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On the Gorenstein locus of some punctual Hilbert schemes

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ABSTRACT

Let k be an algebraically closed field and let $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ be the open locus of the Hilbert scheme $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ corresponding to Gorenstein subschemes. We prove that $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ is irreducible for $d \le 9$. Moreover we also give a complete picture of its singular locus in the same range $d \le 9$. Such a description of the singularities gives some evidence to a conjecture on the nature of the singular points in $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ that we state at the end of the paper.

1. Introduction and notation

Let k be an algebraically closed field and denote by $\mathcal{H}ilb_{p(t)}(\mathbb{P}_k^N)$ the Hilbert scheme parametrizing closed subschemes in $\mathbb{P}_k^N$ with fixed Hilbert polynomial $p(t) \in \mathbb{Q}[t]$. Since Grothendieck's proof in [1] of the existence of $\mathcal{H}ilb_{p(t)}(\mathbb{P}_k^N)$, the problem of finding a useful description of this scheme has attracted the interest of many researchers in algebraic geometry.

One of the first, now well known, results in this direction is due to R. Hartshorne who proved the connectedness of $\mathcal{H}ilb_{p(t)}(\mathbb{P}_k^N)$ in [2]. Other results concerning the singularity of the fat point and the local structure of the Hilbert scheme around it, the radius of the full Hilbert scheme and the smoothness of the lexicographic point were proved by S. Katz, R. Notari and M.L. Spreafico, A.A. Reeves and M. Stillmann (see [3–6]).

There have also been studies of some loci in the Hilbert scheme. Two of the most notable are the description of the locus of codimension 2 arithmetically Cohen–Macaulay subschemes (see [7] for the dimension 0 case and [8] for dimension ≥ 1) and of the locus of codimension 3 arithmetically Gorenstein subschemes (see [9] and [10]).

In the study of punctual Hilbert schemes, the first fundamental result is due to J. Fogarty who proved the irreducibility and smoothness of $\mathcal{H}ilb_d(\mathbb{P}_k^N)$, $d \in \mathbb{N}$, when $N = 2$. The same result holds more generally if one considers subschemes of codimension 2 of any smooth surface (see [7]).

In [11] the author proved that, if d is large with respect to N, $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ is never irreducible. Indeed for every d and N there always exists a generically smooth component of $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ having dimension dN whose general point corresponds to a reduced set of d points but, for d large with respect to $N \ge 3$, there is at least one other component whose general point corresponds to an irreducible scheme of degree d supported on a single point.

In view of these earlier works it is reasonable to consider the irreducibility and smoothness of other naturally occurring loci in $\mathcal{H}ilb_d(\mathbb{P}_k^N)$. E.g: one of the loci that has interested us is the set $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ of points in $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ representing schemes which are Gorenstein. This is an important locus since it includes reduced schemes.

Some results about $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ are known. E.g. since $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ contains all reducible schemes of degree d it follows that $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ contains an open (not necessarily dense) subset of $\mathcal{H}ilb_d(\mathbb{P}_k^N)$. More precisely $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ is actually open since

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its complement in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ coincides in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ with the locus of points over which the relative dualizing sheaf of the universal family is not invertible. Another result, part of the folklore, gives the irreducibility and smoothness of $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ when $N \leq 3$. We provide a proof of this fact in Section 5. In [12] and [13] it is shown that $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is never irreducible for $d \geq 14$ and $N \geq 6$.

These results leave open the question of irreducibility for small d and all $N > 3$. One of our principal results on these matters is the following theorem proved in Section 5.

Theorem A. *Assume the characteristic of k is $p \neq 2, 3$. The locus $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is irreducible for $d \leq 9$.* □

Very recently, in [14], the authors prove both the irreducibility of $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ when $d \leq 7$ and the existence of exactly two components in $\mathcal{H}ilb_8(\mathbb{P}^N_k)$, $N \geq 4$.

In order to prove Theorem A we need to study deformations of some particular local Artinian Gorenstein k-algebras of degree $d \leq 9$ and embedding dimension at least 4. We begin the study of such algebras in Section 2 where we fix the notation and recall some elementary facts. In the final part of Section 2 and in Sections 3 and 4 we give a complete classification of such kind of algebras under suitable restrictions on the characteristic of the base field k.

