An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 43 (2002) 169–178
www.elsevier.com/locate/camwa

The Asymptotic Behavior of Dynamic Rent-Seeking Games

C. CHIARELLA
School of Finance and Economics, University of Technology, Sydney
P.O. Box 123, Broadway, NSW 2007, Australia
carl.chiarella@uts.edu.au

F. SZIDAROVSZKY
Systems and Industrial Engineering Department, University of Arizona
Tucson, AZ 85721-0020, U.S.A.
szidar@tucson.sie.arizona.edu

(Received February 2000; revised and accepted August 2000)

Abstract—Dynamic rent-seeking games with nonlinear cost functions are analyzed. The local asymptotic stability of the solution is first examined. We show that in the absence of a dominant agent, all eigenvalues of the Jacobian are real. Conditions are given for the local asymptotic stability as well as for the local instability of the equilibrium. In the presence of a dominant agent, complex eigenvalues are possible. Simple stability conditions are presented for cases when all eigenvalues are real, and the possibility of limit cycles is analyzed in the case of complex eigenvalues.

Keywords—Oligopoly, Dynamic systems, Stability, Bifurcation.

1. INTRODUCTION

Cournot oligopolies are one of the most frequently studied models in mathematical economics since the middle of the nineteenth century, when the classical work of Cournot [1] was published. The major research areas have included the existence and uniqueness of the equilibrium, the asymptotic properties of the equilibrium in dynamic oligopolies, as well as different variants and extensions of the classical model of Cournot. A comprehensive summary of the most important research findings on single-product models can be found in [2], and the corresponding multi-product models are discussed in [3]. The extended research on oligopoly models has included group-equilibrium problems [4], labor-managed oligopolies [5], oligopsonies [6], multistage [7] and hierarchical models [8].

During the last two decades, increasing attention has been given to the analysis of rent-seeking games. The introductory paper of Tullock [9] has initiated a series of studies on the subject. A typical example of a rent-seeking game could be the process of applying for franchises, e.g., of well-known hamburger chains; these continually come up for renewal, and individuals wishing to bid for these typically would use professional bidding agents who could be viewed as the agents of our model. The basic model can be formulated in the following way.

PII: S0898-1221(01)00281-4

Assume that n agents compete for a rent, which will earn a unit profit for the agent who actually wins the rent. Let x_i denote the effort agent i spends in order to win the rent, and let $c_i(x_i)$ be its cost. The probability of winning the rent for agent i is $x_i/(\sum_{j=1}^n x_j)$, and therefore, its expected profit is given as

$$\varphi_i(x_1,\ldots,x_n) = \frac{x_i}{\sum\limits_{j=1}^{n} x_j} - c_i(x_i). \tag{1}$$

Let $X_i = \Re_+$ be the set of feasible efforts of agent i; then an n-person noncooperative normal-form game $\Gamma = \{n, X_1, \ldots, X_n, \varphi_1, \ldots, \varphi_n\}$ is formally defined, and is called a *rent-seeking game* in which the agents are the players, and X_i and φ_i are the strategy set and payoff function of agent i $(i = 1, 2, \ldots, n)$. The existence of equilibrium of this game has been analyzed by several authors. Decreasing returns in rent-seeking technology as well as increasing returns were analyzed in [10] where the possibility of the agent's reaction function to be upward sloping was also examined. A systematic approach to establish the existence of an equilibrium has been also offered in this paper. Okuguchi [11] has pointed out that rent-seeking games are mathematically equivalent to profit maximizing Cournot oligopolies with hyperbolic price functions. He also proved the existence of a pure symmetric Nash equilibrium, when the agents have identical cost functions. These results have been extended to the nonsymmetric case in [12], where the existence of a unique equilibrium is proved with convex cost functions. The existence and uniqueness of the equilibrium in rent-seeking games is not implied by the corresponding results on the classical Cournot model, since the strategy sets are not compact and hyperbolic price functions are not concave. Therefore, model-oriented special methods have had to be developed.

The dynamic extension of rent-seeking games has been first examined in [13] and later in [14], where the stability of the equilibrium was proved under certain convexity and monotonicity conditions. However, these conditions are not always satisfied for an important class of rent-seeking games. Therefore, a more general, systematic approach is needed to describe stability conditions, and such is the aim of this paper.

There is no previous study investigating unstable equilibria and asymptotic behaviour that is more complex than asymptotical stability for rent-seeking games. Such study has been performed only for the classical Cournot model in [15] and [16] based on bifurcation theory. It is worth mentioning here that the types of dynamical systems we encounter below (as well as those in [15] and [16]) are not unrelated to those encountered in population dynamics in mathematical biology. In this regard, the monograph [17] provides an excellent summary of the relevant mathematical results and techniques.

This paper will provide a systematic stability analysis of rent-seeking games as well as consider the conditions under which the dynamics of such games can give rise to the birth of limit cycles.

