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Smoothing and Rothe's method for Stefan problems in enthalpy form

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Abstract

The classical two-phase Stefan problem as well as its weak variational formulation model the connection between the different phases of the considered material by interface conditions at the occurring free boundary or by a jump of the enthalpy. One way to treat the corresponding discontinuous variational problems consists in its embedding into a family of continuous ones and applying some standard techniques to the chosen approximation problems. The aim of the present paper is to analyze a semi-discretization via Rothe's method and its convergence behavior in dependence of the smoothing parameter. While in Grossmann et al. (Optimization, in preparation) the treatment of the Stefan problem is based on the given variable, i.e. the temperature, here first a transformation via the smoothed enthalpy is applied. Numerical experiments indicate a higher stability of the discretization by Rothe's method. In addition, to avoid inner iterations a frozen coefficient approach as common in literature is used. © 2002 Elsevier Science B.V. All rights reserved.

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1. Formulation of the problem

Parabolic problems with discontinuities require specific techniques for its numerical treatment. One of these methods is based on a parametric embedding of the original problem into a family of smooth ones. However, the original discontinuity causes an asymptotically singular behavior of the embeddings. In particular, some continuity modules cannot be uniformly bounded as the smoothing parameter tends to its supposed limit. This phenomenon requires an appropriate modification of the convergence theory of the methods for the smoothed variational equations.

In the present paper we deal with a discontinuous parabolic problem which e.g. arises from the enthalpy formulation of two-phase Stefan problems and apply a parametric smoothing of the jump

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in the enthalpy. Such a behavior of the enthalpy occurs in phase transitions like melting or freezing and causes naturally a discontinuity of the problem at critical temperatures (cf. [2,5,14,16]).

As essential tool for the analysis as well as for the numerical treatment of the problem we apply Rothe's method of semi-discretization. However, unlike in [8,9] we do not concentrate upon a direct treatment of the equation for the temperature, but study a transformed problem expressed in terms of values of the smoothed enthalpy. Further, to avoid inner iterations we use a frozen coefficient approach instead of the originally implicit scheme by Rothe which would lead to a system of nonlinear elliptic problems.

For given v_0, f, g let us consider the nonlinear parabolic problem

$$\begin{aligned}&\frac{\partial}{\partial t}H(u) - \Delta u = f \quad \text{a.e. in } \Omega_T,\\ &H(u(\cdot,0)) \ni v_0(\cdot) \quad \text{a.e. in } \Omega,\\ &u_{|\Gamma_T} = g,\end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^m$, $m \in \mathbb{N}$, is a bounded region with smooth boundary Γ. Properties assumed for v_0, f, g will be specified later. Further, with $T > 0$ we denote $I := (0,T)$, $\Omega_T := \Omega \times I$ and $\Gamma_T := \Gamma \times I$. The function $H(\cdot)$ is defined by

$$H(z) = \begin{cases} a(z) & \text{for } z < 0,\\ [0,\lambda] & \text{for } z = 0,\\ a(z)+\lambda & \text{for } z > 0 \end{cases} \tag{2}$$

with a monotonically increasing, smooth function $a : \mathbb{R} \to \mathbb{R}$ and some constant $\lambda > 0$. The mapping $H(u(\cdot,\cdot))$ can be interpreted as enthalpy related to the temperature $u(\cdot,\cdot)$ of the underlying heat transfer process and λ stands for the latent heat occurring in phase transition.

Due to the discontinuity of $H(\cdot)$, as a rule, we cannot expect that a classical solution of problem (1) exists. But a related weak formulation provides the framework to extend existence and stability results also to problems with discontinuities. Following Friedman [6], Alt and Luckhaus [1] a pair $u \in W_2^{1,0}(\Omega_T)$, $v \in L_2(\Omega_T)$ is called a weak solution of (1) if it satisfies

$$\begin{aligned}&u - g \in \overset{0}{W}{}_2^{1,0}(\Omega_T),\\ &v \in H(u) \quad \text{a.e. in } \Omega_T,\\ &\int_{\Omega_T}\left(-v\frac{\partial}{\partial t}\varphi + \nabla u \nabla\varphi\right)\mathrm{d}\omega = \int_{\Omega_T} f\varphi\,\mathrm{d}\omega + \int_\Omega v_0(x)\varphi(x,0)\,\mathrm{d}x,\\ &\forall\varphi \in \overset{0}{W}{}_2^{1,1}(\Omega_T) \quad \text{with } \varphi = 0 \quad \text{a.e. in } \Omega\times\{T\}.\end{aligned} \tag{3}$$

Before we continue, let us indicate the connection to the classical formulation of two-phase Stefan problems (cf. [6]). We assume u to be a solution of the considered problem. Suppose Ω_T can be splitted into two parts $\overline{\Omega_T} = \overline{\Omega_T^1} \cup \overline{\Omega_T^2}$ by means of two connected regions

$$\Omega_T^1 := \{(x,t) \in \Omega_T \mid u(x,t) < 0\}, \quad \Omega_T^2 := \{(x,t) \in \Omega_T \mid u(x,t) > 0\}.$$

Suppose further that the set of points in Ω_T where the phase transition takes place is a hyper-surface which can be represented by some smooth function $\Phi : \mathbb{R}^{m+1} \to \mathbb{R}$ via

$$\Gamma^* = \{(x,t) \in \Omega_T \mid \Phi(x,t) = 0\}.$$

Obviously $\Gamma^* = \overline{\Omega_T^1} \cap \overline{\Omega_T^2}$. Hence, it is not a priori known but is a free boundary depending on the solution u.

Let us introduce the sets $\Omega^1 := \{x \in \Omega \mid u(x,0) < 0\}$ and $\Omega^2 := \{x \in \Omega \mid u(x,0) > 0\}$. Further we use the abbreviations $\Gamma_T^i := \partial\Omega_T^i \cap \Gamma_T$, $u_i := u_{|\Omega_T^i}$, $f_i := f_{|\Omega_T^i}$, $g_i := u_{|\Gamma_T^i}$ and $z_i := H^{-1}(v_0)$ on Ω^i for $i = 1,2$. Then the boundary value problem (1) corresponds to the classical two-phase Stefan problem

$$\begin{aligned}
&\frac{\partial}{\partial t}a(u_1) - \Delta u_1 = f_1 \quad \text{on } \Omega_T^1,\\
&u_{1|\Gamma_T^1} = g_1, \quad u_{1|\Omega^1} = z_1,\\
&\frac{\partial}{\partial t}a(u_2) - \Delta u_2 = f_2 \quad \text{on } \Omega_T^2,\\
&u_{2|\Gamma_T^2} = g_2, \quad u_{2|\Omega^2} = z_2,\\
&u_{1|\Gamma^*} = u_{2|\Gamma^*},\\
&\lambda\frac{\partial \Phi}{\partial t} = (\nabla u_1 - \nabla u_2, \nabla\Phi) \quad \text{on } \Gamma^*,
\end{aligned} \tag{4}$$

where $(\cdot,\cdot)$ denotes the standard scalar product in $\mathbb{R}^{m+1}$. Note that a function u which is sufficiently smooth in the subdomains Ω_T^1, Ω_T^2 and satisfies (4) can be completed by an appropriate $v \in H(u)$ to a weak solution of (3). Conversely, if a solution u, v of the weak formulation (3) is regular enough and if the free boundary Γ^* has the indicated properties then the related restrictions u_1, u_2 of u satisfy the classical Stefan problem (4). Indeed, by means of integration by parts applied to the subdomains Ω_T^1 and Ω_T^2 of Ω_T we obtain

$$\begin{aligned}
\int_{\Omega_T}\left(\frac{\partial H}{\partial t}(u) - \Delta u\right)\varphi\,\mathrm{d}\omega &= \int_{\Omega_T^1}\left(\frac{\partial H}{\partial t}(u) - \Delta u\right)\varphi\,\mathrm{d}\omega + \int_{\Omega_T^2}\left(\frac{\partial H}{\partial t}(u) - \Delta u\right)\varphi\,\mathrm{d}\omega\\
&= \int_{\Omega_T}\left(-H(u)\frac{\partial\varphi}{\partial t} + \nabla u\nabla\varphi\right)\mathrm{d}\omega + \int_{\Omega} v_0(x)\varphi(x,0)\,\mathrm{d}x\\
&\quad + \int_{\Gamma^*}(H(u_1)(\mathbf{n}^1,\mathbf{e}^{m+1}) + H(u_2)(\mathbf{n}^2,\mathbf{e}^{m+1}))\varphi\,\mathrm{d}\omega\\
&\quad - \sum_{i=1}^{m}\int_{\Gamma^*}\left(\frac{\partial u_1}{\partial x^i}(\mathbf{n}^1,\mathbf{e}^i) + \frac{\partial u_2}{\partial x^i}(\mathbf{n}^2,\mathbf{e}^i)\right)\varphi\,\mathrm{d}\gamma
\end{aligned} \tag{5}$$

for all $\varphi \in \overset{0}{W}{}_2^{1,1}(\Omega_T)$ satisfying $\varphi(\cdot,T) = 0$, where $\mathbf{n}^i$ denote the outside normals of Ω_T^1 at Γ^* and $\mathbf{e}^j$ $(j = 1,\dots,m+1)$ are the unit vectors of the cartesian coordinate system in space and time. By this setting we have $\mathbf{n}^2 = -\mathbf{n}^1$. According to the assumptions on Γ^* the vector $\mathbf{n} = (\nabla\Phi, (\partial/\partial t)\Phi)^T$ is normal at Γ^* and without loss of generality we can assume, that it points outward of Ω_T^1. Thus

