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ScienceDirect
J. Math. Anal. Appl. 328 (2007) 767–779
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Invex sets and preinvex functions on Riemannian manifolds ☆

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Received 6 January 2006
Available online 3 July 2006
Submitted by A.V. Isaev

Abstract

The concept of a geodesic invex subset of a Riemannian manifold is introduced. Geodesic invex and preinvex functions on a geodesic invex set with respect to particular maps are defined. The relation between geodesic invexity and preinvexity of functions on manifolds is studied. Using proximal subdifferential, certain results concerning extremum points of a non smooth geodesic preinvex function on a geodesic invex set are obtained. The main value inequality and the mean value theorem in invexity analysis are extended to Cartan–Hadamard manifolds.

Keywords: Geodesic invex sets; Geodesic invex functions; Geodesic preinvex functions; Riemannian manifolds

1. Introduction

Convexity plays a vital role in optimization theory. This concept in linear topological spaces relies on the possibility of connecting any two points of the space by the line segment between them. Since convexity is often not enjoyed by the real problems various approaches to the generalization of the usual line segment have been proposed to relax the convexity assumptions.

In 1981 Hanson [9] introduced the concept of invexity, generalizing the difference $x - y$ in the definition of convex function to any function $\eta(x, y)$. Since then numerous articles have appeared in the literature reflecting further generalizations and applications in this category. For a survey

☆ Research partially supported by Center of Excellence for Mathematics, University of Isfahan, Isfahan, Iran.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.05.081

of recent advances in generalized convexity one can consult [14]. Ben-Israel and Mond [5] introduced a new generalization of convex sets and convex functions and Craven [6] called them invex sets and preinvex functions, respectively.

On the other hand, a manifold is not a linear space. Rapcsák [15] and Udriste [16] proposed a generalization of convexity which differs from the others. In this setting the linear space is replaced by a Riemannian manifold and the line segment by a geodesic, see [15,16] and references therein.

The organization of the paper is as follows: in Section 2 some concepts and facts from Riemannian geometry are collected. In Section 3 we define geodesic invex sets, and motivated by [13], we define the concepts of a geodesic invex function. Then geodesic preinvex functions are defined and some examples of these notions on Riemannian manifolds are given.

In Section 4 we study the link between geodesic invexity and geodesic preinvexity for smooth functions. We prove that a differentiable geodesic preinvex function is geodesic invex. Then, applying a natural condition on a geodesic invex function, we investigate whether geodesic invexity implies preinvexity.

In Section 5 we apply the proximal subdifferential of preinvex functions (see [4]). We relax the smoothness condition on geodesic preinvex functions and considering lower semicontinuity we study the question of global minimum of these functions on Riemannian manifolds.

In Section 6 we extend the mean value theorem for differentiable functions defined on invex sets in $\mathbb{R}^n$ to differentiable functions on invex subsets of Riemannian manifolds (see [1,2]).

2. Preliminaries

In this section, we recall some definitions and known results about Riemannian manifolds which will be used throughout the paper. We refer the reader to [10,11] for the standard material of differential geometry.

Throughout this paper M is a C^∞ smooth manifold modelled on a Hilbert space H, either finite dimensional or infinite dimensional, endowed with a Riemannian metric $\langle\cdot,\cdot\rangle_p$ on the tangent space $T_pM \cong H$. The corresponding norm is denoted by $\|\ \|_p$. Let us recall that the length of a piecewise C^1 curve $\gamma : [a,b] \to M$ is defined by

$$L(\gamma) := \int_a^b \left\|\gamma'(t)\right\|_{\gamma(t)} dt.$$

For any two point $p, q \in M$, we define

$$d(p,q) := \inf\bigl\{L(\gamma)\colon\ \gamma \text{ is a piecewise } C^1 \text{ curve joining } p \text{ to } q\bigr\}.$$

Then d is a distance which induces the original topology on M. On every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M. We also recall that a geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ, that is, γ satisfies the equation $\nabla_{d\gamma(t)/dt}\, d\gamma(t)/dt = 0$. Any path γ joining p and q in M such that $L(\gamma) = d(p,q)$ is a geodesic, and it is called a minimal geodesic. The existence theorem for ordinary differential equations implies that for every $v \in TM$ there exist an open interval $J(v)$ containing 0 and exactly one geodesic $\gamma_v : J(v) \to M$ with $d\gamma(0)/dt = v$. This implies that there is an open neighborhood $\tilde{T}M$ of the submanifold M of TM such that for every $v \in \tilde{T}M$ the geodesic $\gamma_v(t)$ is defined for $|t| < 2$. The exponential mapping $\exp : \tilde{T}M \to M$ is then defined as $\exp(v) = J_v(1)$ and the restriction of exp to a fiber

T_pM in $\tilde{T}M$ is denoted by $\exp_p$ for every $p \in M$. We use the parallel transport of vectors along geodesic. Recall that for a given curve $\gamma : I \to M$, a number $t_0 \in I$ and a vector $v_0 \in T_{\gamma(t_0)}M$, there exists exactly one parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = v_0$. Moreover, the mapping defined by $v_0 \mapsto V(t)$ is a linear isometry between the tangent spaces $T_{\gamma(t_0)}M$ and $T_{\gamma(t)}M$, for each $t \in I$. We denote this mapping by $P^t_{t_0,\gamma}$ and we call it the parallel translation from $T_{\gamma(t_0)}M$ to $T_{\gamma(t)}M$ along the curve γ. In the case when γ is a minimizing geodesic and $\gamma(t_0) = x$, $\gamma(t_1) = y$, we denote this mapping by L_{xy}.

