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Asymptotic behavior of bounded solutions for a system of neutral functional differential equations ☆

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Abstract

Consider the system of neutral functional differential equations

$$\begin{cases} (x_1(t) - cx_2(t-r))' = -F(x_1(t)) + G(x_2(t-r)), \\ (x_2(t) - cx_1(t-r))' = -F(x_2(t)) + G(x_1(t-r)), \end{cases}$$

where $r > 0$, $c \in [0, 1)$, $F, G \in C(R^1)$ and F is strictly increasing on R^1. It is shown that if $F(x) \geqslant G(x)$ for all $x \in R^1$ or $F(x) \leqslant G(x)$ for all $x \in R^1$, then every bounded solution of such a system tends to an equilibrium. Our results improve and extend some corresponding ones already known.

Keywords: Asymptotic behavior; Neutral functional differential equation; Positive limit set; Equilibrium

1. Introduction

In [2], Haddock conjectured that each solution of the following scalar neutral functional differential equation

$$\big(x(t) - cx(t-r)\big)' = -ax^{\gamma}(t) + ax^{\gamma}(t-r) \tag{1.1}$$

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tends to a constant as $t \to \infty$, where $c \in (0,1)$, $a \geqslant 0$, $r \geqslant 0$, and $\gamma > 0$. In [5], Wu investigated the following scalar neutral equation

$$\big(x(t) - cx(t-r)\big)' = -F\big(x(t)\big) + F\big(x(t-r)\big), \tag{1.2}$$

where $c \in (0,1)$, $r \geqslant 0$, $F \in C(R^1)$ and $F(x)$ is strictly increasing on R^1. It was shown that each solution of (1.2) converges to a constant as $t \to +\infty$ and hence the conjecture by Haddock [2] has been proven to be true. Later the same problem was investigated in a series of papers by Wu and his collaborators (see, for example, Haddock et al. [3], Krisztin and Wu [4], and Wu [6]) for some scalar neutral equations more general than (1.2). However, to our best knowledge, no results have been obtained for its vector forms. Motivated by this, in this paper, we consider the following system of neutral functional differential equations

$$\begin{cases} (x_1(t) - cx_2(t-r))' = -F(x_1(t)) + G(x_2(t-r)), \\ (x_2(t) - cx_1(t-r))' = -F(x_2(t)) + G(x_1(t-r)), \end{cases} \tag{1.3}$$

where $r > 0$, $c \in [0,1)$, and $F, G \in C(R^1)$. Moreover, it is assumed that F is strictly increasing on R^1.

One can observe that system (1.3) includes the following scalar neutral functional differential equation:

$$\big(x(t) - cx(t-r)\big)' = -F\big(x(t)\big) + G\big(x(t-r)\big), \tag{1.4}$$

as a special case, where $r > 0$, $c \in [0,1)$, and F and G are defined as in (1.3).

We then show that, using some comparison technique and the invariance of positive limit set, when $F(x) \geqslant G(x)$ for all $x \in R^1$ or $F(x) \leqslant G(x)$ for all $x \in R^1$, every bounded solution of (1.3) tends to an equilibrium as $t \to \infty$. This enables us to conclude that every bounded solution of (1.4) also tends to a constant as $t \to \infty$. Therefore, our results improve and extend the corresponding ones of [5], and also include the results of [1] as a special case. It should be noted that our proofs are quite different than that of [1,5].

The paper is organized as follows. In Section 2, we establish some preliminary results, important in the proofs of our main results. In Section 3, we state and prove our main results.

2. Preliminary results

In this section, we will establish several important lemmas which are essential tools in proving our main results in Section 3.

Let us define

$$C = C\big([-r,0], R^2\big), \qquad C_+ = C\big([-r,0], R^2_+\big)$$

and set $K = \{\varphi \in C_+\colon\ \varphi_1(0) - c\varphi_2(-r) \geqslant 0 \text{ and } \varphi_2(0) - c\varphi_1(-r) \geqslant 0\}$. One can observe that K and C_+ are order cones in C. Let $\varphi \in C$. We tacitly assume throughout this section that $\varphi = (\varphi_1, \varphi_2)$.