$\mathbf{n}^1 = \mathbf{n}/|\mathbf{n}|$. Let now u be a solution of (1) and let $v \in H(u)$ be such that u, v solves the weak problem (3). Then from (2), (3) and (5) it follows

$$\int_{\Gamma^*} \lambda \frac{\partial}{\partial t} \Phi\varphi \, \mathrm{d}\gamma + \int_{\Gamma^*} (\nabla u_1 - \nabla u_2, \nabla \Phi)\varphi \, \mathrm{d}\gamma = 0 \quad \forall \varphi \in \overset{0}{W}{}_2^{1,1}(\Omega_T) \tag{6}$$

and according to the smoothness of u this implies that the Stefan condition

$$\lambda \frac{\partial \Phi}{\partial t} = (\nabla u_1 - \nabla u_2, \nabla \Phi) \quad \text{on } \Gamma^* \tag{7}$$

is satisfied. For further studies on Stefan problems we refer e.g. to [4,5,14–19].

2. Regularization and weak formulation

For the sake of simplification of the presentation in the sequel we assume that $H(\cdot)$ has the specific form

$$H(z) = \begin{cases} z & \text{for } z < 0, \\ [0, \lambda] & \text{for } z = 0, \quad z \in \mathbb{R} \\ z + \lambda & \text{for } z > 0 \end{cases} \tag{8}$$

with some constant $\lambda > 0$. In particular, this means that we assume the same heat conduction coefficient in both phases and that the phase transition takes place at $u = 0$. We investigate a penalty type smoothing replacing H by

$$H_\varepsilon(z) = z + \frac{\lambda}{2}\left(1 + \frac{z}{\sqrt{z^2 + \varepsilon}}\right), \quad z \in \mathbb{R}. \tag{9}$$

Here $\varepsilon > 0$ denotes a parameter which controls the approximation of H by H_ε, in particular, for $\varepsilon \to 0+$. In case $H(\cdot)$ has not only a jump but also the one-sided derivatives are discontinuous, then the technique can be applied in a similar way. To illustrate this, let us consider a slightly more general situation with different heat conductivities in the two phases which is characterized by

$$H(z) = \begin{cases} a_1 z & \text{for } z < 0, \\ [0, \lambda] & \text{for } z = 0, \quad z \in \mathbb{R}, \\ a_2 z + \lambda & \text{for } z > 0, \end{cases} \tag{10}$$

with constants $a_2 \geqslant a_1 > 0$. In this case the jump in the enthalpy at $z = 0$ could be smoothed similarly by using the function

$$H_\varepsilon(z) = a_1 z + \frac{a_2 - a_1}{2}(z + \sqrt{z^2 + \varepsilon}) + \frac{\lambda}{2}\left(1 + \frac{z}{\sqrt{z^2 + \varepsilon}}\right). \tag{11}$$

Notice that the maximal monotone graph of the enthalphy $H(\cdot)$ can be represented as the subdifferential of the convex function

$$\beta(s) = \int_0^s a(\xi) \, \mathrm{d}\xi + \lambda \max\{0, s\}, \quad s \in \mathbb{R}. \tag{12}$$

For a further study of the extended smoothing technique (11) in Stefan problems we refer to [20].

Unlike H the mapping H_ε defined by (9) is single valued and even C^∞. So instead of the discontinuous problem (1) we obtain a classical nonlinear parabolic problem

$$\begin{aligned}&\frac{\partial}{\partial t}H_\varepsilon(u_\varepsilon) - \Delta u_\varepsilon = f \quad \text{a.e. in } \Omega_T,\\ &H_\varepsilon(u_\varepsilon(\cdot,0)) = v_0(\cdot) \quad \text{a.e. in } \Omega,\\ &u_{\varepsilon|\Gamma_T} = g\end{aligned} \tag{13}$$

depending on the smoothing parameter $\varepsilon > 0$.

In difference to [8], we transform problem (13) by expressing the temperature u_ε via the smoothed enthalpy $v_\varepsilon := H_\varepsilon(u)$. This yields the nonlinear parabolic boundary value problem

$$\begin{aligned}&\frac{\partial}{\partial t}v_\varepsilon - \nabla(q_\varepsilon(v_\varepsilon)\nabla v_\varepsilon) = f \quad \text{in } \Omega_T,\\ &v_\varepsilon(\cdot,0) = v_0(\cdot) \quad \text{in } \Omega,\\ &v_{\varepsilon|\Gamma_T} = H_\varepsilon(g)\end{aligned} \tag{14}$$

with

$$q_\varepsilon(z) := \frac{1}{H_\varepsilon'(H_\varepsilon^{-1}(z))} \quad \forall z \in \mathbb{R}, \tag{15}$$

which is equivalent to the regularized problem (13). From the properties of H_ε we immediately obtain

$$\frac{\varepsilon^{1/2}}{\varepsilon^{1/2} + \lambda/2} \leqslant q_\varepsilon(z) \leqslant 1, \quad \forall \varepsilon > 0,\ \forall z \in \mathbb{R}. \tag{16}$$

We put

$$\gamma(\varepsilon) := \frac{\varepsilon^{1/2}}{\varepsilon^{1/2} + \lambda/2}.$$

Obviously, $\gamma(\varepsilon)$ behaves asymptotically like $\varepsilon^{1/2}$ as $\varepsilon \to 0+$.

Throughout the paper $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_*$ denote the norms in $L_2(\Omega)$, $W_2^1(\Omega)$ and the dual space $H^{-1} = (W_2^1(\Omega))^*$, respectively, $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ denote the scalar product in $L_2(\Omega)$, and the evaluation of an element of the dual space H^{-1} applied to an element of $W_2^1(\Omega)$, respectively. For brevity we write v instead of v_ε and for any function $z : \Omega \times I \to \mathbb{R}$ we denote by $z(t)$ the function $z(t,\cdot) : \Omega \to \mathbb{R}$ for any fixed time $t \in I$.

In the sequel we concentrate on a weak formulation of problem (14). A function $v \in \overset{0}{W}{}_2^{1,0}(\Omega_T)$ with $(\partial/\partial t)v \in L_2(I, H^{-1})$ is called a weak solution of (14) if

$$\begin{aligned}&\left(\frac{\partial}{\partial t}v(t), \varphi\right) + \langle q_\varepsilon(v(t))\nabla v(t), \nabla\varphi\rangle = \langle f(t), \varphi\rangle, \quad \forall \varphi \in \overset{0}{W}{}_2^1(\Omega),\\ &v(0,\cdot) = v_0(\cdot) \quad \text{in } \Omega\end{aligned} \tag{17}$$

for almost every $t \in I$.

In order to guarantee appropriate a priori estimates for the weak solution throughout the paper we assume that the domain and the occurring boundary data of problem (1) and (3), respectively, satisfy the following conditions (cf. [6,12,14]):

(i) In the case $\Omega \subset \mathbb{R}^m$, $m > 1$ there exist two simply connected domains $G_1, G_2 \subset \mathbb{R}^m$ with $G_1 \subset G_2$ such that $\Omega = G_2 \setminus \bar{G}_1$. Hence, the boundary Γ is split into two separated parts $\Gamma_i := \Gamma \cap \partial G_i$, $i = 1, 2$, which are smooth. In the case $\Omega \subset \mathbb{R}^1$ the sets Γ_1, Γ_2 denote the two endpoints of the interval Ω.