If f is a differentiable map from the manifold M to the manifold N, we shall denote by df_x the differential of f at x.

We also recall that a simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Cartan–Hadamard manifold.

3. Geodesic invex sets and geodesic invex functions

Definition 3.1. Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that for every $x, y \in M$, $\eta(x, y) \in T_yM$. A nonempty subset S of M is said to be geodesic invex with respect to η if for every $x, y \in S$ there exists exactly one geodesic $\alpha_{x,y} : [0, 1] \to M$ such that

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \eta(x, y), \quad \alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0, 1].$$

Recall that a subset S of a Riemannian manifold is called geodesic convex if any two points $x, y \in S$ can be joined by exactly one geodesic of length $d(x, y)$ which belongs entirely to S (see [2,11]).

Remark 3.1. Let M be a Cartan–Hadamard manifold (either finite dimensional or infinite dimensional). On M there exists a natural map η playing the role of the $x - y$ in Euclidean space $\mathbb{R}^n$, for every $x, y \in \mathbb{R}^n$. Indeed, we can define the function η as

$$\eta(p, q) := \alpha'_{p,q}(0), \quad \text{for all } p, q \in M, \tag{1}$$

where $\alpha_{p,q}$ is the unique minimal geodesic joining q to p defined (see [11, p. 253]) as follows:

$$\alpha_{p,q}(t) := \exp_q\bigl(t \exp_q^{-1} p\bigr), \quad \text{for all } t \in [0, 1]. \tag{2}$$

Therefore, every geodesic convex set $S \subseteq M$ is a geodesic invex set with respect to the η defined in (1). Note that the converse does not hold in general, see Example 3.1.

On the other hand, if $S \subseteq M$ is a geodesic invex set with respect to the η defined in (1) then, for every $p, q \in S$ there exists exactly one geodesic $\beta_{p,q} : [0, 1] \to M$ such that $\beta_{p,q}(0) = q$, $\beta'_{p,q}(0) = \eta(p, q)$ and $\beta_{p,q}(t) \in S$ for each $t \in [0, 1]$. Since $\alpha_{p,q}$ defined in (2) and $\beta_{p,q}$ satisfy the same initial conditions, hence by uniqueness of maximal geodesic passing through q with initial velocity $\eta(p, q)$, we have $\alpha_{p,q} = \beta_{p,q}$ on $[0, 1]$. Therefore, S is a geodesic convex set.

Example 3.1. Let M be a Cartan–Hadamard manifold and $x_0, y_0 \in M$, $x_0 \neq y_0$. Let $B(x_0, r_1) \cap B(y_0, r_2) = \phi$ for some $0 < r_1, r_2 < \frac{1}{2}d(x_0, y_0)$, where $B(x, r) = \{y \in M\colon\ d(x, y) < r\}$ is an open ball with the center x and the radius r. We define

$$S := B(x_0, r_1) \cup B(y_0, r_2).$$

Then, S is not a geodesic convex set because, every geodesic curve passing x_0, y_0 is not completely lie in S. Now we define the function $\eta : M \times M \to M$ by

$$\eta(x, y) := \begin{cases} \exp_y^{-1} x, & x, y \in B(x_0, r_1) \text{ or } x, y \in B(y_0, r_2), \\ 0_y, & \text{otherwise.} \end{cases}$$

For every $x, y \in M$ consider the geodesic $\alpha : [0,1] \to M$ defined by

$$\alpha_{x,y}(t) = \exp_y\big(t\eta(x,y)\big), \quad \text{for all } t \in [0,1].$$

Hence,

$$\alpha_{x,y}(0) = y, \qquad \alpha'_{x,y}(0) = \eta(x,y).$$

We show that S is a geodesic invex set with respect to η. Let $x, y \in B(x_0, r_1)$, since $B(x_0, r_1)$ is geodesic convex (see [11, p. 259]) therefore,

$$\alpha_{x,y}(t) = \exp_y\big(t \exp_y^{-1} x\big) \in B(x_0, r_1) \subset S, \quad \text{for all } t \in [0,1].$$

Similarly, for the case $x, y \in B(y_0, r_2)$, we have

$$\alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0,1].$$

If $x \in B(x_0, r_1)$ and $y \in B(y_0, r_2)$ or $x \in B(y_0, r_2)$ and $y \in B(x_0, r_1)$ then, we have

$$\alpha_{x,y}(t) = \exp_y(t0_y) = y \in S, \quad \text{for all } t \in [0,1].$$

Hence, S is a geodesic invex set with respect to η.

