Let $\mathcal{B}'$ be the set of properly embedded arcs in D' with one endpoint on γ and the other endpoint on ∂D, intersecting each α_i in at most one point. Define

$$\ell_{D'}(\gamma) = \max_{\beta \in \mathcal{B}'} \left\{ \left| \beta \cap \left(\bigcup_{i=1}^{r} \alpha_i \right) \right| \right\}.$$

Define $\ell_{D''}(\gamma)$ similarly. Then we define the *level of γ (in D)* to be

$$\ell(\gamma) = \min\{\ell_{D'}(\gamma), \ell_{D''}(\gamma)\}.$$

For A a component of $\mathcal{A}$, define the *level of A (with respect to Δ)* to be

$$\ell(A) = \min\{\ell(\gamma)\},$$

where γ is an arc component of $A \cap \Delta$.

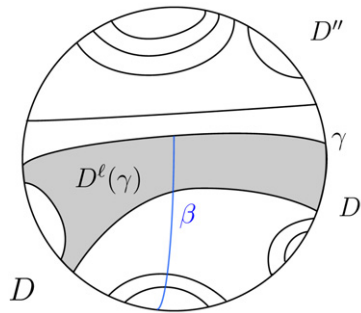Note that an outermost arc for a disk D is a level one arc.

Fig. 7. The level adjacent disk component and an arc $\beta \in \mathcal{B}'$ realizing γ as a level 4 arc in D.

Definition 4.3. Let A be a component of $\mathcal{A}$ and let D be a spanning disk for A such that $D \cap A = \gamma$. Define the *adjacent disk component of γ with respect to D*, denoted $D(\gamma)$, to be the component of D cut along $\mathcal{A}$ that contains γ.

Let γ be an arc in $\mathcal{A} \cap \Delta$, and suppose D is the disk component of Δ containing γ. Define the *level adjacent disk component of γ*, denoted $D^\ell(\gamma)$, to be the adjacent disk component of D' or D'', whichever one is where γ realizes its level (if they both realize the level, choose one). See Fig. 7.

Given a separating incompressible torus T in M, T divides M into two components N_0 and N_1, where $\partial N_0 = \partial N_1 = T$. Since $V \cap T$ is assumed to be a disjoint union of essential annuli, and since T is separating in M, each component of $V \cap T$ has an arbitrarily small neighborhood in V intersecting each of N_0 and N_1 nontrivially. This gives rise to the following definition:

Definition 4.4. Let $\mathcal{A}$ be a disjoint union of annuli properly embedded in a handlebody V. We shall say that $\mathcal{A}$ is *mutually separating* if V cut along $\mathcal{A}$ consists of two (possibly non-connected) 3-manifolds N_0 and N_1 such that each annulus in $\mathcal{A}$ has an arbitrarily small neighborhood in V intersecting both N_0 and N_1 nontrivially.

Remark 4.5. The condition that $\mathcal{A}$ is mutually separating in V is equivalent to the statement that $(\mathcal{A}, \partial \mathcal{A})$ represents the trivial element in $H_2(V, \partial V; \mathbb{Z}/2\mathbb{Z})$.

Note that $\mathcal{A}$ can be mutually separating even if some of its component annuli are themselves nonseparating in V.

Lemma 4.6. *Let $\mathcal{A}$ be a mutually separating disjoint union of essential annuli properly embedded in a handlebody V. Then there is an ordering $A_1, \ldots, A_k$ of the annuli in $\mathcal{A}$ such that for each $1 \leqslant i \leqslant k$, D_i is a spanning disk and γ_i is a spanning arc for A_i, and*

$$D_i \cap A_j = \begin{cases} \text{(a possibly empty set of) arcs properly embedded in } D_i \text{ and parallel in } A_j \text{ to } \gamma_j & \text{if } j < i, \\ \gamma_i \subset \partial D_i & \text{if } j = i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Furthermore, suppose α ($\neq \gamma_i$) is an arc in $\mathcal{A} \cap D_i$ contained in $\partial D_i(\gamma_i)$, so α is parallel in A_j to γ_j for some $j < i$. If $D_i(\gamma_i) \subset N_\varepsilon$, then $D_j(\gamma_j) \subset N_{\varepsilon'}$ where $\{\varepsilon, \varepsilon'\} = \{0, 1\}$.