With this splitting for the boundary function g we assume

$$\inf_{(x,t)\in\Gamma_1\times[0,T]} g(x,t) > 0 \quad \text{and} \quad \sup_{(x,t)\in\Gamma_2\times[0.T]} g(x,t) < 0,$$

i.e. near Γ_1 we suppose a liquid phase while near Γ_2 a solid phase is located.

(ii) Boundary and initial conditions can be extended to a function $\Psi \in C^{2+\beta,1+\beta/2}(\bar{\Omega}_T)$ with some Hölder exponent $\beta > 0$, i.e.

$$\Psi|_{\Gamma_T} = g \quad \text{and} \quad H(\Psi)|_{\Omega\times\{0\}} = v_0,$$

and the right hand side f satisfies $f \in C^{0+\beta,0+\beta/2}(\bar{\Omega}_T)$.

Note that according to assumption (ii) the boundary function g is continuous and according to assumption (i) there exists an extension $b \in L_\infty(I, W_2^1(\Omega))$ with $\frac{\partial}{\partial t} b \in L_\infty(\Omega_T)$ of the boundary function $b_{|\Gamma_T} = H(g)$. Further the smoothness of H_ε guarantees the existence of smooth functions $b_\varepsilon : \bar{\Omega}_T \to \mathbb{R}$ with trace $b_{\varepsilon|\Gamma_T} = H_\varepsilon(g)$ such that $b_\varepsilon \to b$ in $L_\infty(I, W_2^1(\Omega))$ and $(\partial/\partial t) b_\varepsilon \to (\partial/\partial t) b$ in $L_\infty(\Omega_T)$ as $\varepsilon \to 0+$. We notice that the existence of such an extension essentially depends on the fact that no phase transition is allowed at a Dirichlet boundary. If also Neumann conditions are given, then (i) may be relaxed.

We remark that in [12] the existence of the unique solution of (14) in $C^{2+\beta,1+\beta/2}(\bar{\Omega}_T)$ is proved under the given assumptions on g and v_0. Concerning the existence and uniqueness of the weak solution of (17) we refer to [13]. There under slightly milder assumptions the author shows the existence of the unique solution $v \in L_2(\Omega_T) \cap L_\infty(I, H^{-1}(\Omega))$ for the weak formulation of problem (14) without any preceding regularization. For the proof, unlike our approach, in [13] via the unique solution $w \in L_2(\Omega_T) \cap L_\infty(I, H^{-1}(\Omega))$ of

$$\left\langle \frac{\partial}{\partial t} w, \varphi \right\rangle + \langle A(w), \varphi \rangle = \langle \tilde{f}, \varphi \rangle \quad \text{for all } \varphi \in L_2(\Omega),$$

with a functional $\langle A(w), \varphi \rangle := \langle H^{-1}(w), \varphi \rangle$ the author applies a Galerkin technique with the eigenfunctions of $-\Delta$ as basic functions to show that v solves also the original problem. In our approach the uniqueness result of [13] will be of importance.

3. Rothe's method applied to smoothed problems

Next we apply a semi-discretization to derive analytical properties of the given problem as well as to prepare a numerical treatment. Let an equidistant grid $\{t_k\}_{k=0}^N$ with step size $\tau = T/N$ over the time interval I be chosen, i.e. $t_k = k\tau$, $k = 0, \ldots, N$. Then we consider a linearly implicit Euler scheme of (17) with coefficients q_ε frozen at the previous time level. This yields

Find $v^k \in W_2^1(\Omega)$, $\quad v^k|_\Gamma = H_\varepsilon(g(t^k))$, $k = 1,\dots,N$ such that

$$\left\langle \frac{1}{\tau}(v^k - v^{k-1}), \varphi \right\rangle + \langle q^{k-1}\nabla v^k, \nabla\varphi\rangle = \langle f^k, \varphi\rangle, \quad \forall \varphi \in \overset{0}{W}{}_2^1(\Omega), \tag{18}$$

with given initial value v_0, boundary conditions and the abbreviations $q^{k-1} := q_\varepsilon(v^{k-1})andf^k := f(\cdot, t_k)$ for $k = 1,\dots,N$. Due to continuity $g(t^k)$ and f^k are well-defined. The strong formulation of the semi-discretization (18) corresponding to (14) reads as follows

$$\frac{1}{\tau}(v^k - v^{k-1}) - \nabla(q^{k-1}\nabla v^k) = f^k \tag{19}$$

for $k = 1,\dots,N$. With the bilinear forms

$$a_k(\psi, \varphi) := \langle \psi, \varphi\rangle + \tau\langle q^{k-1}\nabla\psi, \nabla\varphi\rangle, \quad \forall \psi, \varphi \in \overset{0}{W}{}_2^1(\Omega),\ k = 1,\dots,N,$$

system (18) is equivalent to

$$a_k(v^k, \varphi) = \tau\langle f^k, \varphi\rangle + \langle v^{k-1}, \varphi\rangle, \quad \forall \varphi \in \overset{0}{W}{}_2^1(\Omega). \tag{20}$$

These problems (20) are linear elliptic and by standard arguments posses unique solutions $v^k \in H^1(\Omega)$, $k = 1,\dots,N$. Indeed, the right-hand sides of (20) are linear and continuous in $\varphi \in \overset{0}{W}{}_2^1$ while the mappings $a_k(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ are bilinear, continuous, and on $\overset{0}{W}{}_2^1(\Omega) \subset H^1(\Omega)$ are elliptic. In particular, we have

$$a_k(\varphi, \psi) \leqslant \max\{1, \tau\}\|\varphi\|_1\|\psi\|_1, \quad \forall \varphi, \psi \in H^1(\Omega),$$

$$\min\{1, \tau\gamma(\varepsilon)\}\|\varphi\|_1^2 \leqslant a_k(\varphi, \varphi), \quad \forall \varphi \in \overset{0}{W}{}_2^1(\Omega). \tag{21}$$

Hence, due to Lax–Milgram lemma the existence and uniqueness of $v^k \in H^1(\Omega)$, $k = 1,\dots,N$ is guaranteed in case of $\varepsilon > 0$. However, as indicated in (21), the ellipticity constant depends on the smoothing parameter $\varepsilon > 0$ and tends to zero for $\varepsilon \to 0+$ even for a fixed step size $\tau > 0$. This gives rise to expect that the elliptic subproblems (20) for finding v^k are asymptotically ill-posed if $\varepsilon \to 0+$.

Now we give some a priori estimates for the discrete solution $\{v^k\}_{k=1}^N$ which we use later to derive a convergence result for the Rothe method. In the sequel c denotes a generic positive constant which is independent of the smoothing parameter ε and which may be different at different places of occurrence.

Lemma 1. *Let* $\{v^k\}_{k=1}^N$ *for any* $N \in \mathbb{N}$ *with step size* $\tau = T/N$ *be the unique solution of* (18) *with initial value* $v^0 = v_0$. *Then the estimates*

1. $\max_{1 \leqslant k \leqslant N} \|v^k\|^2 \leqslant c$,
2. $\tau \sum_{k=1}^N \|\sqrt{q^{k-1}}\nabla v^k\|^2 \leqslant c$,
3. $\sum_{k=1}^N \|v^k - v^{k-1}\|^2 \leqslant c$

hold uniformly for $N \in \mathbb{N}$.

Proof. For the proof we consider a corresponding initial-boundary problem to (14) with homogeneous boundary conditions. Under the regularity assumptions given above this can be derived by introducing the shift $w := v + b_\varepsilon$, where $b_\varepsilon \in L_\infty(I, W_2^1(\Omega))$ with $(\partial/\partial t)b_\varepsilon \in L_\infty(\Omega_T)$ denotes an appropriate extension of the boundary function $b_{\varepsilon|\Gamma_T} = H_\varepsilon(g)$ on the whole domain Ω_T. This way we obtain the parabolic problem

$$\begin{aligned}&\frac{\partial}{\partial t}w - \nabla(q_\varepsilon(w+b_\varepsilon)\nabla(w+b_\varepsilon)) = f - \frac{\partial}{\partial t}b_\varepsilon \quad \text{in } \Omega_T,\\ &w(\cdot,0) = v_0(\cdot) + b_\varepsilon(\cdot,0) \quad \text{in } \Omega,\\ &w|_{\Gamma_T} = 0,\end{aligned} \tag{22}$$

which is equivalent to (14).