Let S be a geodesic convex subset of a finite dimensional Cartan–Hadamard manifold M and $x \in M$. Then, there exists exactly one point $p_S(x) \in S$ such that for each $y \in S$, $d(x, p_S(x)) \leqslant d(x,y)$. The point $p_S(x)$ is called the projection of x onto S (see [8, p. 262]).

Example 3.2. Let S_1 and S_2 be nonempty closed geodesic convex subsets of a finite dimensional Cartan–Hadamard manifold M and $S_1 \cap S_2 = \phi$. We set $S := S_1 \cup S_2$ and define the function $\eta : M \times M \to M$ by

$$\eta(x,y) := \begin{cases} \exp_y^{-1}(p_{S_1}(x)), & y \in S_1,\ x \in M, \\ \exp_y^{-1}(p_{S_2}(x)), & y \in S_2,\ x \in M, \\ 0_y, & \text{otherwise.} \end{cases}$$

Now, for all $x, y \in S$ define

$$\alpha_{x,y}(t) := \exp_y\big(t\eta(x,y)\big), \quad \text{for all } t \in [0,1].$$

Clearly $\alpha_{x,y}(0) = y$, $\alpha'_{x,y}(0) = \eta(x,y)$ and $\alpha_{x,y}(t) \in S$ for all $t \in [0,1]$. Hence, S is a geodesic invex set with respect to η.

In the next example we show that on every Riemannian manifold M there exists a function $\eta : M \times M \to TM$ and a subset S of M which is geodesic invex set with respect to η, but it is not geodesic convex.

Recall that an open geodesic convex subset S of a Riemannian manifold M is called strongly convex if every ε-ball $B(x,\varepsilon)$ in S is a geodesic convex set. For each $x \in M$ there exists a number $\varepsilon = \varepsilon(x) > 0$ such that $B(x,\varepsilon)$ is a strongly geodesic convex set (see [10, p. 84]).

Example 3.3. Let M be a Riemannian manifold and $x_0, y_0 \in M$, $x_0 \neq y_0$ then, there exist two disjoint open balls $B(x_0,\varepsilon)$ and $B(y_0,\varepsilon)$ which are strongly convex (see [10, p. 85]). Set

$$S := B(x_0,\varepsilon) \cup B(y_0,\varepsilon).$$

Clearly S is not a geodesic convex set. Now we define the function $\eta : M \times M \to TM$ by

$$\eta(x, y) := \begin{cases} \alpha'_{x,y}(0), & x, y \in B(x_0, \varepsilon) \text{ or } x, y \in B(y_0, \varepsilon), \\ 0_y, & \text{otherwise}, \end{cases}$$

where $\alpha_{x,y}$ for $x, y \in S$ is the unique geodesic joining x and y such that $\alpha_{x,y}(0) = y$, $\alpha_{x,y}(1) = x$. Clearly S is a geodesic invex set with respect to η.

A real differentiable function f defined on manifold M is said to be η-invex (see [13]) if for every $x, y \in M$,

$$f(x) - f(y) \geqslant df_y\big(\eta(x, y)\big).$$

Now we define the invexity of a function f which is defined on an open geodesic invex subset of a Riemannian manifold.

Definition 3.2. Let M be a Riemannian manifold, S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$ and f be a real differentiable function on S. We say that f is η-invex on S if the following inequality holds

$$f(x) - f(y) \geqslant df_y\big(\eta(x, y)\big), \quad \text{for all } x, y \in S.$$

The definition of a preinvex function on $\mathbb{R}^n$ is given in [17]. See also [12,18] for properties of preinvex functions. Now we extend this notion to Riemannian manifolds and study some of its properties in this setting.

Definition 3.3. Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic invex set with respect to $\eta : M \times M \to TM$. We say that a function $f : S \to \mathbb{R}$ is geodesic η-preinvex if for every $x, y \in S$,

$$f\big(\alpha_{x,y}(t)\big) \leqslant tf(x) + (1 - t)f(y), \quad \text{for all } t \in [0, 1], \tag{3}$$

where $\alpha_{x,y}$ is the unique geodesic defined in Definition 3.1. If the inequality (3) is strict then, we say that f is a strictly geodesic η-preinvex function.

Proposition 3.1. *Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic invex set with respect to $\eta : M \times M \to TM$. Suppose that the function $f : S \to \mathbb{R}$ is geodesic η-preinvex then, we have*:

(i) *Every lower section of f defined by*

$$S(f, \lambda) := \big\{x \in S\colon\ f(x) \leqslant \lambda\big\}, \quad \lambda \in \mathbb{R},$$

is a geodesic invex set with respect to η.

(ii) *The set K of solutions of problem*

$$\begin{aligned} (P) \quad & \min \quad f(x) \\ & \textit{s.t.} \quad x \in S, \end{aligned}$$

is a geodesic invex set with respect to η. Moreover, if f is a strictly geodesic η-preinvex function, then M contains at most one point.