We now define several orderings as follows. $\varphi \leqslant_K \psi$ iff $\psi - \varphi \in K$, $\varphi <_K \psi$ iff $\psi - \varphi \in K\backslash\{0\}$, $\varphi \ll_K \psi$ iff $\psi - \varphi \in \operatorname{Int} K$, $\varphi \leqslant_K A$ iff $\varphi \leqslant_K \psi$ for any $\psi \in A$, $\varphi <_K A$ iff $\varphi <_K \psi$ for any $\psi \in A$, $\varphi \ll_K A$ iff $\varphi \ll_K \psi$ for any $\psi \in A$, where $\varphi, \psi \in C$ and $A \subseteq C$. Notations such as $\psi \geqslant_K \varphi$ and $\psi \gg_K \varphi$ can be defined analogously.

Let us define $\hat{\alpha} = ((\hat{\alpha})_1, (\hat{\alpha})_2)$, where $(\hat{\alpha})_i(\theta) = \alpha$, $i = 1, 2$, $\theta \in [-r, 0]$. In what follows, we assume that $\varphi \in C$ and use $x_t(\varphi)$ $(x(t,\varphi))$ to denote the solution of (1.3).

We need the following elementary result whose proof is contained in [1].

Lemma 2.1. *For any constants a and $x_0 \in R^1$, the initial value problem*

$$\begin{cases} x'(t) = -F(x(t)) + a, \\ x(t_0) = x_0 \end{cases} \tag{2.1}$$

has a unique solution $x(t)$ on $[0, +\infty)$.

Lemma 2.2. *Let $r > 0$ be given and $a, b \in C([t_0, t_0 + r])$. For any constant $x_0 \in R^1$, the initial value problem*

$$\begin{cases} x'(t) = -F(x(t) + a(t)) + b(t), \\ x(t_0) = x_0 \end{cases} \tag{2.2}$$

has a unique solution $x(t)$ on $[t_0, t_0 + r]$.

Proof. Since $a, b \in C([t_0, t_0 + r])$, there exist $m_1, m_2, n_1, n_2 \in R^1$ such that

$$F(u + m_1) + m_2 \leqslant F\big(x + a(t)\big) + b(t) \leqslant F(x + n_1) + n_2 \quad \text{for } t \in [t_0, t_0 + r] \text{ and } x \in R^1.$$

Therefore, by comparison theorem and Lemma 2.1, $x(t)$ exists and is unique on $[t_0, t_0 + r]$. The proof is now complete. □

Lemma 2.3. *Let $\varphi \in C$. Then $x_t(\varphi)$ exists and is unique on R^1_+.*

Proof. First we will show that $x_t(\varphi)$ exists and is unique on $[0, r]$. We only prove that $x_1(t, \varphi)$ exists and is unique on $[0, r]$, the proof that $x_2(t, \varphi)$ exists and is unique on $[0, r]$ being similar. Indeed let $a(t) = c\varphi_1(t - r)$ and $b(t) = \varphi_2(t - r))$ for $t \in [0, r]$. Consider the solution $y(t)$ of the following system:

$$\begin{cases} y'(t) = -F(y(t) + a(t)) + b(t), \\ y(0) = \varphi_1(0) - c\varphi_2(-r). \end{cases} \tag{2.3}$$

By Lemma 2.2, $y(t)$ exists and is unique on $[0, r]$. Since $x_1(t, \varphi) - c\varphi_2(t - r)$ satisfies (2.3), $x_1(t, \varphi)$ exists and is unique on $[0, r]$. Therefore, $x_t(\varphi)$ exists and is unique on $[0, r]$. It follows from induction that $x_t(\varphi)$ exists and is unique on R^1_+. The proof is complete. □

Below, we call $G \geqslant F$ (or $G \leqslant F$) if $G(x) \geqslant F(x)$ for all $x \in R^1$ (or $G(x) \leqslant F(x)$ for all $x \in R^1$).