Now at each time level $t = t_k$ we choose in (18) the test function $\varphi = w^k := v^k - b^k$ where $b^k := b_\varepsilon(\cdot, t^k)$, $k = 1,\ldots,N$ and obtain

$$\frac{1}{\tau}\langle v^k - v^{k-1}, w^k\rangle + \langle q^{k-1}\nabla v^k, \nabla w^k\rangle = \langle f^k, w^k\rangle, \quad k = 1,\ldots,N. \tag{23}$$

Using the definition of w^k and rearranging yields

$$\langle w^k - w^{k-1}, w^k\rangle + \tau\langle q^{k-1}\nabla w^k, \nabla w^k\rangle = \tau\langle F^k, w^k\rangle - \tau\langle q^{k-1}\nabla b^k, \nabla w^k\rangle \tag{24}$$

for $k=1,\ldots,N$ where $F^k := f^k - \frac{1}{\tau}(b^k - b^{k-1})$. Now, we indicate that the properties of scalar products imply

$$\begin{aligned}\langle w^k - w^{k-1}, w^k\rangle - \tfrac{1}{2}(\|w^k\|^2 - \|w^{k-1}\|^2) &= \langle w^k - w^{k-1}, w^k\rangle - \tfrac{1}{2}\langle w^k - w^{k-1}, w^k + w^{k-1}\rangle\\ &= \tfrac{1}{2}\|w^k - w^{k-1}\|^2 \geqslant 0.\end{aligned} \tag{25}$$

Hence, we have

$$\langle w^k - w^{k-1}, w^k\rangle \geqslant \tfrac{1}{2}(\|w^k\|^2 - \|w^{k-1}\|^2). \tag{26}$$

Let $II := \langle q^{k-1}\nabla w^k, \nabla w^k\rangle + \langle q^{k-1}\nabla b^k, \nabla w^k\rangle$. Using the positivity of $q_\varepsilon(\cdot)$. We may rewrite this expression in the following way:

$$II = \langle \sqrt{q^{k-1}}\nabla w^k, \sqrt{q^{k-1}}\nabla w^k\rangle + \langle \sqrt{q^{k-1}}\nabla b^k, \sqrt{q^{k-1}}\nabla w^k\rangle.$$

The assumed boundedness of q_ε and ∇b_ε and the Schwarz inequality yield

$$\begin{aligned}II &\geqslant (\|\sqrt{q^{k-1}}\nabla w^k\| - \|\sqrt{q^{k-1}}\nabla b^k\|)\|\sqrt{q^{k-1}}\nabla w^k\|\\ &\geqslant (\|\sqrt{q^{k-1}}\nabla w^k\| - C_0)\|\sqrt{q^{k-1}}\nabla w^k\|\end{aligned} \tag{27}$$

with some constant $C_0 > 0$. Then (23) and (27) result in

$$\langle w^k - w^{k-1}, w^k\rangle \leqslant \tau\langle F^k, w^k\rangle + \tau(C_0 - \|\sqrt{q^{k-1}}\nabla w^k\|)\|\sqrt{q^{k-1}}\nabla w^k\|.$$

Now (25), the assumptions on F, the fact that $\max_{s\in\mathbb{R}}(C_0 - s)s = C_0^2/4$ holds and Cauchy's inequality lead to

$$\frac{\|w^k\|^2}{2} - \frac{\|w^{k-1}\|^2}{2} + \frac{1}{2}\|w^k - w^{k-1}\|^2 \leqslant \frac{\tau}{2}\|F^k\|^2 + \frac{\tau}{2}\|w^k\|^2 + \tau\frac{C_0^2}{4} \leqslant \tau c + \frac{\tau}{2}\|w^k\|^2 \tag{28}$$

for $k=1,\dots,N$. Thus summing over the first j steps we obtain

$$\frac{1}{2}\|w^j\|^2+\sum_{k=1}^{j}\|w^k-w^{k-1}\|^2 \leqslant c+\sum_{k=1}^{j}\frac{1}{2}\|w^k\|^2\tau, \quad j=1,\dots,N$$

and Gronwall's Lemma yields

$$\max_{j=1,\dots,N}\|w^j\| \leqslant c \quad \text{and} \quad \sum_{k=1}^{j}\|w^k-w^{k-1}\|^2 \leqslant c, \;\; j=1,\dots N. \tag{29}$$

From these estimates and the properties of b_ε the first and third assertion of the lemma follow immediately.

To show the second assertion we use (23), (26) and (27), the assumptions on F and sum up over the first j steps for arbitrary $j\in\{1,\dots,N\}$. This leads to

$$\frac{1}{2}\|w^j\|^2+\tau\sum_{k=1}^{j}(\|\sqrt{q^{k-1}}\nabla w^k\|-C_0)\|\sqrt{q^{k-1}}\nabla w^k\| \leqslant \tau c\sum_{k=1}^{j}\|w^k\|+\frac{1}{2}\|w^0\|^2.$$

Applying (29) results in

$$\tau\sum_{k=1}^{j}(\|\sqrt{q^{k-1}}\nabla w^k\|-C_0)\|\sqrt{q^{k-1}}\nabla w^k\| \leqslant c. \tag{30}$$

With

$$s_j:=\left(\tau\sum_{k=1}^{j}\|\sqrt{q^{k-1}}\nabla w^k\|^2\right)^{1/2},$$

Cauchy's inequality applied to (30) provides

$$s_j^2 \leqslant c(1+s_j), \quad j=1,\dots,N.$$

Thus some constant $c>0$ exists such that $\max_{j=1,\dots,N}s_j \leqslant c$. Finally, together with $v^k=w^k+b^k$, the assumptions on q_ε and b_ε and the parallelogram equality we obtain

$$\tau\sum_{k=1}^{j}\|\sqrt{q^{k-1}}\nabla v^k\|^2 \leqslant 2\tau\sum_{k=1}^{j}(\|\sqrt{q^{k-1}}\nabla w^k\|^2+\|\sqrt{q^{k-1}}\nabla b^k\|^2) \leqslant c, \quad j=1,\dots,N. \qquad \square$$

So far we have shown some properties of the semi-discretization problem at the time levels $t=t^k$. To obtain an approximation in the original domain Ω_T and prove convergence results we assign two extensions to the discrete solution $\{v^k\}_{k=1}^N$. Let us introduce a piecewise linear extension, called *Rothe function* (cf. [10]), by

$$v^{(N)}(t):=v^{k-1}+(v^k-v^{k-1})\frac{t-t^{k-1}}{\tau} \quad \forall t\in[t^{k-1},t^k], \quad k=1,\dots,N \text{ in } \Omega \tag{31}$$

and a piecewise constant function

$$\bar{v}^{(N)}(t):=v^k \quad \forall t\in(t^{k-1},t^k], \;\; k=1,\dots N \text{ in } \Omega.$$

As already shown, for any $N \in \mathbb{N}$ the solution $\{v^k\}_{k=1}^N$ of the semi-discretization is unique and contained in $\overset{0}{W}{}_2^1(\Omega)$. Thus $v^{(N)}$ and $\bar{v}^{(N)}$ are uniquely defined and by construction we have

1. $\dfrac{\partial}{\partial t} v^{(N)} = \dfrac{v^k - v^{k-1}}{\tau}$,
2. $\bar{v}^{(N)}(t) - \bar{v}^{(N)}(t-\tau) = v^k - v^{k-1}, \qquad t \in (t^{k-1}, t^k),\ k = 1, \dots, N.$
3. $\bar{v}^{(N)}(t) - v^{(N)}(t) = (v^k - v^{k-1})\frac{t^k - t}{\tau}$,

Further, applying Lemma 1 and $\tau = T/N$ we derive

$$\|\bar{v}^{(N)} - v^{(N)}\|^2_{L_2(\Omega_T)} = \sum_{k=1}^N \int_{t^{k-1}}^{t^k} \|v^k - v^{k-1}\|^2 \frac{(t^k - t)^2}{\tau^2}\,\mathrm{d}t = \frac{\tau}{3}\sum_{k=1}^N \|v^k - v^{k-1}\|^2 \leqslant \frac{c}{N}.$$

Thus

$$\|\bar{v}^{(N)} - v^{(N)}\|^2_{L_2(\Omega_T)} \leqslant \frac{c}{N} \to 0 \quad \text{for } N \to \infty. \tag{32}$$

For any $N \in \mathbb{N}$ the sequence $\{v^k\}_{k=0}^N$ is contained in $\overset{0}{W}{}_2^1(\Omega)$. This combined with the a priori estimates of Lemma 1 implies $\bar{v}^{(N)} \in L_2(I, \overset{0}{W}{}_2^1(\Omega))$ for all $N \in \mathbb{N}$. Further, we obtain $v^{(N)} \in W_2^1(\Omega_T)$ for all $N \in \mathbb{N}$ directly from the following result.