Proof. (i) Let $x, y \in S(f, \lambda)$. Since S is a geodesic invex set with respect to η, there exists exactly one geodesic $\alpha_{x,y} : [0, 1] \to M$ such that

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \eta(x, y), \quad \alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0, 1].$$

By the geodesic η-preinvexity of f we have

$$f\big(\alpha_{x,y}(t)\big) \leqslant tf(x) + (1 - t)f(y) \leqslant \lambda, \quad \text{for all } t \in [0, 1].$$

Therefore, $\alpha_{x,y}(t) \in S(f, \lambda)$ for all $t \in [0, 1]$.

(ii) If $\alpha := \inf_{x\in S} f(x)$ then, it is obvious that $K = \bigcap_{\lambda>\alpha} S(f, \lambda)$, that is, K is an intersection of geodesic invex sets with respect to η which is also a geodesic invex set with respect to η. If f is a strictly geodesic η-preinvex function and $x, y \in K$ then, by the geodesic invexity of K with respect to η there exists exactly one geodesic $\beta_{x,y} : [0, 1] \to M$ such that

$$\beta_{x,y}(0) = y, \quad \beta'_{x,y}(0) = \eta(x, y), \quad \beta_{x,y}(t) \in K, \quad \text{for all } t \in [0, 1].$$

Since f is a strictly η-preinvex function, we have

$$\alpha = f\big(\beta_{x,y}(t)\big) < tf(x) + (1 - t)f(y) \leqslant \alpha, \quad \text{for all } t \in [0, 1],$$

which is impossible. □

The following proposition is a generalization of Lemma 4 in [3].

Proposition 3.2. *Let M be a complete Riemannian manifold, $S \subseteq M$ be a geodesic invex set with respect to $\eta : M \times M \to TM$ and $F : S \times S \to \mathbb{R}$ be a continuous geodesic (η, η)-preinvex function, that is, F is η-preinvex with respect to each variable. Then, the function $\psi : S \to \mathbb{R}$ defined by*

$$\psi(x) = \inf_{y\in S} F(x, y),$$

is geodesic η-preinvex.

Proof. Let $x_0, x_1 \in S$ and $\varepsilon > 0$ is given. Since S is a geodesic invex set with respect to η, there exists exactly one geodesic $\alpha_{x_0,x_1} : [0, 1] \to M$ such that

$$\alpha_{x_0,x_1}(0) = x_1, \quad \alpha'_{x_0,x_1}(0) = \eta(x_0, x_1), \quad \alpha_{x_0,x_1}(t) \in S, \quad \text{for all } t \in [0, 1].$$

By the definition of ψ, there exist $y_0, y_1 \in S$ such that

$$F(x_1, y_1) < \psi(x_1) + \varepsilon, \qquad F(x_0, y_0) < \psi(x_0) + \varepsilon.$$

By the geodesic invexity of S with respect to η, there exists exactly one geodesic β_{y_0,y_1}: $[0, 1] \to M$ such that

$$\beta_{y_0,y_1}(0) = y_1, \quad \beta'_{y_0,y_1}(0) = \eta(y_0, y_1), \quad \beta_{y_0,y_1}(t) \in S, \quad \text{for all } t \in [0, 1].$$

It is clear that the curve $\gamma = (\alpha_{x_0,x_1}, \beta_{y_0,y_1}) : [0, 1] \to M$ is a geodesic in $S \times S$, with $\gamma(0) = (x_1, y_1)$ such that for each $t \in [0, 1]$ we have $\gamma(t) = (\alpha_{x_0,x_1}(t), \beta_{y_0,y_1}(t)) \in S \times S$ and

$$\gamma'(0) = \big(\alpha'_{x_0,x_1}(0), \beta'_{y_0,y_1}(0)\big) = \big(\eta(x_0, x_1), \eta(y_0, y_1)\big).$$

By the definition of ψ and the (η, η)-preinvexity of F we have

$$\begin{aligned}\psi\big(\alpha_{x_0,x_1}(t)\big) &= \inf_{y\in S} F\big(\alpha_{x_0,x_1}(t), y\big) \leqslant F\big(\alpha_{x_0,x_1}(t), \beta_{y_0,y_1}(t)\big)\\ &\leqslant tF(x_1, y_1) + (1-t)F(x_0, y_0)\\ &\leqslant t\big(\psi(x_1)+\varepsilon\big) + (1-t)\psi\big((x_0)+\varepsilon\big)\\ &= t\psi(x_1) + (1-t)\psi(x_0) + \varepsilon.\end{aligned}$$

Therefore,

$$\psi\big(\alpha(t)\big) \leqslant t\psi(x_1) + (1-t)\psi(x_0). \qquad \square$$

4. Preinvexity and differentiability

Motivated by [12] we introduce a condition on the function $\eta: M \times M \to TM$ which verifies the relation between geodesic invexity and preinvexity of functions on geodesic invex subsets of Riemannian manifolds.

Definition 4.1. Let M be a Riemannian manifold. We say that the function $\eta: M \times M \to TM$ satisfies the condition (C), if for each $x, y \in M$ and for the geodesic $\alpha: [0, 1] \to M$ satisfying $\alpha(0) = y$, $\alpha'(0) = \eta(x, y)$, we have

(C_1) $P^0_{t,\alpha}\big[\eta\big(y, \alpha(t)\big)\big] = -t\eta(x, y)$,

(C_2) $P^0_{t,\alpha}\big[\eta\big(x, \alpha(t)\big)\big] = (1-t)\eta(x, y)$,

for all $t \in [0, 1]$.