Lemma 2.4. *Let $G \geqslant F$, $\varphi \in C$, $\alpha \in R^1$ and $\varphi \geqslant_K \hat{\alpha}$. Then $x_t(\varphi) \geqslant_K \hat{\alpha}$ for $t \geqslant 0$.*

Proof. Let $y_1(t) = x_1(t, \varphi) - cx_2(t - r, \varphi)$ and $y_2(t) = x_2(t, \varphi) - cx_1(t - r, \varphi)$ for $t \geqslant 0$. Next we will show that $y_1(t) \geqslant (1 - c)\alpha$ for $t \in [0, r]$. Similarly, $y_2(t) \geqslant (1 - c)\alpha$ for $t \in [0, r]$. Otherwise, there exists $t_1 \in (0, r]$ such that $y_1(t_1) < (1 - c)\alpha$. By differential mean value theorem, there exists $t_2 \in (0, t_1)$ such that

$$y_1(t_2) < (1 - c)\alpha \quad \text{and} \quad y_1'(t_2) < 0.$$

From (1.3), we obtain

$$y_1'(t_2) = -F\big(x_1(t_2, \varphi)\big) + G\big(x_2(t_2 - r, \varphi)\big).$$

Again since $y_1(t_2) < (1-c)\alpha$, that is,

$$x_1(t_2,\varphi) < cx_2(t_2-r,\varphi) + (1-c)\alpha \leqslant x_2(t_2-r,\varphi),$$

it follows that $y_1'(t_2) \geqslant 0$, which yields a contradiction. Hence,

$$x_i(t,\varphi) \geqslant \alpha \quad \text{for } t \in [0,r],$$

where $i \in \{1,2\}$. Thus, $x_t(\varphi) \geqslant_K \hat{\alpha}$ for $t \in [0,r]$. Therefore, by induction, $x_t(\varphi) \geqslant_K \hat{\alpha}$ for $t \geqslant 0$. This completes the proof of the lemma. □

Remark 2.1. A similar conclusion of Lemma 2.4 holds for the case $G \leqslant F$.

Lemma 2.5. *Let $G \geqslant F$ and $\varphi \geqslant \hat{\alpha}$. Then either $x_t(\varphi) \gg_K \hat{\alpha}$ or $x_t(\varphi) = \hat{\alpha}$ for $t \geqslant 5r$.*

Proof. Let $y_1(t) = x_1(t,\varphi) - cx_2(t-r,\varphi)$ and $y_2(t) = x_2(t,\varphi) - cx_1(t-r,\varphi)$. We next distinguish two cases to finish the proof.

Case 1. $y_1(t) = (1-c)\alpha$ for $t \in [0,3r]$. From (1.3), we have

$$y_1'(t) = -F\big(x_1(t,\varphi)\big) + G\big(x_2(t-r,\varphi)\big),$$

which yields

$$G\big(x_2(t-r,\varphi)\big) = F\big(x_1(t,\varphi)\big) \quad \text{for } t \in [0,3r].$$

Thus,

$$x_2(t-r,\varphi) \leqslant x_1(t,\varphi) \quad \text{for } t \in [0,3r].$$

Therefore,

$$y_1(t) = x_1(t,\varphi) - cx_2(t-r,\varphi) \geqslant (1-c)x_1(t,\varphi)$$

for $t \in [0,3r]$. From Lemma 2.4 and the fact that $y_1(t) = \alpha(1-c)$ for $t \in [0,3r]$, it follows that $x_1(t,\varphi) = \alpha$ for $t \in [0,3r]$, and hence,

$$x_2(t-r,\varphi) = \alpha \quad \text{for } t \in [0,3r].$$

Therefore,

$$x_t(\varphi) = \hat{\alpha} \quad \text{for } t \in [r,2r].$$

Consequently,

$$x_t(\varphi) = \hat{\alpha} \quad \text{for } t \geqslant r.$$

Case 2. $y_1(t_1) > (1-c)\alpha$ for some $t_1 \in [0,3r]$. Next we will prove that $y_1(t) > (1-c)\alpha$ for $t \in [t_1,\infty)$. Consider the solution $z(t)$ of the following system:

$$\begin{cases} z'(t) = -F(z(t) + cx_2(t-r)) + F(x_2(t-r)), \\ z(t_1) = (1-c)\alpha. \end{cases} \tag{2.4}$$