Lemma 2 (cf. [7]). *Let $G \subset \mathbb{R}^m$ be a subset such that $\bar{G} = \bigcup_{i=1}^M \bar{G}_i$ and all G_i of the partition are bounded regions with $\partial G_i \in C^{0,1}$ and int $G_i \cap$ int $G_j = \emptyset$ for $i \neq j$. Let further $u : G \to \mathbb{R}$ be a function, such that $u \in C(\bar{G})$ and $u|_{G_i} \in C^1$. Then $u \in W_2^1(G)$.*

The a priori estimates for the discrete solution $\{v^k\}_{k=1}^N$ as stated in Lemma 1 permit adequate a priori estimates for the Rothe function and for $\bar{v}^{(N)}$ as well.

Lemma 3. *Under assumptions* (i) *and* (ii) *for arbitrary $N \in \mathbb{N}$ the following estimates are satisfied*:

1. $\|\frac{\partial}{\partial t} v^{(N)}\|_{L_2(I, H^{-1})} \leqslant c$,
2. $\|v^{(N)}\|^2_{L_2(\Omega_T)} \leqslant c$,
3. $\|\nabla v^{(N)}\|^2_{L_2(\Omega_T)} \leqslant c\gamma(\varepsilon)^{-1}$.

Proof. Using (16) and (18) and the assumptions on f and b_ε we have

$$\begin{aligned}\left|\left(\frac{\partial}{\partial t} v^{(N)}(t), \varphi\right)\right| &\leqslant |\langle f^k, \varphi\rangle| + |\langle q^{k-1}\nabla(v^k + b^k), \nabla\varphi\rangle| \\ &\leqslant c\|\varphi\|_1 + \|\sqrt{q^{k-1}}\nabla v^k\|\,\|\nabla\varphi\| \\ &\leqslant (c + \|\sqrt{q^{k-1}}\nabla v^k\|)\|\varphi\|_1 \end{aligned} \tag{33}$$

for all $t \in (t^{k-1}, t^k)$, $k = 1, \ldots, N$ and arbitrary $\varphi \in \overset{0}{W}{}^1_2(\Omega)$. This estimate combined with Lemma 1 and Cauchy's inequality yields

$$\left\|\frac{\partial}{\partial t}v^{(N)}\right\|_{L_2(I,H^{-1})} = \int_0^T \left\|\frac{\partial}{\partial t}v^{(N))}\right\|_* \mathrm{d}t = \int_0^T \sup_{\|\varphi\| \leqslant 1} \left|\left(\frac{\partial}{\partial t}v^{(N)}, \varphi\right)\right| \mathrm{d}t$$
$$\leqslant c + \tau\sum_{k=1}^{N}\|\sqrt{q^{k-1}}\nabla v^k\|$$
$$\leqslant c\left(1 + \left[\tau\sum_{k=1}^{N}\|\sqrt{q^{k-1}}\nabla v^k\|^2\right]^{1/2}\right) \leqslant c.$$

To prove the second assertion we apply Lemma 1 and obtain

$$\|v^{(N)}\|^2_{L_2(\Omega_T)} = \sum_{k=1}^{N}\int_{t^{k-1}}^{t^k}\left\|v^{k-1} + (v^k - v^{k-1})\frac{t - t^{k-1}}{\tau}\right\|^2 \mathrm{d}t$$
$$\leqslant \tau\left(\sum_{k=1}^{N}\|v^{k-1}\|^2 + \frac{1}{3}\sum_{k=1}^{N}\|v^k - v^{k-1}\|^2\right) \leqslant c.$$

By a similar argumentation using Lemma 1 and (16) it follows

$$\|\nabla v^{(N)}\|^2_{L_2(\Omega_T)} \leqslant \tau\left(\frac{4}{3}\sum_{k=1}^{N}\|\nabla v^{k-1}\|^2 + \sum_{k=1}^{N}\nabla v^k\|^2\right) \leqslant c\gamma(\varepsilon)^{-1}. \qquad \square$$

In [11] the relative compactness of the set $\{v^{(N)}\}_{N\in\mathbb{N}}$ in $L_2(\Omega_T)$ is proved means of the compactness theorem by Riesz–Kolmogorov (cf. [21]) which implies the existence of a convergent subsequence of $\{v^{(N)}\}_{N\in\mathbb{N}}$. Such an approach could be used here too, but we will derive the desired result directly from a well-known embedding result of Rellich–Kondrashov type for Sobolev spaces as given.

Theorem 1 (see Lions [13]). *Let $V_1 \subset V \subset V_2$ denote a triple of Banach spaces such that V_i, $i = 1, 2$ are reflexive and the embedding $V_1 \to V$ is compact. Then the space*

$$W := \left\{v \in L_2(I, V_1)\,\middle|\, \frac{\partial}{\partial t}v \in L_2(I, V_2)\right\}$$

equipped with the norm $\|v\|_W := \|v\|_{L_2(I,V_1)} + \|v\|_{L_2(I,V_2)}$ is a Banach space and the embedding $\imath : W \to L_2(I, V)$ is compact.

In the following part the dependence on the smoothing parameter $\varepsilon > 0$ is investigated, therefore we write $\{v_\varepsilon^{(N)}\}_{N\in\mathbb{N}}$ instead of $\{v^{(N)}\}_{N\in\mathbb{N}}$.

Lemma 4. *For any fixed $\varepsilon > 0$ the sequence $\{v_\varepsilon^{(N)}\}_{N\in\mathbb{N}}$ possesses a convergent subsequence in $L_2(\Omega_T)$.*

Proof. Recall that the sequence $\{v_\varepsilon^{(N)}\}_{N\in\mathbb{N}}$ is contained in $W_2^1(\Omega_T)$. Now we set $V_1 := W_2^1(\Omega)$, $V = L_2(\Omega)$ and $V_2 = H^{-1}$. By assumption we have $\partial\Omega_T \in C^{0,1}$. Thus we can apply the well-known fact that $V_1 \subset V \subset V_2$ forms an evolution triple and the embedding $W_2^1(\Omega) \to L_2(\Omega)$ is compact (cf. [21]). Hence, all requirements of Theorem 1 are fulfilled and we obtain that the embedding of $W = \{v \in L_2(I, W_2^1(\Omega)) \,|\, (\partial/\partial t)v \in L_2(I, H^{-1})\}$ in $L_2(\Omega_T)$ is compact. Since, according to Lemma 3, the set $\{v_\varepsilon^{(N)}\}$ is bounded in W for every fixed value of the smoothing parameter $\varepsilon > 0$ there exists a subsequence $\{v_\varepsilon^{(N)}\}_N$ which converges to an element $v_\varepsilon \in L_2(\Omega_T)$. □

Now we can formulate the main result on convergence for the semi-discretization scheme.

Theorem 2. *Let assumptions* (i) *and* (ii) *be satisfied. Then for every $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists the unique sequence of discrete solutions $\{v^k\}_{k=0}^N$ of* (18). *Further*

$$\|v_\varepsilon^{(N)} - v_\varepsilon\|_{L_2(\Omega_T)} \to 0 \quad \textit{for } N \to \infty$$

and

$$\|\bar{v}_\varepsilon^{(N)} - v_\varepsilon\|_{L_2(\Omega_T)} \to 0 \quad \textit{for } N \to \infty$$

hold, where $v_\varepsilon \in L_2(\Omega_T) \cap L_\infty(I, H^{-1}(\Omega))$ is the unique weak solution of (17).

Proof. From Lemma 4 we already know that there exists a subsequence $\{v_\varepsilon^{(N)}\}_N$ which converges to some v_ε in $L_2(\Omega_T)$. The first estimate of Lemma 3 implies that $(\partial/\partial t)v_\varepsilon^{(N)}$ is uniformly bounded in $L_2(I, H^{-1})$ for $N \in \mathbb{N}$. Thus $(\partial/\partial t)v_\varepsilon \in L_2(\Omega_T)$ exists and $(\partial/\partial t)v_\varepsilon^{(N)} \to (\partial/\partial t)v_\varepsilon$ weakly in $L_2(I, H^{-1})$.