In the following example we show that the function η defined in Example 3.1 satisfies the condition (C).

Example 4.1. Let $S \subseteq M$ and η be the same as the one given in Example 3.1, we show that η satisfies the condition (C_1). If $x \in B(x_0, r_1)$ and $y \in B(y_0, r_2)$ or $x \in B(y_0, r_2)$ and $y \in B(x_0, r_1)$ then, the result is trivially hold. Suppose that $x, y \in B(x_0, r_1)$ or $x, y \in B(y_0, r_2)$. Fix $t \in [0, 1]$ and set $z := \exp_y(t\eta(x, y))$, $v := t\eta(x, y)$ and $w := \eta(y, z) = \exp_z^{-1} y$. Consider the geodesic $\beta(s) := \exp_y(sv)$, $s \in [0, 1]$, hence, $\beta(0) = y$, $\beta(1) = z$ and $\beta'(0) = v$. By the definition of parallel translation it is clear that

$$L_{yz}(v) = P^1_{0,\beta}(v) = \beta'(1). \tag{4}$$

Now, consider the geodesic $\gamma(s) := \exp_z(sw)$, $s \in [0, 1]$, hence, $\gamma(0) = z$, $\gamma(1) = y$ and $\gamma'(0) = w$. Indeed, $\gamma(s) = \beta(1-s)$ and so

$$\gamma'(0) = -\beta'(1). \tag{5}$$

By (4) and (5) and the definition of η we have $\eta(y, z) = \gamma'(0)$ and

$$L_{yz}(v) = \beta'(1) = -\gamma'(0) = -\eta(y, z).$$

Therefore,

$$L_{zy}\big(\eta(y, z)\big) = -v = -t\eta(x, y).$$

Hence, the condition (C_1) is satisfied.

Now, we show that η satisfies the condition (C_2). Let

$$\bar{\alpha}_{x,y}(s) := \alpha_{x,y}(1-s), \quad \text{for all } s \in [0,1],$$

be the reverse curve of $\alpha_{x,y}$, hence $z = \bar{\alpha}_{x,y}(1-t)$. By a similar proof as above we have

$$L_{zx}\big[\eta(x,z)\big] = P^0_{(1-t),\bar{\alpha}_{x,y}}\big[\eta(x,z)\big] = -(1-t)\eta(y,x)$$
$$= -(1-t)\exp_y^{-1} x.$$

By property $L_{xy} o L_{zx} = L_{zy}$ of parallel translation we get

$$L_{xy} o L_{zx}\big[\eta(x,z)\big] = L_{xy}\big[-(1-t)\exp_y^{-1} x\big] = -(1-t)L_{xy}\big[\exp_y^{-1} x\big]$$
$$= (1-t)\eta(x,y).$$

Hence,

$$L_{zy}\big[\eta(x,z)\big] = (1-t)\eta(x,y).$$

Therefore, the condition (C_2) is also hold.

Theorem 4.1. *Let M be a Riemannian manifold and S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$. Assume that $f : S \to \mathbb{R}$ is a differentiable and geodesic η-preinvex function. Then, f is a geodesic η-invex function.*

Proof. By the geodesic invexity of S with respect to η, for every $x, y \in S$ there exists exactly one geodesic $\alpha_{x,y} : [0,1] \to M$ such that

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \eta(x,y), \quad \alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0,1].$$

Since f is geodesic η-preinvex for $t \in (0,1)$, we have

$$f\big(\alpha_{x,y}(t)\big) \leqslant t f(x) + (1-t) f(y),$$

which implies

$$f\big(\alpha_{x,y}(t)\big) - f(y) \leqslant t\big(f(x) - f(y)\big).$$

Divide by t to obtain

$$\frac{1}{t}\big[f\big(\alpha_{x,y}(t)\big) - f(y)\big] \leqslant f(x) - f(y).$$

Taking the limit as $t \to 0$, we have

$$df_{\alpha_{x,y}(0)}\big(\alpha'_{x,y}(0)\big) \leqslant f(x) - f(y).$$

Therefore, $df_y(\eta(x,y)) \leqslant f(x) - f(y)$. □

Theorem 4.2. *Let M be a Riemannian manifold and S be an open subset of M which is also geodesic invex with respect to $\eta : M \times M \to TM$. Suppose that the function $f : S \to \mathbb{R}$ is differentiable. If f is geodesic η-invex on S and η satisfies the condition* (C) *then, f is geodesic η-preinvex on S.*

Proof. By the geodesic invexity of S with respect to η, for every $x, y \in S$ there exists exactly one geodesic $\alpha_{x,y} : [0,1] \to M$ such that

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \eta(x,y), \quad \alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0,1].$$