2. THE DYNAMIC MODEL

The existence of equilibrium is usually examined by determining the best response functions of each player and investigating its analytic properties. For each agent i, denote $Q_i = \sum_{j \neq i} x_j$. For each $Q_i \geq 0$, the best response of agent i can be obtained as

$$x_i(Q_i) = \underset{x_i \geq 0}{\operatorname{argmax}} \left\{ \frac{x_i}{x_i + Q_i} - c_i(x_i) \right\}. \tag{2}$$

We assume that for each agent i, the cost function c_i is twice continuously differentiable. If $x_i(Q_i) > 0$, then the first- and second-order conditions can be written as

$$\frac{Q_i}{(x_i(Q_i) + Q_i)^2} - c_i'(x_i(Q_i)) = 0, \tag{3}$$

and

$$\frac{-2Q_i}{(x_i(Q_i)+Q_i)^3} - c_i''(x_i(Q_i)) < 0. \tag{4}$$

Notice that relation (4) also guarantees that the left-hand side of equation (3) is strictly monotonic, and therefore, for each Q_i the quantity $x_i(Q_i)$ is unique, and hence, $x_i(Q_i)$ is a properly defined function.

Let now $(\bar{x}_1, \dots, \bar{x}_n)$ be a positive equilibrium that satisfies both the first- and second-order conditions (3) and (4). Thus, with the notation $\bar{Q}_i = \sum_{l \neq i} \bar{x}_l$ and $\bar{s} = \sum_{j=1}^{n} \bar{x}_j$, the first-order condition (3) may be rewritten as

$$\bar{Q}_i = c_i'(\bar{x}_i)\bar{s}^2, \tag{5}$$

whilst the second-order condition (4) may be expressed as

$$c_i''(\bar{x}_i) > \frac{-2\bar{Q}_i}{\bar{s}^3}. \tag{6}$$

We can easily show that the last two relations characterise the equilibrium efforts and cost derivatives. Select any arbitrary positive numbers $\bar{x}_1, \dots, \bar{x}_n$, and form the quantities $\bar{s} = \sum_{i=1}^{n} \bar{x}_i$ and (for all i), $\bar{Q}_i = \sum_{l \neq i} \bar{x}_l$. In addition, select any sequence $\bar{c}_1'', \dots, \bar{c}_n''$ of real numbers that satisfy inequality (6). Then, there is a rent-seeking game in which $(\bar{x}_1, \dots, \bar{x}_n)$ is an equilibrium and for all i, $c_i''(\bar{x}_i) = \bar{c}_i''$, since for each i there is a strictly increasing cost function with given first and second derivatives at a single point.

Assume next that the time scale is continuous, and at each time period $t \geq 0$, each agent adjusts its effort proportionally to its marginal profit. The resulting dynamic model has the form

$$x_i = k_i \left(\frac{Q_i}{(x_i+Q_i)^2} - c_i'(x_i) \right), \tag{7}$$

where $k_i > 0$ is given for each $i = 1, 2, \dots, n$. The value of k_i shows how fast agent i follows the profit changes. In the economic literature, k_i is usually called the speed of adjustment. Notice that any positive equilibrium $(\bar{x}_1, \dots, \bar{x}_n)$ is an equilibrium of this dynamic system as well as the consequence of the first-order conditions (5). Assume that the initial vector $\underline{x}(0)$ is selected from a neighbourhood of this equilibrium. In the following sections, the local stability and the dynamic behaviour of the solution trajectory $\underline{x}(t)$ will be investigated.

3. STABILITY ANALYSIS

In order to analyze the local asymptotic behavior of the solution of the dynamical system (7), its Jacobian at the equilibrium $\underline{J}$ has to be first determined. Simple differentiation shows that at the equilibrium

$$\begin{aligned} \underline{J} &= \frac{1}{\bar{s}^3} \begin{bmatrix} k_1\left(-2\bar{Q}_1 - \bar{s}^3 c_1''(\bar{x}_1)\right) & k_1\left(\bar{x}_1 - \bar{Q}_1\right) & \cdots & k_1\left(\bar{x}_1 - \bar{Q}_1\right) \\ k_2\left(\bar{x}_2 - \bar{Q}_2\right) & k_2\left(-2\bar{Q}_2 - \bar{s}^3 c_2''(\bar{x}_2)\right) & \cdots & k_2\left(\bar{x}_2 - \bar{Q}_2\right) \\ \vdots & \vdots & \ddots & \vdots \\ k_n\left(\bar{x}_n - \bar{Q}_n\right) & k_n\left(\bar{x}_n - \bar{Q}_n\right) & \cdots & k_n\left(-2\bar{Q}_n - \bar{s}^3 c_n''(\bar{x}_n)\right) \end{bmatrix} \\ &= \frac{1}{\bar{s}^3}\left(\underline{D} + \underline{a}\,\underline{1}^\top\right), \end{aligned} \tag{8}$$

with

$$\begin{aligned} \underline{D} &= \operatorname{diag}\left(k_1\left(-\bar{s} - \bar{s}^3 c_1''(\bar{x}_1)\right), \dots, k_n\left(-\bar{s} - \bar{s}^3 c_n''(\bar{x}_n)\right)\right), \\ \underline{a} &= \left(k_1\left(\bar{x}_1 - \bar{Q}_1\right), \dots, k_n\left(\bar{x}_n - \bar{Q}_n\right)\right)^\top, \qquad \text{and} \qquad \underline{1}^\top = (1, \dots, 1). \end{aligned} \tag{9}$$