It remains to prove that the limit function $v_\varepsilon \in L_2(\Omega_T)$ solves the weakly formulated problem (18). Since $v_\varepsilon^{(N)}(0) = v_0$ holds for arbitrary $N \in \mathbb{N}$ the limit function v_ε satisfies the initial condition. Now we rewrite (18) in the form

$$\int_0^T \left(\frac{\partial}{\partial t} v_\varepsilon^{(N)}, \varphi\right) \mathrm{d}t + \int_0^T \langle \bar{q}_\varepsilon^{(N)}(v_\varepsilon^{(N)}(t-\tau))\nabla \bar{v}_\varepsilon^{(N)}, \nabla\varphi\rangle \,\mathrm{d}t = \int_0^T \langle \bar{f}^{(N)}(t), \varphi\rangle \,\mathrm{d}t$$

with the step functions

$$\bar{q}_\varepsilon^{(N)}(x,t,u) = q_\varepsilon(u(x,t^k)), \quad \bar{f}^{(N)}(x,t) = f(x,t^k) \quad \forall t \in (t^{k-1}, t^k],\ k = 1,\ldots,\ N,$$

where $\bar{q}_\varepsilon^{(N)}(x,0,u) = q_\varepsilon(u(x,0))$ and $\bar{f}^N(x,0) = f(x,0)$. Applying (32), the convergence $\{v_\varepsilon^{(N)}\}_N \to v_\varepsilon$ in $L_2(\Omega_T)$ and $\bar{v}_\varepsilon^{(N)} \in L_2(I, \overset{0}{W}{}_2^1(\Omega))$ for all $N \in \mathbb{N}$ results in $\bar{v}_\varepsilon^{(N)} \to v_\varepsilon$ in $L_2(\Omega_T)$ According to Lemma 3 we have $\nabla v_\varepsilon^N \to \nabla v_\varepsilon$ weakly in $L_2(\Omega_T)$. Further, (32) yields $\nabla \bar{v}_\varepsilon^N \rightharpoonup \nabla v_\varepsilon$ in $L_2(\Omega_T)$. Thus $\bar{q}_\varepsilon^{(N)}(v_\varepsilon^{(N)}(t-\tau))\nabla \bar{v}_\varepsilon^{(N)} \rightharpoonup q_\varepsilon(v_\varepsilon)\nabla v_\varepsilon$ in $L_2(\Omega_T)$.

The above convergence results and $\bar{f}^{(N)} \to f$ a.e in Ω_T provide that v_ε is a weak solution of (17). On the other hand according to [13] the weak formulation of problem (14) possesses a unique solution $v_\varepsilon \in L_2(\Omega_T) \cap L_\infty(I, H^{-1}(\Omega))$. Therefore v_ε is the unique weak solution of (17). Moreover, not only a subsequence but the total sequence $\{v_\varepsilon^{(N)}\}$ converges to v_ε (cf. [21]). □

4. Numerical results

In this section we report some results obtained in numerical tests, namely cases of a one-dimensional problem with a known exact solution and the two-dimensional wedge solidification problem taken from [3]. The numerical treatment of the semi-discretization requires a further discretization of the spatial variables. For the sake of simplicity in our computer experiments we applied fixed uniform grids in space and the occurring divergence form of the spatial derivatives have been approximated on uniform grids $\{x_i\}_{i=0}^{M}$ by the standard difference scheme

$$(qv')'(x_i) \approx h^{-2}(q_{i+1/2}(v_{i+1} - v_i) - q_{i-1/2}(v_i - v_{i-1}))$$

with $q_{i\pm 1/2} := (q_{i\pm 1} + q_i)/2$.

Example 1. First, we deal with a simple one-dimensional test problem with a known analytical solution. The construction of this problems rests on the classical formulation of the two-phase Stefan problem with a prescribed melting front $s(t)$, $t \in [0, T]$ which separates the frozen and the liquid zones

$$\Omega_T^1 \ := \ \{(x,t) \in (0,1) \times (0,T) : x < s(t)\}$$

and

$$\Omega_T^2 \ = \ \{(x,t) \in (0,1) \times (0,T) : x > s(t)\}, \tag{34}$$

respectively. Now, the Stefan drift equation (7) in the one-dimensional case has the form

$$\dot{s}(t) = \frac{1}{\lambda}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x}\right)(s(t), t), \quad t \in (0, T]. \tag{35}$$

With a given differentiable function $s: I \to \mathbb{R}$ we choose

$$\begin{aligned} u_1(x,t) &= 2\alpha_0(x - s(t)) + \alpha_1(x - s(t))^2, \quad (x,t) \in \Omega_T^1, \\ u_2(x,t) &= \alpha_0(x - s(t)) + \alpha_1(x - s(t))^2, \quad (x,t) \in \Omega_T^2. \end{aligned} \tag{36}$$

To obtain the solution given above the right-hand side f of the parabolic problem (1) has been selected according to

$$f(x,t) = \begin{cases} -2(\alpha_0 + \alpha_1(x - s(t))\dot{s}(t) - 2\alpha_1, & (x,t) \in \Omega_T^1, \\ -(\alpha_0 + 2\alpha_1(x - s(t))\dot{s}(t) - 2\alpha_1, & (x,t) \in \Omega_T^2 \end{cases}. \tag{37}$$

Table 1

M	N	ε	δ_2	δ_∞
100	100	1.0	0.0478	0.2354
		0.1	0.0134	0.0359
		0.01	0.0247	0.2246
500	500	0.01	0.0065	0.0163
		0.001	0.0042	0.0307
100	1000	0.001	0.0662	0.4580
1000	1000	0.001	0.0027	0.0119

In addition, the boundary conditions are chosen consistently to (36) by

$$\left.\begin{aligned} u(0,t) &= -2\alpha_0 s(t) + \alpha_1 s(t)^2, \\ u(1,t) &= \alpha_0(1-s(t)) + \alpha_1(1-s(t))^2, \end{aligned}\right. \quad t \in (0,T]. \tag{38}$$

The enthalpy $v(\cdot,0)$ at the starting time $t=0$ is for fixed regularization parameter $\varepsilon > 0$ defined by

$$v(x,0) = H_\varepsilon(u(x,0)), \quad x \in [0,1] \tag{39}$$

with $u(\cdot,\cdot)$ continuously extended from (36). In our calculations we used $\lambda = 20$, $T = 0.5$ and

$$s(t) = 0.5 + 0.25\frac{t}{T}, \quad t \in [0,T]. \tag{40}$$

Then (35) holds if the parameter $\alpha_0 > 0$ satisfies $\alpha_0 T = 0.25\lambda$. Table 1 reports the obtained accuracies

$$\delta_2 := \frac{1}{\sqrt{N+1}\sqrt{M+1}}\sum_{i=0}^{M}\sum_{k=0}^{N}|u(x_i,t_k) - u_{i,k}|^2, \quad \delta_\infty := \max_{0\leqslant i\leqslant M, 0\leqslant k\leqslant N}|u(x_i,t_k) - u_{i,k}|,$$

i.e. discrete L_2-norms and maximum norms, respectively, for various numbers M, N of grid points in space and time and smoothing parameters ε.

As expected, for small parameters $\varepsilon > 0$ the rapid change of the smoothed enthalpy function near the melting temperature requires dense grid points close to the transition zone. Hence, an spatial grid locally adapted to the moving boundary has to be included to make the method practically efficient.

In Fig. 1 we give a surface plot of the temperature and enthalpy, respectively, over the space time cylinder Ω_T for $N = M = 50$ and $\varepsilon = 0.1$. For the same data Fig. 2 shows the level lines $u = 0.3l$, $l = 0, \pm 1, \pm 2, \ldots$ of the temperature. Temperature levels below and equal to zero are given as solid lines while positive temperatures are indicated by dashed lines. Finally, in this example Fig. 3 shows the distribution of errors over Ω_T.

As one can see, the absolute error reaches its maximum along the free boundary. This effect is reasonable because of the poor approximation of the discontinuity of the enthalpy in this zone.