By Lemma 2.2 and induction, $z(t)$ exists and is unique on $[t_1,\infty]$. We will show that $z(t) \geqslant (1-c)\alpha$ for $t \geqslant t_1$. Otherwise, there exists $t_2 > t_1$ such that

$$z'(t_2) < 0 \quad \text{and} \quad z(t_2) < (1-c)\alpha.$$

By Lemma 2.4,

$$z(t_2) + cx_2(t-r) < x_2(t-r).$$

Therefore, from (2.4), we obtain

$$z'(t_2) \leqslant 0,$$

which is a contradiction. Again from (1.3), we have

$$y_1'(t) = -F\big(x_1(t,\varphi)\big) + G\big(x_2(t-r,\varphi)\big) \geqslant -F\big(y_1(t) + cx_2(t-r)\big) + F\big(x_2(t-r,\varphi)\big)$$

and $y_1(t_1) > (1-c)\alpha$. Therefore, from the standard comparison theorem and the existence and uniqueness of the solution of (2.4), we have

$$y_1(t) > z(t) \geqslant (1-c)\alpha \quad \text{for } t \geqslant t_1.$$

We claim that there exists $t_1' \in [0, 3r]$ such that $y_2(t_1') > (1-c)\alpha$. Suppose the above assertion is false. Then $y_2(t) = (1-c)\alpha$ for $t \in [0, 3r]$. Using a similar argument to that of Case 1, we can obtain

$$x_t(\varphi) = \hat{\alpha} \quad \text{for } t \geqslant r.$$

Thus, $y_1(t) = (1-c)\alpha$ for $t \geqslant 2r$, a contradiction. This contradiction establishes the above assertion, which, together with the above discussion in Case 2, implies that

$$y_2(t) > (1-c)\alpha \quad \text{for } t \geqslant t_2.$$

Therefore,

$$y_i(t) > (1-c)\alpha \quad \text{for } t \geqslant 3r,$$

where $i \in \{1, 2\}$. It follows that $x_i(t,\varphi) > \alpha$ for $t \geqslant 3r$, where $i \in \{1, 2\}$. Consequently,

$$x_t(\varphi) \gg_K \hat{\alpha} \quad \text{for } t \geqslant 5r.$$

The proof of the lemma is now complete. □

Remark 2.2. A similar conclusion of Lemma 2.5 holds for the case $G \leqslant F$.

3. Main results and their proofs

Before stating and proving our main results, we need some definitions and notations.

Let $\varphi \in C$. We define $O(\varphi) = \{x_t(\varphi)\colon\ t \geqslant 0\}$. If $O(\varphi)$ is bounded, then $\overline{O(\varphi)}$ is compact in C, where $\overline{O(\varphi)}$ denotes the closure of $O(\varphi)$, and in this case we define

$$\omega(\varphi) = \bigcap_{t \geqslant 0} \overline{O\big(x_t(\varphi)\big)}.$$

One can observe that $\omega(x)$ is nonempty, compact and invariant.

Our main results are the following.