Since the signs of the real parts of the eigenvalues determine the asymptotic behavior of the solution, we ignore the positive factor $1/\bar{s}^3$. The characteristic polynomial of $\underline{D} + \underline{a}\,\underline{1}^\top$ can be written as

$$\begin{aligned}\det\left(\underline{D} - \lambda \underline{I} + \underline{a}\,\underline{1}^\top\right) &= \det(\underline{D} - \lambda \underline{I}) \det\left(\underline{I} + (\underline{D} - \lambda \underline{I})^{-1} \underline{a}\,\underline{1}^\top\right) \\ &= \prod_{i=1}^{n} \left(k_i\left(-\bar{s} - \bar{s}^3 c_i''(\bar{x}_i)\right) - \lambda\right) \left[1 + \sum_{i=1}^{n} \frac{k_i\left(\bar{x}_i - \bar{Q}_i\right)}{k_i\left(\bar{s} - \bar{s}^3 c_i''(\bar{x}_i)\right) - \lambda}\right].\end{aligned} \tag{10}$$

In order to simplify the notation, let $d_i = k_i(-\bar{s} - \bar{s}^3 c_i''(\bar{x}_i))$ and $e_i = k_i(\bar{x}_i - \bar{Q}_i)$ for $i = 1, 2, \dots, n$, and assume that the different d_i values are numbered so that $d_1 < d_2 < \cdots < d_r$. Let I_l denote the set of agents with the same d_l value ($l = 1, 2, \dots, r$), and let m_l denote the number of agents in set I_l. Then, the eigenvalues are the solutions of the equation

$$\prod_{l=1}^{r} (d_l - \lambda)^{m_l} \left[1 + \sum_{l=1}^{r} \frac{\gamma_l}{d_l - \lambda}\right] = 0, \tag{11}$$

where $\gamma_l = \sum_{i \in I_l} e_i$. The eigenvalues can be obtained in the following way. If $m_l > 1$, then $\lambda = d_l$ is an eigenvalue with multiplicity $m_l - 1$, and if $m_l = 1$, then the corresponding factor cancels, so in this case $\lambda = d_l$ is not an eigenvalue ($l = 1, 2, \dots, r$). In this way, we have $m_1 + \cdots + m_r - r = n - r$ eigenvalues (included with multiplicities). The other eigenvalues are the solutions of the nonlinear equation

$$1 + \sum_{l=1}^{r} \frac{\gamma_l}{d_l - \lambda} = 0. \tag{12}$$

Since this is equivalent to a polynomial equation of degree r, it has r real (or complex) roots.

The foregoing discussion may be summarized in the following theorem.

THEOREM 1.

(i) *Assume that with some l, $m_l \geq 2$ and $d_l > 0$. Then, the equilibrium is locally unstable.*

(ii) *Suppose that for all l such that $m_l \geq 2$, $d_l < 0$. Let $\lambda_1, \dots, \lambda_r$ denote the roots of equation (12).*

If for at least one i,

$$\operatorname{Re} \lambda_i > 0, \tag{13}$$

then the equilibrium is locally unstable. If for all i, $\operatorname{Re} \lambda_i < 0$, then the equilibrium is locally asymptotically stable.

We note that in the second case of Theorem 1 when for all roots, $\operatorname{Re} \lambda_i \leq 0$ and for at least one root $\operatorname{Re} \lambda_i = 0$, no conclusion can be given based only on analysis of the eigenvalues, since the system is nonlinear. In order to apply the above theorem, the locations of the roots of equation (12) should be determined. We will consider two major cases.