Proof. Let Δ be a complete system of meridian disks for V, such that $\mathcal{A} \cap \Delta$ is transverse and consists of spanning arcs in the annuli (see Remark 4.1). We will order the annuli in $\mathcal{A}$ and construct the disks D_i using the level of the annuli with respect to Δ.

First suppose A is a component of $\mathcal{A}$ which is level one. That is, there is a level one arc γ in $A \cap \Delta$. Label A as A_1, γ as γ_1, and take D_1 to be the level adjacent disk component of γ (i.e. the outermost disk cut off by γ of the component of Δ containing γ). Continue this for all level one annuli which are components of $\mathcal{A}$. We obtain a list of annuli $A_1, \ldots, A_{k_1}$ with respective spanning arcs $\gamma_1, \ldots, \gamma_{k_1}$, and respective disks $D_1, \ldots, D_{k_1}$. Note that each D_i for $1 \leqslant i \leqslant k_1$ is contained in N_0 or N_1, thereby satisfying the conclusion of the lemma.

Continuing in an inductive manner, suppose that we have compiled a list $A_1, \ldots, A_{k_{m-1}}$ of annuli of level $\leqslant m-1$, along with their corresponding spanning arcs $\gamma_1, \ldots, \gamma_{k_{m-1}}$ and disks $D_1, \ldots, D_{k_{m-1}}$. Proceeding as before, let A be
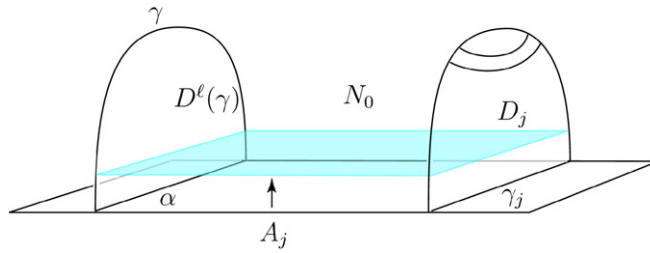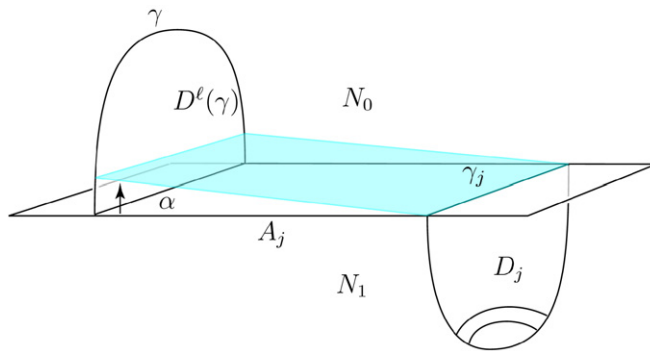
Fig. 8. $D_j(\gamma_j)$ is in N_0.

Fig. 9. $D_j(\gamma_j)$ is in N_1.

a level m annulus in $\mathcal{A}$, and let γ be a level m arc in $A \cap \Delta$. Set $A = A_{k_{m-1}+1}$, and set $\gamma = \gamma_{k_{m-1}+1}$. It remains to construct $D_{k_{m-1}+1}$.

Let D be the component of Δ containing γ. Assume without loss of generality that $D^\ell(\gamma)$ is contained in N_0. Let α be an arc in $\partial D^\ell(\gamma) \cap \mathcal{A}$ other than γ. Then $\ell(\alpha) < m$ and hence α is a spanning arc of some annulus A_j, $j \leqslant k_{m-1}$. The annulus A_j has corresponding spanning arc γ_j. First assume that α is the only such arc other than γ contained in $\partial D^\ell(\gamma) \cap \mathcal{A}$.

Assume first that $D_j(\gamma_j)$ is contained in N_0. Since α and γ_j are spanning arcs of A_j, they cut off a rectangle in A_j (if $\alpha = \gamma_j$, then this rectangle is simply the arc α). Taking that rectangle and attaching D_j and $D^\ell(\gamma)$ gives a disk, which can be pushed slightly into N_0. Take this disk as $D_{k_{m-1}+1}$. Note that α has been eliminated as an arc of intersection in $D_{k_{m-1}+1}$ (see Fig. 8). Since the disk D_j satisfies the conclusion of the lemma, this implies $D_{k_{m-1}+1}$ does as well.