Theorem 3.1. *Let $G(x) \geqslant F(x)$ for all $x \in R^1$ and $\varphi \in C$. If $O(\varphi)$ is bounded, then there exists $\alpha^* \in R^1$ such that $\omega(\varphi) = \{\hat{\alpha}^*\}$.*

Proof. Let $\alpha^* = \sup\{\alpha \in R^1\colon \hat{\alpha} \leqslant_K \omega(\varphi)\}$. Since $\omega(\varphi)$ is compact, we obtain $\alpha^* \in R^1$. We will show that $\omega(\varphi) = \{\hat{\alpha}^*\}$. Otherwise, $\omega(\varphi)\backslash\{\hat{\alpha}^*\} \neq \phi$. According to the invariance of $\omega(\varphi)$, we have $x_{5r}(\omega(\varphi)) = \omega(\varphi)$. It follows that

$$x_{5r}\big(\omega(\varphi)\big)\backslash\big\{\hat{\alpha}^*\big\} \neq \phi$$

and hence there exists $\psi \in \omega(\varphi)$ such that

$$x_{5r}(\psi) >_K \hat{\alpha}^*.$$

Hence, from Lemma 2.5 and the fact that $\psi \geqslant_K \hat{\alpha}^*$, we obtain

$$x_{5r}(\psi) \gg_K \hat{\alpha}^*.$$

Therefore, there exists $\alpha^{**} > \alpha^*$ such that

$$x_{5r}(\psi) \gg_K \hat{\alpha}^{**}.$$

Again by the invariance of $\omega(\varphi)$ and its definition, there exists $t_1 > 0$ such that

$$x_{t_1}(\varphi) \geqslant_K \hat{\alpha}^{**} \gg_K \hat{\alpha}^*.$$

By Lemma 2.4,

$$x_t\big(x_{t_1}(\varphi)\big) \geqslant_K \hat{\alpha}^{**} \gg_K \hat{\alpha}^* \quad \text{for } t \geqslant 0.$$

Thus,

$$\omega(\varphi) \geqslant_K \hat{\alpha}^{**} \gg_K \hat{\alpha}^*.$$

This contradicts the definition of α^*. The proof of the theorem is now complete. □

Theorem 3.2. *Let $G(x) \leqslant F(x)$ for all $x \in R^1$ and $\varphi \in C$. If $O(\varphi)$ is bounded, then there exists $\alpha^* \in R^1$ such that $\omega(\varphi) = \{\hat{\alpha}^*\}$.*

Proof. By a similar argument to that in the proof of Theorem 3.1, the conclusion of Theorem 3.2 follows immediately by applying Remarks 2.1 and 2.2. □

Putting Theorems 3.1 and 3.2 together, we obtain the following result.

Corollary 3.1. *Let $G = F$ and $\varphi \in C$. Then there exists $\alpha^* \in R^1$ such that $\omega(\varphi) = \{\hat{\alpha}^*\}$.*

Proof. From Lemma 2.4 and Remark 2.1, it follows that $O(\varphi)$ is bounded. Therefore, by Theorems 3.1 or 3.2, the conclusion of Corollary 3.1 holds. □

Remark 3.1. If $G \leqslant F$ (or $G \geqslant F$), then by Theorems 3.1 and 3.2, each bounded solution of (1.4) tends to a constant as $t \to +\infty$, which extends and improves the main theorem in [1,5].

References

[1] Tongren Ding, Asymptotic behavior of solutions of some retarded differential equations, Sci. Sinica (A) 24 (1981) 939–945.

[2] J.R. Haddock, Functional differential equations for which each constant function is a solution: A narrative, in: Proc. of the 11th Internat. Conf. on Nonlinear Oscillations, János Bolyai Math. Soc., Budapest, 1987, pp. 86–93.

[3] J.R. Haddock, M.N. Nkashama, J. Wu, Asymptotic constancy for linear neutral Volterra integrodifferential equations, Tôhoku Math. J. 41 (1989) 689–710.
[4] T. Krisztin, J. Wu, Asymptotic periodicity, monotonicity, and oscillation of solutions of scalar neutral functional differential equations, J. Math. Anal. Appl. 199 (1996) 502–525.
[5] J. Wu, On J.R. Haddock's conjecture, Appl. Anal. 33 (1989) 127–137.
[6] J. Wu, Asymptotic periodicity of solutions to a class of neutral functional differential equations, Proc. Amer. Math. Soc. 113 (1991) 355–363.