CASE A. Assume that for all l,

$$\gamma_l \leq 0. \tag{14}$$

Notice that if for some l the value of γ_l is zero, then the corresponding term is missing from the left-hand side of equation (12) without changing its structure. Therefore, in the following discussion we may assume that inequality (14) is strict for all l. Introduce the function

$$g(\lambda) = \sum_{l=1}^{r} \frac{\gamma_l}{d_l - \lambda}. \tag{15}$$

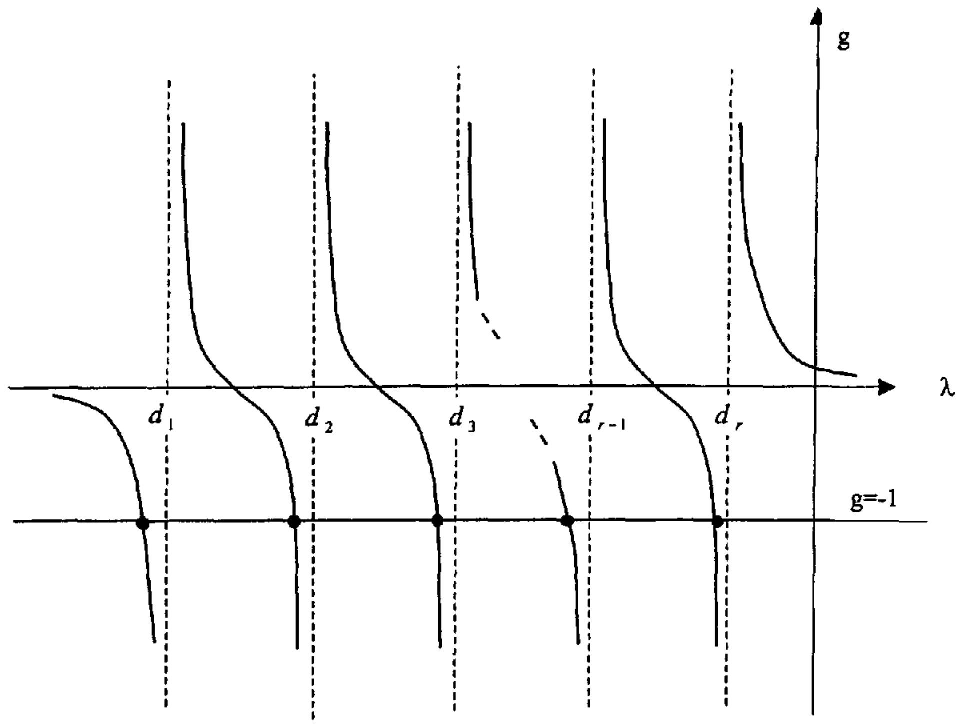


Figure 1. The shape of $g(\lambda)$ in Case A.

It is easy to see that

$$\lim_{\lambda\to d_l+0} g(\lambda) = \infty, \qquad \lim_{\lambda\to d_l-0} g(\lambda) = -\infty, \qquad \lim_{\lambda\to\pm\infty} g(\lambda) = 0, \qquad \text{and} \qquad g'(\lambda) < 0, \tag{16}$$

for all λ, except at the poles. The graph of $g(\lambda)$ is illustrated in Figure 1. Since equation (12) can be written as $g(\lambda) = -1$, we see that all roots are real: one before d_1, and one between each d_i and d_{i+1} $(i = 1, 2, \dots, r-1)$.

Hence, we may state the following result.

THEOREM 2. *Consider the case in which $\gamma_l \le 0$ for all l.*

(i) *If for all l, $d_l \le 0$, then all roots are negative implying the local asymptotic stability of the equilibrium.*
(ii) *Assume that $d_r > 0$ and all $d_i < 0$ $(i = 1, 2, \dots, r-1)$. If $g(0) < -1$, then the equilibrium is locally asymptotically stable, and if $g(0) > -1$, then the equilibrium is locally unstable.*
(iii) *If $d_r > 0$ and $d_{r-1} \ge 0$, then the equilibrium is locally unstable regardless of the signs of the other d_i values $(i = 1, 2, \dots, r-2)$.*

We note again that in the case when the largest root is zero, no conclusion can be given, since the system is nonlinear.

CASE B. Suppose that for some l^*,

$$\gamma_{l^*} > 0. \tag{17}$$

We first show that this relation can hold for only one value of l^*. Notice, first, that there is at most one dominant agent p such that $\bar{x}_p > \bar{Q}_p$. If set I_a does not contain this agent, then

$$\gamma_a = \sum_{i\in I_a} k_i \left(\bar{x}_i - \bar{Q}_i\right) \le 0,$$

showing that agent p must belong to set I_{l^*}. We show next that necessarily $d_{l^*} < 0$. The second-order conditions (6) imply that for the dominant agent p,

$$-\bar{s} - \bar{s}^3 c_p''(\bar{x}_p) < -\bar{s} + 2\bar{Q}_p = \bar{Q}_p - \bar{x}_p,$$

and by multiplying this inequality by k_p we have

$$d_{l^*} = k_p \left(-\bar{s} - \bar{s}^3 c_p''(\bar{x}_p)\right) < k_p \left(\bar{Q}_p - \bar{x}_p\right) = -e_p.$$