Now assume that $D_j(\gamma_j)$ is contained in N_1. If $\alpha = \gamma_j$, set $D_{k_{m-1}+1} = D^\ell(\gamma) \cup D_j$. If $\alpha \neq \gamma_j$, as above take a rectangle in A_j bounded by α and γ_j. The rectangle, along with $D^\ell(\gamma)$ and D_j, forms a disk. Isotope the disk by pushing the subdisk formed by $D^\ell(\gamma)$ and the rectangle slightly into N_0, keeping D_j fixed. Take the resulting disk as $D_{k_{m-1}+1}$ (see Fig. 9). As before, $D_{k_{m-1}+1}$ satisfies the conclusion of the lemma since D_j does.

Now suppose α' is an arc in addition to γ and α in $\partial D^\ell(\gamma) \cap \mathcal{A}$. If α' is not a spanning arc of the same annulus component of $\mathcal{A}$ as α, then the above construction applies similarly to α' to obtain $D_{k_{m-1}+1}$. If, however, α' is a spanning arc of the same annulus as α, then a slight modification of the above argument is needed in order to attach a disk to $\partial D^\ell(\gamma)$ at α'. Let A_j be the annulus containing α and α', and, as above, consider the disk D_j. As before, attach D_j to $D^\ell(\gamma)$ along a rectangle between α and γ_j, and then isotope appropriately off of A_j. Then for α', take a parallel copy D'_j of D_j in V cut along A_j, and a rectangle from α' to the arc in $\partial D'_j$ parallel to γ_j, and push off into N_0 as before, depending on whether $D_j(\gamma_j)$ is in N_0 or N_1. By choosing D'_j to be on the appropriate side of D_j, we can ensure that the rectangle between α' and D'_j is disjoint from the rectangle between α and D_j. Repeating this process for any additional arcs in $\partial D^\ell(\gamma) \cap \mathcal{A}$ (except for γ), we obtain the desired disk $D_{k_{m-1}+1}$ (see Fig. 10).   □
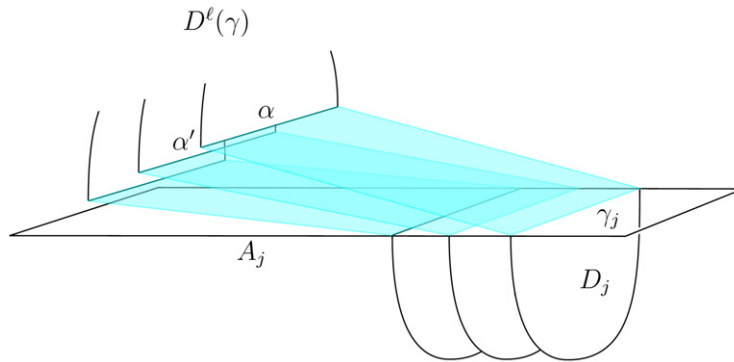
Fig. 10. $\partial D^{\ell}(\gamma) \cap \mathcal{A}$ contains more than one arc other than γ.
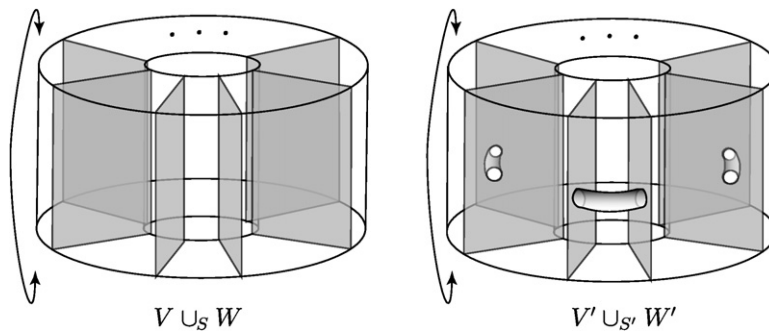

Fig. 11. $V \cup_S W$ and $V' \cup_{S'} W'$ inside $T \times I$.

5. Proof of the main theorem

Proof of Theorem 1.1. Note that both $W \cap T$ and $V \cap T$ are nonempty since T is incompressible and cannot be contained in a handlebody. Let $\lambda_1, \ldots, \lambda_k$ be spanning arcs of the components of $W \cap T$. Attach k 1-handles to V so that their cores are $\lambda_1, \ldots, \lambda_k$, respectively. As discussed in Remark 3.3 this yields a k-times stabilization $V' \cup_{S'} W'$ of $V \cup_S W$.