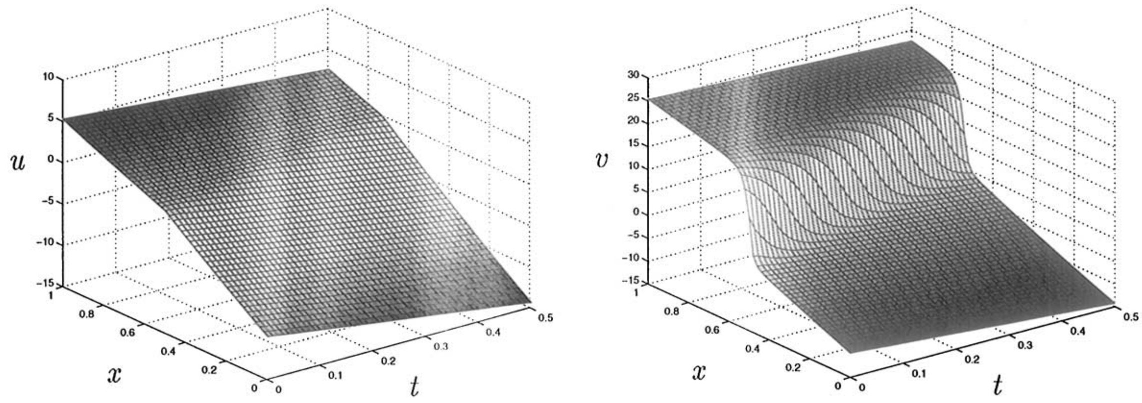


Fig. 1.

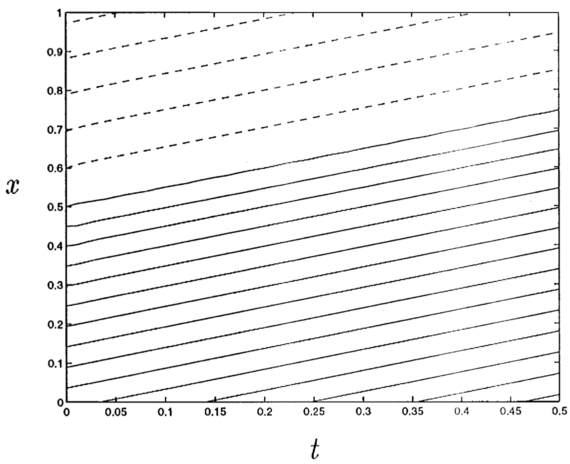


Fig. 2.

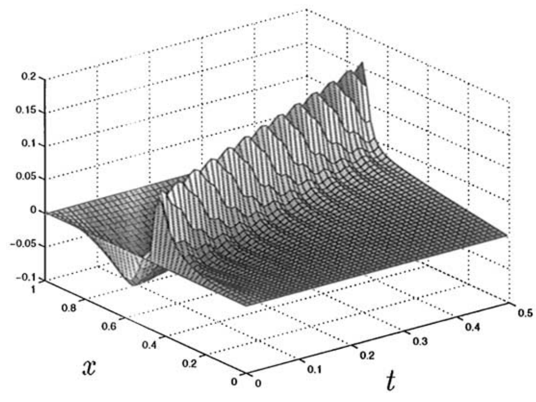
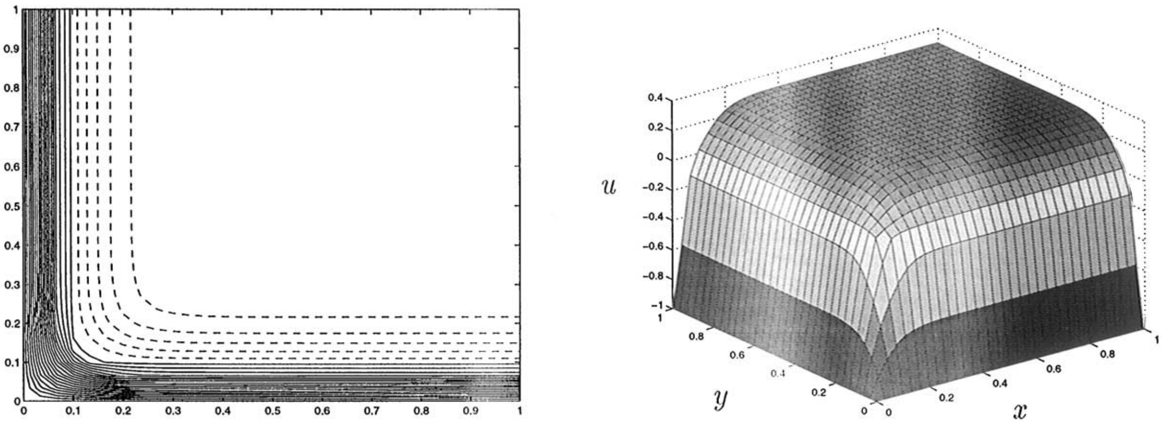
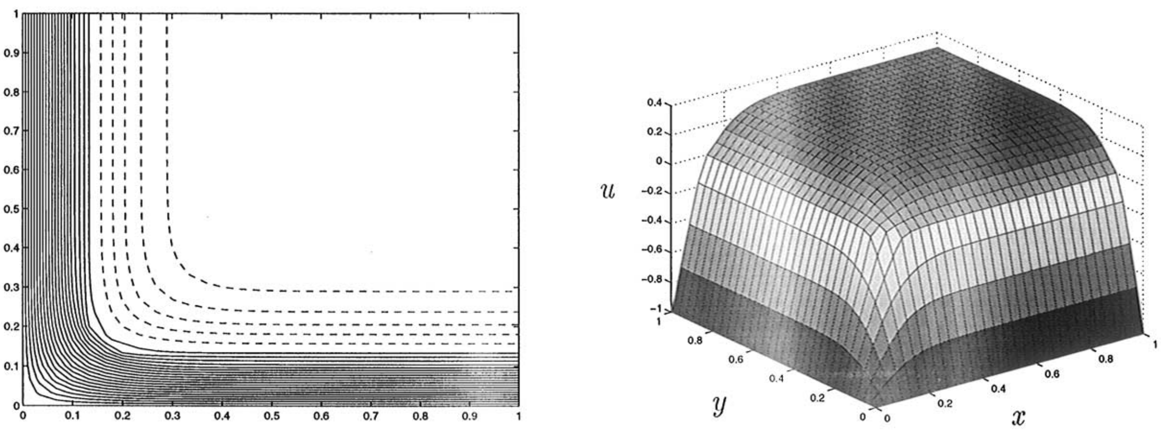


Fig. 3.



$t_c = 0.005$

Fig. 4.



$t_c = 0.01$

Fig. 5.

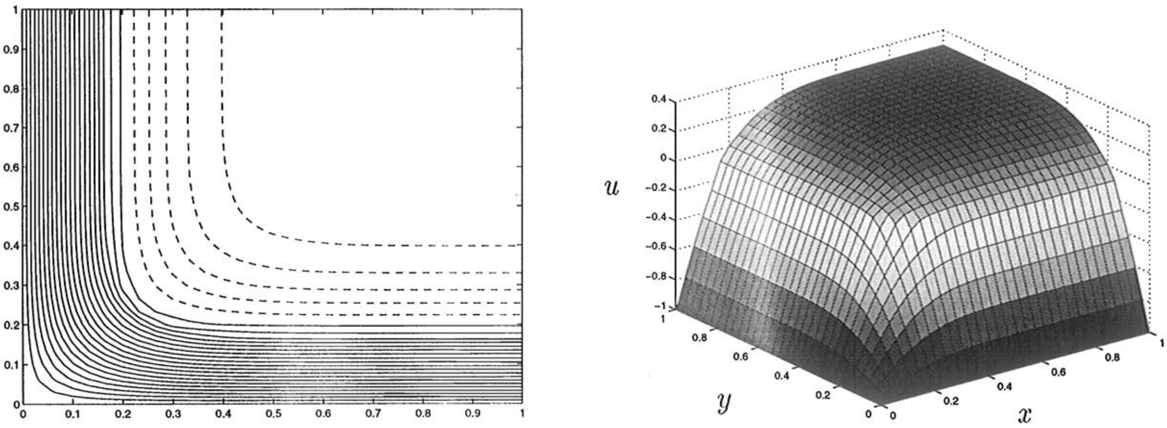
Example 2. As two-dimensional test problem we deal with the wedge solidification as considered in [3]. With $\Omega = (0,1)^2 \subset \mathbb{R}^2$ and $\lambda = 0.5$ the initial temperature is defined by