Since for all other agents $i \in I_{l^*}$, $e_i \le 0$, we see that

$$d_{l^*} < -\sum_{i \in I_{l^*}} e_i = -\gamma_{l^*} < 0. \tag{18}$$

For the sake of simplicity assume that $l^* = r$. We still keep the assumption that $d_1 < d_2 < \dots < d_{r-1}$, and note that now d_r is not necessarily the largest among the d_l values. In this case, function g satisfies the following properties:

$$\begin{aligned} &\lim_{\lambda \to d_l + 0} g(\lambda) = \infty, \qquad \lim_{\lambda \to d_l - 0} g(\lambda) = -\infty, \qquad l = 1, 2, \dots, r-1, \\ &\lim_{\lambda \to d_r + 0} g(\lambda) = -\infty, \quad \lim_{\lambda \to d_r - 0} g(\lambda) = \infty, \end{aligned} \tag{19}$$

and

$$\lim_{\lambda \to \pm\infty} g(\lambda) = 0.$$

Notice that no condition can be given on the sign of the derivative $g'(\lambda)$ in this case. The shape of function g is illustrated in Figure 2, where d_r is either the largest, or the smallest among the d_l values, or it is in the middle.

The above discussion allows us to state the following.

THEOREM 3. *Consider the case in which $\gamma_{l^*} > 0$ for some l^*.*

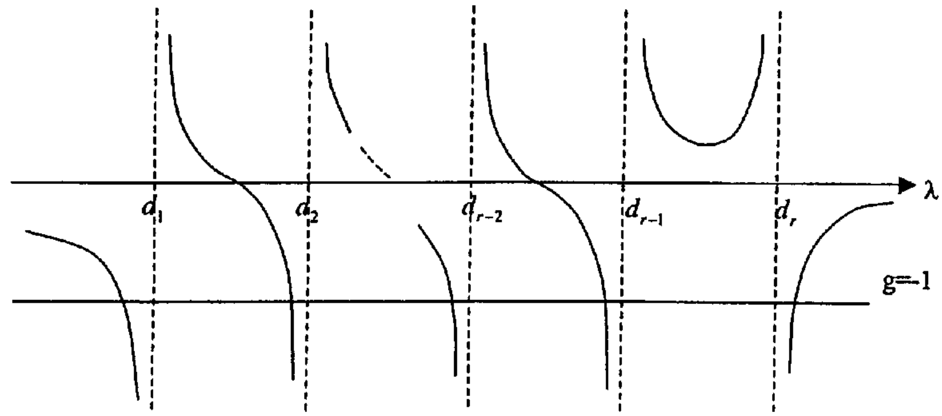
(i) *Assume first that $d_r > d_l$ ($l = 1, 2, \dots, r-1$). If $d_r \ge 0$, then the equilibrium is unstable. If $d_r < 0$, then in the case of $g(0) > -1$ the equilibrium is locally asymptotically stable, and if $g(0) < -1$, the equilibrium is locally unstable.*

(ii) *Assume now that $d_r < d_l$ ($l = 1, 2, \dots, r-1$) and there are real roots between d_r and d_1. If $d_{r-2} \ge 0$, then the equilibrium is unstable. If $d_{r-2} < 0 < d_{r-1}$, then in the case of $g(0) < -1$ the equilibrium is locally asymptotically stable, and if $g(0) > -1$, the equilibrium is locally unstable. If $d_{r-1} \le 0$, then the equilibrium is locally asymptotically stable.*

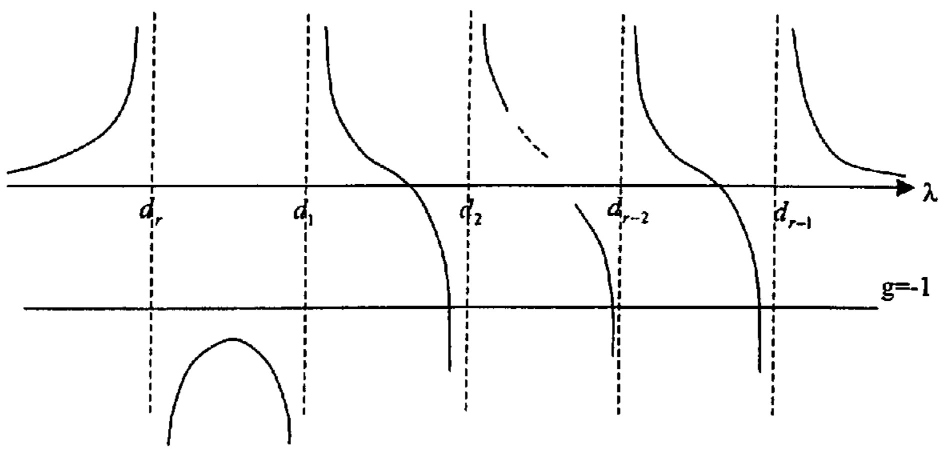
(iii) *Assume next that $d_{r-2} < d_r < d_{r-1}$, and there are real roots between d_{r-2} and d_{r-1}. If $d_{r-2} \ge 0$, then the equilibrium is locally unstable. If $d_{r-2} < 0 < d_{r-1}$, then in the case when the roots between d_{r-2} and d_{r-1} are negative, the equilibrium is locally asymptotically stable, and if at least one root between d_{r-2} and d_{r-1} is positive, the equilibrium is locally unstable. If $d_{r-1} \le 0$, then the equilibrium is locally asymptotically stable.*