Let $T \times I$ be a regular neighborhood of T such that $T \times \{1/2\} = T$. Then $V \cup_S W$ and $V' \cup_{S'} W'$ can be considered identical except for inside $T \times I$.

A *tube* τ in a handlebody V is a regular neighborhood $D \times I$ of a compressing disk D of V, so that $\partial D \times I \subset \partial V$. We refer to $D \times \{0\}$ and $D \times \{1\}$ as the *feet* of τ. Observe that two parallel annuli connected by a tube in M is isotopic to a dual picture, as in Fig. 12. Performing this isotopy in $T \times I$ then yields a configuration of $V' \cup_{S'} W' \cap (T \times I)$ as in the first part of Fig. 13. If we take a top–down view, we obtain the schematic picture in the second part of Fig. 13. The tubes running through $T \times I$ are the "spokes" in this schematic.
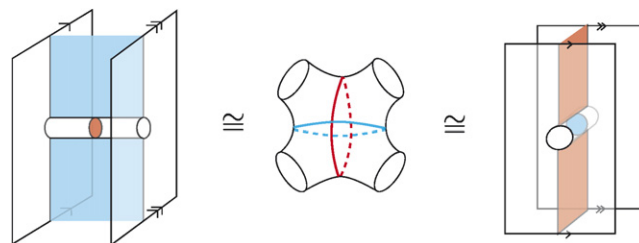

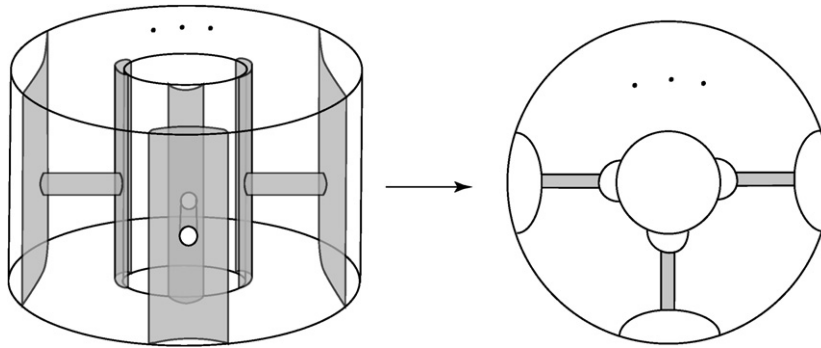Fig. 12. Isotopic pictures of annuli connected by a tube.

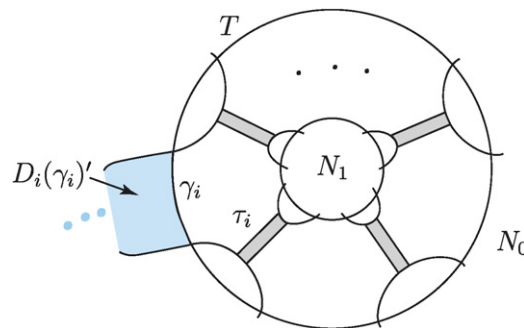Fig. 13. A schematic for $V' \cup_{S'} W'$ inside $T \times I$.

Fig. 14. $D_i(\gamma_i)'$, τ_i, and γ_i.

Let A be the disjoint union of annuli in $V \cap T$. Apply Lemma 4.6 to obtain an ordering $A_1, \ldots, A_k$ of the annuli in A, spanning arcs $\gamma_1, \ldots, \gamma_k$ of the annuli, and spanning disks $D_1, \ldots, D_k$ satisfying the conclusion of the lemma. Note that these disks can be chosen to miss the stabilizations added above, thus they exist for V'.

As in the proof of Lemma 4.6, we may assume that any arcs in $D_i \cap \mathcal{A}$ parallel in A_j to γ_j for $j < i$ are arbitrarily close to γ_j in A_j. Choose the D_i so that they intersect $T \times I$ in $\gamma_i \times I$, $1 \leqslant i \leqslant k$. For notational purposes, if $D_i(\gamma_i)$ is the adjacent disk component of γ_i with respect to D_i, set $D_i(\gamma_i)' = D_i(\gamma_i) \cap (\overline{M - (T \times I)})$. Let τ_i, $1 \leqslant i \leqslant k$, be the tubes in W' resulting from the stabilizations of $V \cup_S W$ as in Fig. 13, ordered so that τ_i is the tube immediately counterclockwise in the schematic from γ_i. See Fig. 14.