$$u(x,y) = 0.3, \quad \forall (x,y) \in (0,1)^2.$$

The boundary conditions are of mixed type

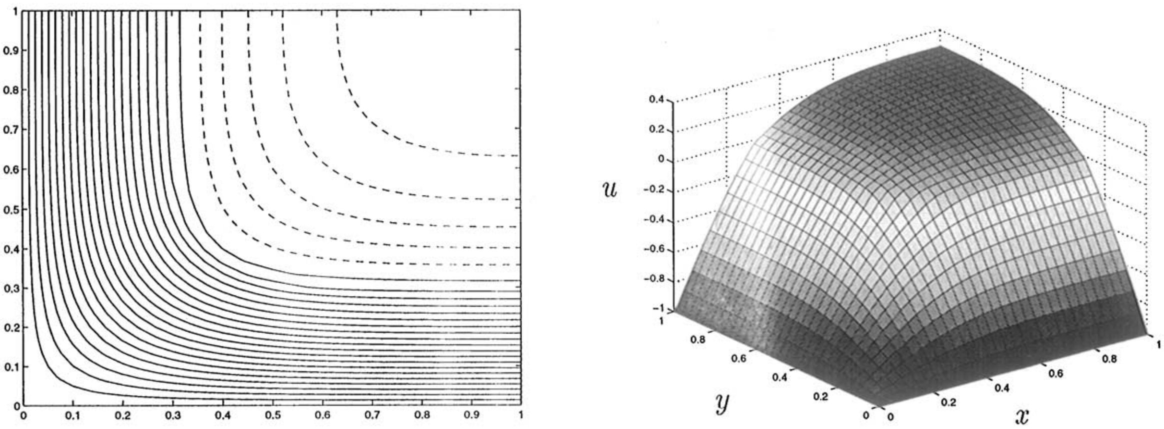
$$\left.\begin{aligned} u(x,0,t) &= u(0,y,t) = -1, \quad x,y \in [0,1], \\ u_y(x,1,t) &= u_x(1,y,t) = 0, \quad x,y \in (0,1), \end{aligned}\right. \quad t \in (0,T]. \tag{41}$$

The boundary conditions can be transferred to Dirichlet type only if one embeds the problem into the larger domain $\tilde{\Omega} := (0,2)^2$. Then the Neumann conditions in (41) are just a consequence of symmetry.



$t_c = 0.02$

Fig. 6.

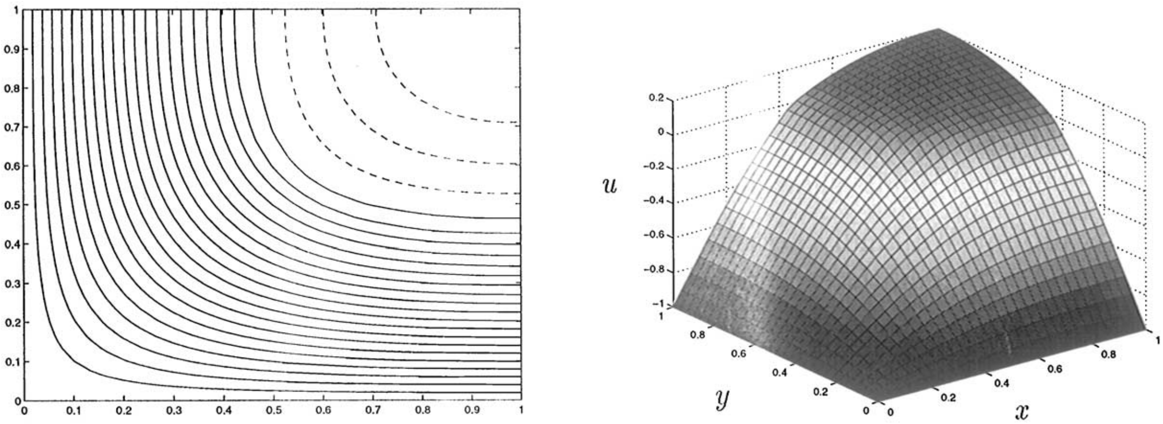


$t_c = 0.05$

Fig. 7.

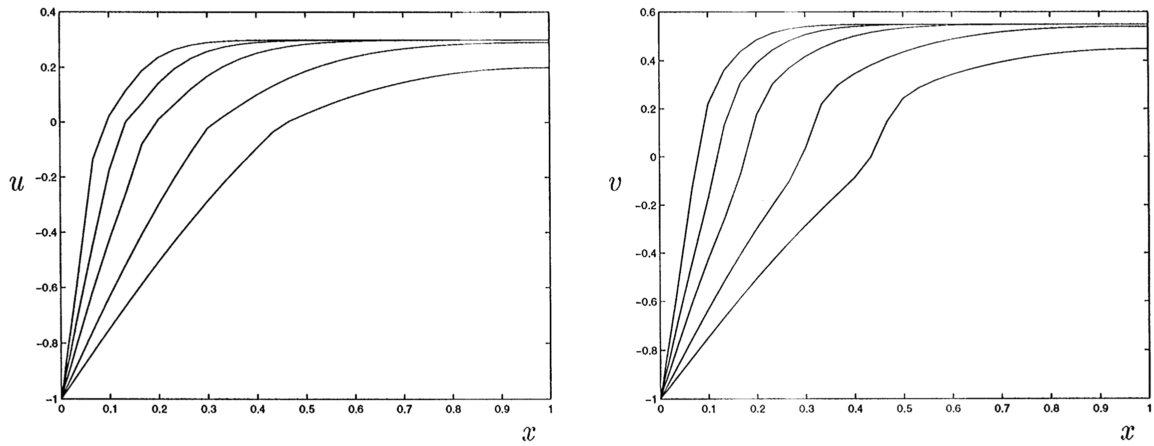
Unlike in [3] in the present paper out main emphasis is on the analysis of the smoothing technique itself. So, as already mentioned, for the sake of simplicity of the implementation an equidistributed grid has been chosen. To make the proposed smoothing method more efficient, however the grid must be adapted to the specific situation of phase change problems as proposed e.g. in [3]. To enable an easier comparison with the result in [3] we have chosen the same discretization parameters $M = 30$, $N = 80$. As regularization parameter we have selected $eps = 0.01$.

In Figs. 4–8 we give the obtained level curves at levels $u = 0.05l$, $l = 0, \pm 1, \pm 2, \ldots$ and surface plots for the temperature $u(\cdot, \cdot, t_c)$ at the selected time steps $t_c \in \{0.005, 0.01, 0.02, 0.05, 0.1\}$. As before, temperature levels below and equal to zero are given as solid lines while positive temperatures



$t_c = 0.1$

Fig. 8.



$u(x, 1, t_c)$ $v(x, 1, t_c)$

Fig. 9.

are indicated by dashed ones. Fig. 9 shows the graphs of the temperature and of the enthalpy along the upper boundary $y = 1$ at the considered time levels $t_c \in \{0.005, 0.01, 0.02, 0.05, 0.1\}$. Finally, in Fig. 10 the position of the melting front along the spatial diagonal $x = y$ and the development of temperature at the central point $(x, y) = (0.5, 0.5)$ are given. Overall, the obtained results show a good coincidence with those from [3].

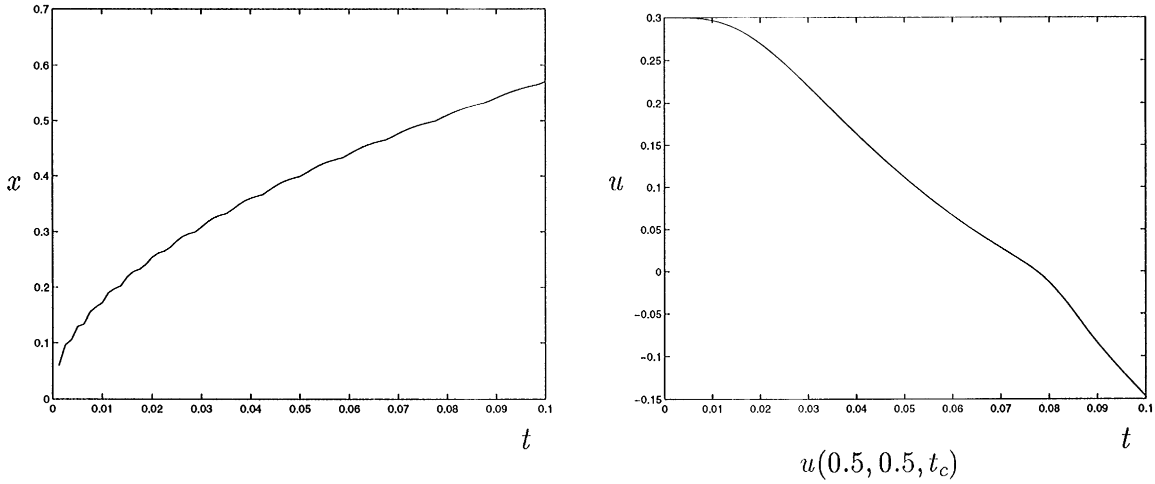


Fig. 10. Melting front.

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