Fix $t \in [0,1]$ and set $\bar{x} := \alpha_{x,y}(t)$. Then, we have

$$f(x) - f(\bar{x}) \geqslant df_{\bar{x}}\big(\eta(x,\bar{x})\big), \tag{6}$$

$$f(y) - f(\bar{x}) \geqslant df_{\bar{x}}\big(\eta(y,\bar{x})\big). \tag{7}$$

Now, multiplying (6) and (7) by t and $(1-t)$, respectively, and adding we have

$$tf(x) + (1-t)f(y) - f(\bar{x}) \geqslant df_{\bar{x}}\big(t\eta(x,\bar{x}) + (1-t)\eta(y,\bar{x})\big).$$

By the condition (C),

$$t\eta(x,\bar{x}) + (1-t)\eta(y,\bar{x}) = t(1-t)P^t_{0,\alpha_{x,y}}\big[\eta(x,y)\big] + (1-t)(-t)P^t_{0,\alpha_{x,y}}\big[\eta(x,y)\big] = 0.$$

Hence,

$$tf(x) + (1-t)f(y) \geqslant f(\bar{x}). \qquad \square$$

5. Preinvexity and semicontinuity

We recall the definition of a proximal subdifferential of a function defined on a Riemannian manifold and refer the reader to [4,7] for the discussion of proximal calculus on such a manifold.

Definition 5.1. Let M be a Riemannian manifold and $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function. A point $\zeta \in T_yM$ is a proximal subgradient of f at $y \in \operatorname{dom}(f)$, if there exist positive numbers δ and σ such that

$$f(x) \geqslant f(y) + \big\langle \zeta, \exp_y^{-1} x \big\rangle_y - \sigma d(y,x)^2, \quad \text{for all } x \in B(y,\delta),$$

where $\operatorname{dom}(f) := \{x \in M\colon\ f(x) < \infty\}$.

The set of all proximal subgradients of f at $y \in M$ is denoted by $\partial_p f(y)$ and is called the proximal subdifferential of f at y.

Before going into a study of semicontinuous preinvex functions, let us prove the following theorem which will be useful in the sequel.

Theorem 5.1. *Let M be a Riemannian manifold and S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$. Suppose the function $f : S \to \mathbb{R}$ is geodesic η-preinvex. If $\bar{x} \in S$ is a local optimal solution to the problem*

$$\begin{array}{lll} (P) & \min & f(x) \\ & s.t. & x \in S, \end{array}$$

then $\bar{x}$ is a global minimum in (P).

Proof. Suppose that $\bar{x} \in S$ is a local minimum. Then, there is a neighborhood $N_\varepsilon(\bar{x})$ such that

$$f(\bar{x}) \leqslant f(x), \quad \text{for all } x \in S \cap N_\varepsilon(\bar{x}). \tag{8}$$

If $\bar{x}$ is not a global minimum of f then, there exists a point $x^* \in S$ such that

$$f(x^*) < f(\bar{x}).$$

Since S is a geodesic invex set with respect to η, there exits exactly one geodesic α such that

$$\alpha(0) = \bar{x}, \quad \alpha'(0) = \eta(x^*, \bar{x}), \quad \alpha(t) \in S, \quad \text{for all } t \in [0, 1].$$

If we choose $\varepsilon > 0$ small enough such that $d(\alpha(t), \bar{x}) < \varepsilon$ then, $\alpha(t) \in N_\varepsilon(\bar{x})$. By the geodesic η-preinvexity of f we have

$$f\big(\alpha(t)\big) \leqslant t f(x^*) + (1-t) f(\bar{x}) < f(\bar{x}), \quad \text{for all } t \in (0, 1).$$

Therefore, for each $\alpha(t) \in S \cap N_\varepsilon(\bar{x})$, $f(\alpha(t)) < f(\bar{x})$ which is a contradiction to (8). This complete the proof. □

Theorem 5.2. *Let M be a Cartan–Hadamard manifold and S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$ with $\eta(x, y) \neq 0$ for $x \neq y$. Suppose that $f : S \to (-\infty, +\infty]$ is a lower semicontinuous geodesic η-preinvex function. Let $y \in \operatorname{dom}(f)$ and $\zeta \in \partial_p f(y)$. Then, there exists a number $\delta > 0$ such that*

$$f(x) \geqslant f(y) + \langle \zeta, \eta(x, y)\rangle_y, \quad \text{for all } x \in S \cap B(y, \delta). \tag{9}$$

Proof. By the definition of $\partial_p f(y)$, there are positive numbers δ and σ such that

$$f(x) \geqslant f(y) + \langle \zeta, \exp_y^{-1} x\rangle_y - \sigma d(x, y)^2, \quad \text{for all } x \in B(y, \delta). \tag{10}$$

Fix $x \in S \cap B(y, \delta)$. Since S is a geodesic invex set with respect to η, there exits exactly one geodesic $\alpha_{x,y} : [0, 1] \to M$ such that

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \eta(x, y), \quad \alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0, 1].$$

Since M is a Cartan–Hadamard manifold then, $\alpha_{x,y}(t) = \exp_y(t\eta(x, y))$, for each $t \in [0, 1]$ (see [11, p. 253]). If we choose $t_0 = \frac{\delta}{\|\eta(x,y)\|_y}$ then,

$$\exp_y\big(t\eta(x, y)\big) \in S \cap B(y, \delta), \quad \text{for all } t \in (0, t_0).$$