We now use the disks D_i to isotope the tubes τ_i as follows. First, consider τ_1 and the disk D_1. Note that D_1 is the adjacent disk component $D_1(\gamma_1)$ and lies in either N_0 or N_1. Assume first that $D_1 \subset N_0$. As D_1 intersects $\mathcal{A}$ only at γ_1, isotope the foot of τ_1 lying on $T \times \{0\}$ across a regular neighborhood of $D_1(\gamma_1)'$, and then down the tube immediately clockwise from τ_1 in the schematic. The result is that τ_1 is isotoped so that its core is $\gamma_1 \times \{1\}$ in $T \times I$. Push τ_i slightly into N_1 (we say that τ_1 is *adjacent* to $\gamma_1 \times \{1\}$ after this isotopy). See Fig. 15.

If, on the other hand, $D_1 \subset N_1$, then by a symmetric argument the other foot of τ_1 can be isotoped through a neighborhood of $D_1(\gamma_1)'$ so that it is adjacent to $\gamma_1 \times \{0\}$.

Given $2 \leqslant i \leqslant k - 1$, isotope all tubes τ_j, $j < i$, to lie adjacent to $\gamma_j \times \{0\}$ or $\gamma_j \times \{1\}$ in $T \times I$ as above (note that this depends on whether or not $D_j(\gamma_j)$ is in N_1 or N_0). To isotope τ_i, assume that $D_i(\gamma_i)$ is in N_0 (as above a symmetric argument applies if $D_i(\gamma_i)$ is in N_1). By Lemma 4.6, each of the arcs $\alpha_1, \ldots, \alpha_m$ in $(\partial D_i(\gamma_i) \cap \mathcal{A}) - \gamma_i$ is parallel in A_j to γ_j for some $j < i$. Let $j_1, \ldots, j_n$ be this set of indices. For these j_l, the latter conclusion of Lemma 4.6 implies that $D_{j_l}(\gamma_{j_l})$ is contained in N_1. Hence, τ_{j_l} is adjacent to $\gamma_{j_l} \times \{0\}$. Since we chose the arcs $\alpha_1, \ldots, \alpha_m$ to lie arbitrarily close to the spanning arc γ_{j_l} of the annulus in which they lie, $D_i(\gamma_i)$ may be isotoped so that it meets $\partial \tau_{j_1} \cup \cdots \cup \partial \tau_{j_n}$ in place of $\alpha_1 \cup \cdots \cup \alpha_m$. The upshot is that $D_i(\gamma_i)'$ is a disk in V' that meets T only in the arc γ_i. Thus, as before, keeping one foot fixed isotope τ_i through a regular neighborhood of $D_i(\gamma_i)'$ until τ_i is in $T \times I$ and the other foot of τ_i has reached the next section clockwise in the schematic picture (see Fig. 16).
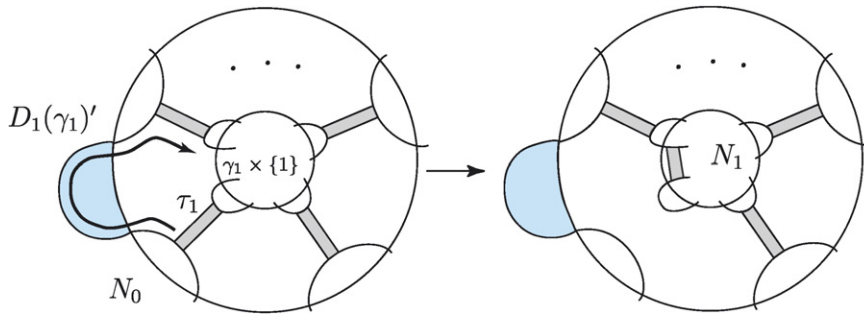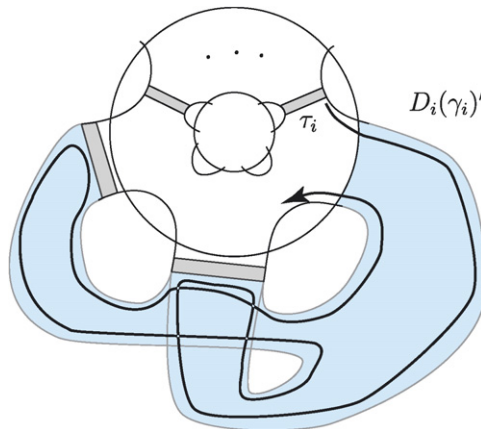
Fig. 15. Isotoping τ_1.