(iv) *Assume, finally, that $d_{i-1} < d_r < d_i$ with some $i \le r-2$, and there are real roots between d_{i-1} and d_i. If $d_{r-2} \ge 0$, then the equilibrium is locally unstable. If $d_{r-2} < 0 < d_{r-1}$, then in the case of $g(0) < -1$ the equilibrium is locally asymptotically stable, and if $g(0) > -1$, the equilibrium is locally unstable. If $d_{r-1} \le 0$, then the equilibrium is locally asymptotically stable.*

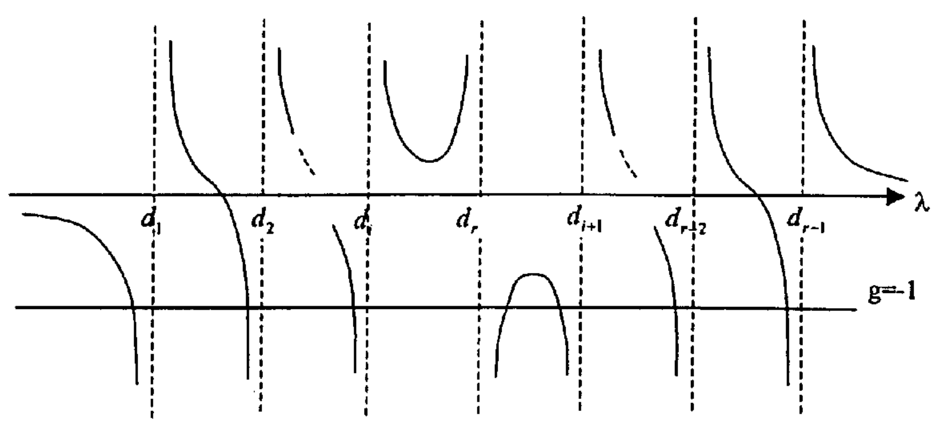
Notice that the theorem does not apply for cases when the largest real root is zero or there are no roots in the intervals indicated. In every case there are always $(r-2)$ real roots, and



(a) The situation d_r is largest.



(b) The situation d_r is smallest.



(c) The situation d_r is in the middle.

Figure 2. The shape of $g(\lambda)$ in Case B.

therefore, either all roots are real, or there is exactly one complex conjugate pair of roots. The next example shows that both possibilities might occur.

EXAMPLE 1. Consider the special case when $r = 2$, $k_1 = k_2 = 1$, $d_2 = -1$, $d_1 = 0$. Let $2\bar{x}_2 - \bar{s} = \gamma > 0$ be a parameter. From equation (18) we must have $\gamma < 1$. Furthermore,

$$2\bar{s}_1 - m_1\bar{s} = 2(\bar{s} - \bar{x}_2) - m_1\bar{s} = -(m_1 - 2)\bar{s} - 2\bar{x}_2 = -(m_1 - 1)\bar{s} - \gamma.$$

By introducing the notation $\alpha = (m_1 - 1)\bar{s}$ we see that α can take on any nonnegative value. Then, equation (12) has the form

$$\frac{-\alpha-\gamma}{0-\lambda} + \frac{\gamma}{-1-\lambda} = -1,$$

which simplifies as

$$\lambda^2 + \lambda(1+\alpha) + (\alpha+\gamma) = 0.$$

The discriminant is

$$D = (1+\alpha)^2 - 4(\alpha+\gamma) = (1-\alpha)^2 - 4\gamma,$$

showing that the cases of both real and complex roots are possible. ∎

We have seen in the above discussion that Case B may occur only in the case of a dominant agent such that $\bar{x}_p > \bar{Q}_p$. This inequality shows that at the equilibrium, agent p spends more effort than all other agents combined. In the case of an asymptotically stable equilibrium, this is the case for all t large enough. This assumption is therefore not very realistic, since antitrust constraints usually prohibit this kind of behaviour in the long run.