By the geodesic η-preinvexity of f, we have

$$f\big(\exp_y\big(t\eta(x, y)\big)\big) \leqslant t f(x) + (1-t) f(y), \quad \text{for all } t \in (0, t_0). \tag{11}$$

By combining (10) and (11) for each $t \in (0, t_0)$ we get

$$\begin{aligned} f\big(\exp_y\big(t\eta(x, y)\big)\big) &\geqslant f(y) + \big\langle \zeta, \exp_y^{-1} \exp_y\big(t\eta(x, y)\big)\big\rangle_y - \sigma d\big(\exp_y\big(t\eta(x, y)\big), y\big)^2 \\ &= f(y) + \big\langle \zeta, t\eta(x, y)\big\rangle_y - \sigma d\big(\exp_y\big(t\eta(x, y)\big), y\big)^2. \end{aligned} \tag{12}$$

Since M is a Cartan–Hadamard manifold, for each $t \in (0, t_0)$,

$$d\big(\exp_y\big(t\eta(x, y)\big), y\big)^2 = \big\|t\eta(x, y)\big\|_y^2 = t^2 \big\|\eta(x, y)\big\|_y^2. \tag{13}$$

By (11) and (13) we have

$$\begin{aligned} t f(x) + (1-t) f(y) &\geqslant f\big(\exp_y\big(t\eta(x, y)\big)\big) \\ &\geqslant f(y) + \big\langle \zeta, t\eta(x, y)\big\rangle_y - t^2 \big\|\eta(x, y)\big\|_y^2. \end{aligned}$$

Hence,

$$t\big(f(x)-f(y)\big) \geqslant t\big\langle \zeta, \eta(x,y)\big\rangle_y - t^2\big\|\eta(x,y)\big\|_y^2,$$

divide by t to obtain

$$f(x)-f(y) \geqslant \big\langle \zeta, \eta(x,y)\big\rangle_y - t\big\|\eta(x,y)\big\|^2.$$

Taking limit as $t \to 0$ we obtain

$$f(x)-f(y) \geqslant \big\langle \zeta, \eta(x,y)\big\rangle_y.$$

Since $x \in S \cap B(y,\delta)$ is arbitrary, thus (9) holds and the proof is complete. □

Corollary 5.1. *Let M be a Cartan–Hadamard manifold and S be an open subset of M which is geodesic invex with respect to $\eta : M \times M \to TM$. Suppose that $f : S \to \mathbb{R}$ is a lower semicontinuous geodesic η-preinvex function. Let $y \in S$ and $0 \in \partial_p f(y)$. Then, y is a global minimum of f.*

Proof. By Theorem 5.2, y is a local minimum of f and hence by Theorem 5.1, y is a global minimum of f. □

It should be noted that if S is a subset of a Riemannian manifold M and $f : S \to (-\infty, +\infty]$ is a lower semicontinuous function which has a local minimum at $y \in S$ then, $0 \in \partial_p f(y)$ (see [4]).

6. Mean value theorem

T. Antczak in [1] proved the mean value inequality and the mean value theorem in invexity analysis. Now we extend these notions to Cartan–Hadamard manifolds.

Definition 6.1. Let S be a nonempty subset of a Riemannian manifold M which is geodesic invex with respect to $\eta : M \times M \to TM$ and x and u be two arbitrary points of S. Let $\alpha : [0,1] \to M$ be the unique geodesic such that

$$\alpha(0) = u, \quad \alpha'(0) = \eta(x,u), \quad \alpha(t) \in S, \quad \text{for all } t \in [0,1].$$

A set P_{uv} is said to be a closed η-path joining the two points u and $v := \alpha(1)$ if

$$P_{uv} := \big\{y:\ y = \alpha(t);\ t \in [0,1]\big\}.$$

An open η-path joining the points u and v is a set of the form

$$P_{uv}^0 := \big\{y:\ y = \alpha(t);\ t \in (0,1)\big\}.$$

If $u = v$ we set $P_{uv}^0 := \phi$.

Example 6.1. Suppose that M is a Cartan–Hadamard manifold and S is the geodesic invex set with respect to η defined in Example 3.1. Let x and u be two arbitrary points of S and $\alpha(t) := \exp_u(t \exp_u^{-1} x)$. Then:

(i) If $x, u \in B(x_0, r_1)$ or $x, u \in B(y_0, r_2)$, then $P_{uv} = P_{ux}$ is the unique geodesic with end points u and x and velocity vector $\eta(x,y) = \exp_u^{-1} x$.

(ii) If $u \in B(x_0, r_1)$, $x \in B(y_0, r_2)$ or $u \in B(y_0, r_2)$, $x \in B(x_0, r_1)$, then $P_{uv} = P_{uu} = \{u\}$, and $P^0_{uv} = \phi$.

Example 6.2. Suppose that M is a Cartan–Hadamard manifold and S is the geodesic invex set with respect to η defined in Example 3.2. Let x and u be two arbitrary points of S and $\alpha(t) := \exp_u(t\eta(x,u))$. Then, for $u \in S_1$, $x \in S_2$ we have $v = \alpha(1) = P_{S_1}(x)$ and P_{uv} is the unique geodesic with end points u and $P_{S_1}(x)$.