Fig. 16. Isotoping τ_i through $D_i(\gamma_i)'$.

Having isotoped the foot of τ_i through to the next section of the schematic, one of two things can occur. If the tube τ_s originating in that section has not yet been isotoped (i.e. $s > i$), then as before the foot of τ_i may be isotoped along τ_s causing τ_i to become adjacent to $\gamma_i \times \{1\}$ as desired. If, on the other hand, $s < i$ so that τ_s has already been isotoped, then τ_s lies adjacent to $\gamma_s \times \{\varepsilon\}$ for $\varepsilon = 0$ or 1. But this implies that $D_s(\gamma_s)'$ meets $\gamma_s \times \{1 - \varepsilon\}$ along its boundary (i.e. $D_s(\gamma_s) \subset N_0$ if $\varepsilon = 1$ or $D_s(\gamma_s) \subset N_1$ if $\varepsilon = 0$). Lemma 4.6 ensures that any arc α in $(\partial D_s(\gamma_s) \cap \mathcal{A}) - \gamma_s$ is parallel in A_j to γ_j for some $j < s$, so by the same argument as above, the corresponding components of $(\partial D_s(\gamma_s) \cap \mathcal{A}) - \gamma_s$ lie on already isotoped tubes. Therefore, we may continue to isotope τ_i across to the next section of the schematic, either by sliding the foot of τ_i across τ_s or through a regular neighborhood of $D_s(\gamma_s)'$.

Continue sliding across sections in this manner until reaching a tube $\tau_{s'}$ that has not been isotoped (i.e. $s' > i$). Slide the foot of τ_i down $\tau_{s'}$, and then back through all the previous sections, this time on the other side. That is, if the foot of τ_i was isotoped initially along $T \times \{0\}$ and across the isotoped τ_s, then after having been isotoped down $\tau_{s'}$, τ_i can be isotoped along $T \times \{1\}$ and through a regular neighborhood of $D_s(\gamma_s)'$. Continue the isotopy so that τ_i becomes adjacent to $\gamma_i \times \{1\}$ as desired (see Fig. 17). Having done this process for each of the tubes τ_i, $1 \leqslant i \leqslant k - 1$, we leave the remaining tube τ_k unmoved.

Suppose that $D_k(\gamma_k)$ is in N_0 (again, a symmetric argument applies if $D_k(\gamma_k)$ is in N_1). As above, we can take $D_k(\gamma_k)'$ to be completely in N_0 so that it meets T only at γ_k. That is, $D_k(\gamma_k)'$ is a spanning disk for A_k contained in N_0. Each of the tubes τ_i, $1 \leqslant i \leqslant k - 1$, lies adjacent to either $\gamma_i \times \{0\}$ or $\gamma_i \times \{1\}$. In the former case, there is a disk between $\partial \tau_i$ and $\gamma_i \times \{0\}$ which is a spanning disk for A_i contained completely in N_0. In the latter case, as was argued above, $D_i(\gamma_i)'$ is a spanning disk for A_i contained completely in N_0. Each of the annuli $W \cap T$ clearly has a spanning disk contained in N_1, as indicated in Fig. 18. Thus, by Lemma 3.2, $V' \cup_{S'} W'$ is an amalgamation along T. $\quad\square$
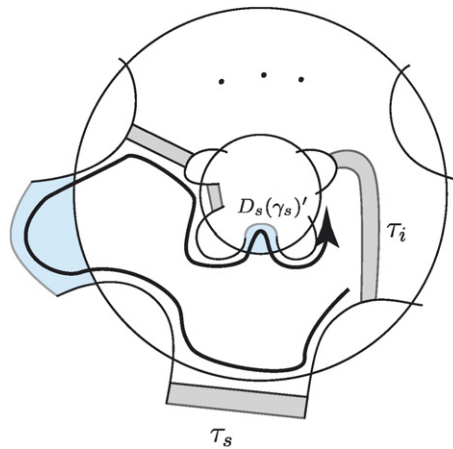
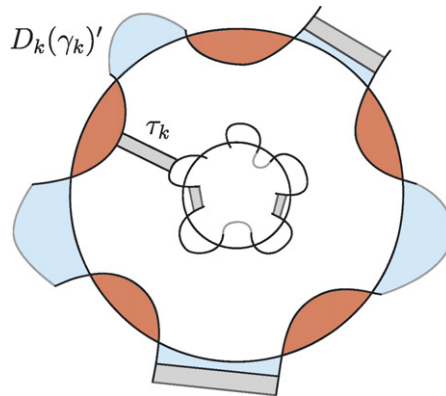Fig. 17. Isotoping τ_i to be adjacent to $\gamma_i \times \{1\}$.