4. THE EMERGENCE OF LIMIT CYCLES

We have seen in the previous section that there are many plausible situations in which the equilibrium of the rent-seeking game may be locally unstable. The question arises as to what is the fate of the dynamic rent-seeking game in such situations. One possibility is that the outcome of the game cycles around the equilibrium. This could, for instance, be the case if the speeds of reaction, k_i, are relatively large so that there is continual overshooting of the equilibrium. Such could be the situation in the case of a locally unstable equilibrium if we could demonstrate the existence of limit cycle motion by applying the Hopf bifurcation theorem (see, for example, [18]). To this end we need to determine the regions in the parameter space where pure complex eigenvalues exist. Letting $\lambda = i\alpha$, then equation (12) has the form

$$\sum_{l=1}^{r-1} \frac{\gamma_l}{d_l - i\alpha} + \frac{\gamma_r}{d_r - i\alpha} = -1, \tag{20}$$

where $\gamma_l < 0$ ($i = 1, 2, \dots, r-1$), and $\gamma_r > 0$. By equating the real and imaginary parts, we have

$$\sum_{l=1}^{r-1} \frac{\gamma_l d_l}{d_l^2 + \alpha^2} + \frac{\gamma_r d_r}{d_r^2 + \alpha^2} = -1, \qquad \text{and} \qquad \sum_{l=1}^{r-1} \frac{\gamma_l i\alpha}{d_l^2 + \alpha^2} + \frac{\gamma_r i\alpha}{d_r^2 + \alpha^2} = 0. \tag{21}$$

Since we assume that $\alpha \neq 0$, the second equation implies that

$$\gamma_r = -\left(d_r^2 + \alpha^2\right) \sum_{l=1}^{r-1} \frac{\gamma_l}{d_l^2 + \alpha^2}, \tag{22}$$

and combining this relation with the first equation of (21) leads to a simple equation for α^2

$$\sum_{l=1}^{r-1} \frac{\gamma_l (d_l - d_r)}{d_l^2 + \alpha^2} = -1. \tag{23}$$

Notice that this is a polynomial equation of degree $r-1$ for α^2, which can be solved by routine methods (see, for example, [19]). Notice that in case (i) of Theorem 3, the left-hand side is always positive, so no real root exists. This observation coincides with the corresponding results when we saw that all roots of equation (12) are real, so no pure complex root exists. The existence of real roots in equation (23) depends on the signs and the orders of magnitudes of the differences $d_l - d_r$. The following example illustrates the possibility of real roots and hence the possibility of limit cycles.

EXAMPLE 2. Consider the case of $r = 3$, $\gamma_1 = -a$, $\gamma_2 = -b$ (where a and b are positive parameters), $\gamma_3 = \gamma > 0$ is the bifurcation variable, and $d_1 = -2$, $d_2 = 1$, $d_3 = -1$. Then, equation (20) has the form

$$\frac{-a}{-2-\lambda} + \frac{-b}{1-\lambda} + \frac{\gamma}{-1-\lambda} = -1. \tag{24}$$

If $\lambda = i\alpha$, then

$$\frac{-a}{-2-i\alpha} + \frac{-b}{1-i\alpha} + \frac{\gamma}{-1-i\alpha} = -1.$$

Equating the real and imaginary parts,

$$\frac{2a}{4+\alpha^2} + \frac{-b}{1+\alpha^2} + \frac{-\gamma}{1+\alpha^2} = -1,$$

and

$$\frac{-ai\alpha}{4+\alpha^2} + \frac{-bi\alpha}{1+\alpha^2} + \frac{\gamma i\alpha}{1+\alpha^2} = 0.$$

Notice that we are interested in the case of $\alpha \neq 0$. So, solving both equations for $\gamma/(1+\alpha^2)$, we get

$$\frac{\gamma}{1+\alpha^2} = \frac{2a}{4+\alpha^2} - \frac{b}{1+\alpha^2} + 1 = \frac{a}{4+\alpha^2} + \frac{b}{1+\alpha^2}. \tag{25}$$

A fourth-order equation is therefore obtained for α

$$\alpha^4 + \alpha^2(5 + a - 2b) + (4 + a - 8b) = 0. \tag{26}$$

Notice that this equation will have positive roots if $4+a-8b < 0$. In order to satisfy relation (18), we have to guarantee that $\gamma < 1$, that is,

$$\frac{\gamma}{1+\alpha^2} = \frac{a}{4+\alpha^2} + \frac{b}{1+\alpha^2} < \frac{1}{1+\alpha^2},$$

which can be rewritten as

$$\alpha^2(a + b - 1) < 4 - a - 4b.$$

For example, by selecting $a = 0.15$, $b = 0.75$, this inequality is satisfied for all $\alpha^2 \geq 0$; furthermore, from equation (26) we have

$$\alpha^2 \approx 0.451, \qquad \text{and} \qquad \alpha^2 \approx -4.101,$$

from which we see that the real roots are

$$\alpha_{1,2} \approx \pm 0.672.$$

Substituting these values into equation (25), we see that $\gamma \approx 0.799$. By differentiating equation (24) with respect to γ, we have the derivative value

$$\frac{d\lambda}{d\gamma} = \frac{1/(1+\lambda)}{-a/(-2-\lambda)^2 - b/(1-\lambda)^2 + \gamma/(1+\lambda)^2},$$

which has the approximating value $(0.468 + 0.719i)$ at $\lambda = i\alpha$ with the above values of α and γ. Hence, the real part is nonzero showing the existence of a limit cycle. ∎

We have established the possibility of limit cycle motion in the case of local instability. Our analysis does not enable us to say anything about the stability of such limit cycles. In order to do so, we would need to specify in some detail the nonlinear structure of the dynamic rent-seeking game and apply normal form theory (see [18]). However, very often these calculations are intractable and it becomes necessary to resort to numerical methods as in [15]. Such analysis warrants a paper in its own right, and we leave this task for future research.