Theorem 6.1 *(Mean value inequality). Let M be a Cartan–Hadamard manifold and $S \subseteq M$ be a geodesic invex set with respect to $\eta : M \times M \to TM$ such that $\eta(a,b) \neq 0$ for all $a, b \in S$, $a \neq b$. Suppose that $\alpha_{b,a}(t) = \exp_a(t\eta(b,a))$ for every $a, b \in S$, $t \in [0,1]$, and $c = \alpha_{b,a}(1)$. Then, a necessary and sufficient condition for a function $f : S \to \mathbb{R}$ to be geodesic η-preinvex is that the inequality*

$$f(x) \leqslant f(a) + \frac{f(b) - f(a)}{\langle \eta(b,a), \eta(b,a)\rangle_a} \bigl\langle \exp_a^{-1} x, \eta(b,a)\bigr\rangle_a \tag{14}$$

holds, for each $x \in P_{ca}$.

Proof. *Necessity*. Let $f : S \to \mathbb{R}$ be a geodesic η-preinvex function, $a, b \in S$ and $x \in P_{ca}$. If $x = a$ or $x = c$, then (14) trivially holds. If $x \in P^0_{ca}$, then $x := \exp_a(t\eta(b,a))$, for some $t \in (0,1)$. By the geodesic η-invexity of S it follows that $x \in S$ and

$$t = \frac{\langle \exp_a^{-1} x, \eta(b,a)\rangle_a}{\langle \eta(b,a), \eta(b,a)\rangle_a}.$$

Since f is a geodesic η-preinvex function on S, we have

$$\begin{aligned} f(x) &= f\bigl(\exp_a\bigl(t\eta(b,a)\bigr)\bigr) \\ &\leqslant tf(b) + (1-t)f(a) \\ &= f(a) + t\bigl[f(b) - f(a)\bigr] \\ &= f(a) + \frac{f(b) - f(a)}{\langle \eta(b,a), \eta(b,a)\rangle_a} \bigl\langle \exp_a^{-1} x, \eta(b,a)\bigr\rangle_a. \end{aligned}$$

Sufficiency. Assume that the Mean value inequality (14) holds. Let $a, b \in S$ and $x := \exp_a(t\eta(b,a))$, for some $t \in [0,1]$. Then, $x \in S$ and we have

$$\begin{aligned} f(x) &= f\bigl(\exp_a\bigl(t\eta(b,a)\bigr)\bigr) \\ &\leqslant f(a) + \frac{f(b) - f(a)}{\langle \eta(b,a), \eta(b,a)\rangle_a} \bigl\langle \exp_a^{-1} x, \eta(b,a)\bigr\rangle_a \\ &= f(a) + \frac{f(b) - f(a)}{\langle \eta(b,a), \eta(b,a)\rangle_a} \bigl\langle \exp_a^{-1}\bigl(\exp_a\bigl(t\eta(b,a)\bigr)\bigr), \eta(b,a)\bigr\rangle_a \\ &= f(a) + \frac{f(b) - f(a)}{\langle \eta(b,a), \eta(b,a)\rangle_a} t\bigl\langle \eta(b,a), \eta(b,a)\bigr\rangle_a \\ &= tf(b) + (1-t)f(a). \end{aligned}$$

We conclude that f is a geodesic η-preinvex function. □

Theorem 6.2 *(Mean value theorem). Let M be a Cartan–Hadamard manifold and $S \subseteq M$ be a nonempty, open geodesic invex set with respect to $\eta : M \times M \to TM$. Suppose that $f : S \to \mathbb{R}$ is differentiable on S. Then, for every a, $b \in S$ there exists $c \in P^0_{ab}$ such that*

$$f\big(\exp_a\big(\eta(b,a)\big)\big) - f(a) = df_c\big[(d\exp_a)_u\big(\eta(b,a)\big)\big],$$

where $u := t_0\eta(b,a)$, $t_0 \in (0,1)$, and $(d\exp_a)_u : T_u(T_aM) \cong T_aM \to T_cM$ is the differential of $\exp_a$ at u.

Proof. We define the function $g : [0,1] \to \mathbb{R}$ as follows:

$$g(t) := f\big(\exp_a\big(t\eta(b,a)\big)\big) - f(a) - t\big[f\big(\exp_a\big(\eta(b,a)\big)\big) - f(a)\big]. \tag{15}$$

Since $g(1) = g(0) = 0$ then, using Rolle's theorem, it follows that there exists $t_0 \in (0,1)$ such that $g'(t_0) = 0$. Let $c := \exp_a(t_0\eta(b,a))$ then, by (15), we have

$$0 = g'(t_0) = df_c\big[(d\exp_a)_u\big(\eta(b,a)\big)\big] - f\big(\exp_a\big(\eta(b,a)\big)\big) - f(a).$$

Since $t_0 \in (0,1)$, by definition $c \in P^0_{ab}$ and the proof is complete. □

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