Fig. 18. Spanning disks for each of the annuli in $V' \cap T$ and $W' \cap T$.

Dehn twisting and Corollary 1.3. Consider the torus T in M as the product $S^1 \times S^1$, so that a point on T can be written in the form (x, y). Define the map $r_\theta : S^1 \to S^1$ to be a rotation of S^1 of angle θ.

Definition 5.1. A *Dehn twist along T* is a homeomorphism $h : M \to M$ such that in a product neighborhood $T \times I$ of T,

$$h(x, y, t) = \big(r_{2\pi pt}(x), r_{2\pi qt}(y), t\big),$$

where p and q are relatively prime integers, and $h|\overline{M - (T \times I)}$ is the identity.

Remark 5.2. Any homeomorphism of M to itself which is the identity outside a product neighborhood $T \times I$ is isotopic to a power of a Dehn twist along T.

Proof of Corollary 1.3. Suppose that $V \cup_S W$ and $P \cup_\Sigma Q$ are Heegaard splittings of M such that $V \cup_S W$ can be isotoped so that $V \cap T$ consists of k annuli, and $P \cup_\Sigma Q$ is obtained from $V \cup_S W$ by any power of a Dehn twist along T. Theorem 1.1 implies that both splittings are isotopic to amalgamations along T after at most k stabilizations. Let $V' \cup_{S'} W'$ and $P' \cup_{\Sigma'} Q'$ be $V \cup_S W$ and $P \cup_\Sigma Q$ stabilized k times, respectively. Let $J : M \times I \to M$ be the isotopy of $V' \cup_{S'} W'$ constructed in the proof of Theorem 1.1, and let $h : M \to M$ be a power of a Dehn twist along T. Note that $J \circ (h \times id) = h \circ J$, which implies that $J((P' \cup_{\Sigma'} Q'), 1)$ is obtained from $J((V' \cup_{S'} W'), 1)$ by a power of a Dehn twist along T. It is a simple exercise to observe that a power of a Dehn twist along T of a Heegaard splitting which is an amalgamation along T is isotopic in $T \times I$ to the original splitting. This implies that $V' \cup_{S'} W'$ and $P' \cup_{\Sigma'} Q'$ are isotopic. $\square$

6. Counting annuli in handlebodies

In this section we adapt arguments from [13] to prove Corollary 1.2. Recall a theorem of Jaco and Shalen and also Johannson (see, for example, [6]) that states in a compact, orientable, irreducible 3-manifold M there exists a disjoint union of incompressible tori Θ, unique up to isotopy, such that each component of M cut along Θ either is a Seifert fibered space or is atoroidal. This decomposition of M is called the *JSJ decomposition of M*.

Definition 6.1. A component T of Θ is called a *canonical torus in the JSJ decomposition of M*.

Any incompressible torus in M is isotopic either to a canonical torus in the JSJ decomposition or to a torus contained completely in one of the Seifert fibered components. The following definitions are taken from Section 4 of [13].

Definition 6.2. Let $\mathcal{A}$ be a disjoint union of properly embedded essential annuli in a handlebody V. A component Z of V cut along $\mathcal{A}$ is called *toral* if Z is a solid torus.

Define the *complexity* of a toral component Z to be $c(Z) = |\partial V \cap Z| - \varepsilon$, where $\varepsilon = 1$ if the annuli $\mathcal{A} \cap Z$ are longitudes of Z and $\varepsilon = 0$ otherwise.

The complexity of a union of toral components is defined to be the sum of the complexities of the individual components.

Definition 6.3. Suppose $V \cup_S W$ is a Heegaard splitting of M and that N is a Seifert fibered component obtained by cutting M along the JSJ decomposition of M. N is called *aligned* with respect to $V \cup_S W$ if S intersects ∂N only in fibers.