5. CONCLUSIONS

In this paper, nonlinear rent-seeking games were analyzed where the cost functions were assumed to be nonlinear. A dynamic adjustment process in which each agent adjusts its output proportionally to its marginal profit was examined. The local asymptotic stability of the equilibrium was first examined. In the absence of a dominant agent, it turns out that all eigenvalues are real. Conditions for the local asymptotic stability and also for the local instability of the equilibrium were then derived. In the presence of a dominant agent, complex eigenvalues are also possible. Simple conditions were presented for the stability and also for the instability of the equilibrium in the case of real eigenvalues. If complex eigenvalues are possible, then we have shown the possibility of limit cycles via the use of the Hopf bifurcation theorem.

Future research could focus on the specific nature of nonlinearities and analyse the stability of the limit cycle motion that emerges when the equilibrium becomes locally unstable. A further avenue of research would be how agents come to learn the optimal effort of all other agents. Here, we have assumed that each agent knows instantaneously the optimal effort of all other agents. However, it would be more realistic to assume that this is learnt with some time delay. Such learning could, for example, be modelled using the continuously distributed lags employed in [15] and [16] to analyse such learning in oligopoly models.

REFERENCES

1. A. Cournot, *Recherches sur les Principles Mathèmatiques de la Thèorie de Richesses*, Hachette, Paris, (1938); *Researches into the Mathematical Principles of the Theory of Wealth*, Kelley, New York, (1960).
2. K. Okuguchi, *Expectations and Stability in Oligopoly Models*, Springer-Verlag, Berlin, (1976).
3. K. Okuguchi and F. Szidarovszky, *The Theory of Oligopoly with Multi-Product Firms*, 2nd edition, Springer-Verlag, Berlin, (1999).
4. F. Szidarovszky and S. Yakowitz, Contributions to Cournot oligopoly theory, *Journal of Economic Theory* **28**, 51–70, (1982).
5. B. Ward, The firm in Illyria: Market syndactilism, *American Economic Review* **48**, 566–589, (1958).
6. C. Chiarella and K. Okuguchi, A dynamic analysis of Cournot duopoly in imperfectly competitive product and factor markets, *Keio Economic Studies* **XXXIV**, 21–33, (1997).
7. K. Okuguchi and T. Yamazaki, Two-stage Cournot oligopolies with industry-wide externalities, *Seoul Journal of Economics* **9** (1), 1–15, (1996).
8. K. Okuguchi and T. Yamazaki, Bertrand and hierarchical Stackelberg oligopolies with product differentiation, *Keio Economic Studies* **XXXI**, 75–80, (1994).
9. G. Tullock, Efficient rent-seeking, In *Toward a Theory of Rent-Seeking Society*, (Edited by J.M. Buchanan *et al.*), Texas A&M Press, College Station, TX, (1980).
10. J.D. Pérez-Castrillo and T. Verdier, A general analysis of rent-seeking games, *Public Choice* **73**, 335–350, (1992).
11. K. Okuguchi, Decreasing returns and existence of Nash equilibrium in rent-seeking games, Mimeo, Department of Economics, Nanzan University, Nagoya, Japan, (1995).
12. F. Szidarovszky and K. Okuguchi, On the existence and uniqueness of pure Nash equilibrium in rent-seeking games, *Games and Econ. Behavior* **18**, 135–140, (1997).
13. F. Szidarovszky and K. Okuguchi, On a dynamic rent-seeking game, Mimeo, Department of Economics, Nanzan University, Nagoya, Japan, (1997).
14. L. Xu and F. Szidarovszky, The stability of dynamic rent-seeking games, *International Game Theory Review* **1** (1), 87–102, (1999).
15. C. Chiarella and A. Khomin, An analysis of the complex behaviour of nonlinear oligopoly models with time lags, *Chaos, Solitons & Fractals* **7** (12), 2049–2065, (1996).
16. C. Chiarella and F. Szidarovszky, The birth of limit cycles in nonlinear oligopolies with continuously distributed information lags, In *Modeling Uncertainty*, (Edited by M. Dror), Kluwer Academic Publishers, Dordrecht (to appear).
17. J.M. Cushing, *Integro-Differential Equations and Delay Models in Population Dynamics*, Springer-Verlag, Berlin, (1977).
18. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York, (1983).
19. F. Szidarovszky and S. Yakowitz, *Principles and Procedures of Numerical Analysis*, Plenum Press, New York, (1978).