The following theorem, due to Waldhausen, characterizes incompressible, ∂-incompressible surfaces in Seifert fibered spaces. The interested reader is referred to [6], Theorem VI.32 for a proof.

Theorem 6.4. *A properly embedded incompressible and ∂-incompressible two-sided surface in an orientable Seifert fibered space N can be properly isotoped so that either it is vertical (a union of fibers) or it is horizontal (transverse to the fibering). If it is horizontal, then N is the union of two copies of an I-bundle ξ over a surface E, glued along their ∂I-bundles ξ', and the incompressible surface consists of parallel copies of ξ'.*

We now prove the first of three results that will establish an upper bound for the number of annuli in $V \cap T$. This proof can be found in the proof of a more general result, Theorem 4.7 in [13]. We include it here for completeness.

Lemma 6.5. *If T is a separating canonical torus in the JSJ decomposition of M and $V \cup_S W$ is a Heegaard splitting of M isotoped so that $V \cap T$ consists of a minimal number of essential annuli, then each component of $V \cap T$ intersects at most one toral component of V cut along $V \cap T$.*

Proof. Suppose N is a component of M cut along the JSJ decomposition of M, and that T is a component of ∂N. Suppose N contains a toral component of V cut along T. The incompressibility of T and the hypothesis that $|V \cap T|$ is minimal imply that N contains an essential annulus, and hence is Seifert fibered. If N is non-aligned, then the annulus cannot be vertical. By Theorem 6.4, the annulus must therefore be horizontal. Moreover, Theorem 6.4 implies that N is an I-bundle over an annulus or a Möbius band. In both cases, N can be refibered to be aligned. Note that this follows since T is separating, and hence does meet N on both sides.

Hence, if two adjacent components of V cut along T were toral, they would have to be in aligned Seifert fibered spaces. But this implies that T is incident to Seifert fibered spaces that are aligned on both sides, implying T is not a canonical torus in the JSJ decomposition. $\quad\square$

Lemma 6.6. *Let $\mathcal{A}$ be a disjoint union of essential annuli in a handlebody V, and let Z be the union of toral components of V cut along $\mathcal{A}$. Suppose that at most one of the components of V cut along $\mathcal{A}$ on either side of a component of $\mathcal{A}$ is toral, and let α be the number of annuli which do not meet toral components (on either side). Then $c(Z) + \alpha \leqslant 2g - 2$, where g is the genus of ∂V.*

This is Lemma 4.4 in [13].

Lemma 6.7. *Let T be a canonical torus in the JSJ decomposition of M and let $V \cup_S W$ be a Heegaard splitting of M isotoped so that $V \cap T$ consists of a minimal number k of essential annuli. Then $k \leqslant 4g - 4$.*

Proof. Let α be the number of annuli in $\mathcal{A} = V \cap T$ which are not incident to toral components of V cut along $\mathcal{A}$ on either side, and let β be the number of annuli which meet a toral component. Then $k = \alpha + \beta$. Let $Z_1, \ldots, Z_n$ be the toral components of V cut along $\mathcal{A}$, and let β_i be the number of components of $\mathcal{A}$ incident to Z_i, $1 \leqslant i \leqslant n$. Then $c(Z_i) = \beta_i - 1$ if $Z_i \cap \mathcal{A}$ are longitudes, and $c(Z_i) = \beta_i$ otherwise. Since by Lemma 6.5 no two adjacent components of V cut along $\mathcal{A}$ are toral, we have

$$\beta = \sum_{i=1}^{n} \beta_i \leqslant \sum_{i=1}^{n} \big(c(Z_i) + 1\big) = c(Z) + n,$$

where $Z = \bigcup_{i=1}^{n} Z_i$.

Now, observe that for each toral component Z_i, we have $c(Z_i) \geqslant 1$. For if some Z_i were such that $c(Z_i) = 0$, then Z_i would have one longitudinal annulus on its boundary in ∂V, implying that the sole annulus in $Z_i \cap \mathcal{A}$ is boundary parallel and hence not essential. Therefore, $n \leqslant c(Z)$, and from the above we conclude that

$$\beta \leqslant 2c(Z).$$

Hence,

$$k = \alpha + \beta \leqslant \alpha + 2c(Z) \leqslant 2\alpha + 2c(Z) \leqslant 4g - 4$$

by Lemma 6.6. $\quad\square$

The proof of Corollary 1.2 follows readily from Theorem 1.1 and Lemma 6.7.

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