The problem of classifying local Artinian k-algebras is classical. It is completely solved for $d \leq 6$ (see [15–17] when $\mathrm{char}(k) > 3$ and [18] without any restriction on the characteristic). When $d \geq 7$ it is classically known that such algebras have moduli and their parameter spaces have been the object of deep study (again see [16,12] and [19]).

Let us now restrict to the *Gorenstein* k-algebras. Their classification in degree $d = 7$ is obtained combining the results proved in Section 3 with the result classically proved in [20] (see also [17]). In degrees $d = 8, 9$ a complete classification can be done using, in addition to those papers, also the work [21] on the classification of nets of conics (see also the unpublished paper [22]).

Then we turn our attention to the singularities of the Hilbert scheme. We are aware of some scattered results about the existence of singular points on $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ that can be found in [3,17]. In Section 5 we also prove the following

Theorem B. *If $d \leq 8$, then $X \in \mathrm{Sing}(\mathcal{H}ilb^G_d(\mathbb{P}^N_k))$ if and only if the corresponding scheme X has embedding dimension 4 at least at one of its points. If $d = 9$ a complete description of $\mathrm{Sing}(\mathcal{H}ilb^G_d(\mathbb{P}^N_k))$ can also be given.* □

At the end of Section 5 we give some evidence to a conjecture on the singularities of the component of $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ containing points representing reduced schemes for each d.

In order to prove the above theorem we have to combine on one hand the study of the hierarchy of local Artinian Gorenstein k-algebras of degree $d \leq 9$ and embedding dimension at least 4, on the other the properties of the G-fat point $X \subseteq \mathbb{P}^{d-2}_k$ proved in [17].

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Notation. In what follows k is an algebraically closed field. We denote its characteristic by $\mathrm{char}(k)$.

Recall that a Cohen–Macaulay local ring R is one for which $\dim(R) = \mathrm{depth}(R)$. If, in addition, the injective dimension of R is finite then R is called Gorenstein (equivalently, if $\mathrm{Ext}^i_R(M, R) = 0$ for each R-module M and $i > \dim(R)$). An arbitrary ring R is called Cohen–Macaulay (resp. Gorenstein) if $R_{\mathfrak{M}}$ is Cohen–Macaulay (resp. Gorenstein) for every maximal ideal $\mathfrak{M} \subseteq R$.

All the schemes X are separated and of finite type over k. A scheme X is Cohen–Macaulay (resp. Gorenstein) if for each point $x \in X$ the ring $\mathcal{O}_{X,x}$ is Cohen–Macaulay (resp. Gorenstein). The scheme X is Gorenstein if and only if it is Cohen–Macaulay and its dualizing sheaf ω_X is invertible.

For each numerical polynomial $p(t) \in \mathbb{Q}[t]$ of degree at most n we denote by $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$ the Hilbert scheme of closed subschemes of $\mathbb{P}^N_k$ with Hilbert polynomial $p(t)$. With abuse of notation we will denote by the same symbol both a point in $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$ and the corresponding subscheme of $\mathbb{P}^N_k$. In particular we will say that X is obstructed (resp. unobstructed) in $\mathbb{P}^N_k$ if the corresponding point is singular (resp. non-singular) in $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$.

Moreover we denote by $\mathcal{H}ilb^G_{p(t)}(\mathbb{P}^N_k)$ the locus of points representing Gorenstein schemes. This is an open subset of $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$, though not necessarily dense.

If $X \subseteq \mathbb{P}^N_k$ we will denote by $\mathfrak{I}_X$ its sheaf of ideals in $\mathcal{O}_{\mathbb{P}^N_k}$ and we define the normal sheaf of X in $\mathbb{P}^N_k$ as $\mathcal{N}_X := (\mathfrak{I}_X/\mathfrak{I}^2_X)\check{} := \mathcal{H}om_X\left(\mathfrak{I}_X/\mathfrak{I}^2_X, \mathcal{O}_X\right)$. If we wish to stress the fixed embedding $X \subseteq \mathbb{P}^N_k$ we will write $\mathcal{N}_{X|\mathbb{P}^N_k}$ instead of $\mathcal{N}_X$. If $X \in \mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$, the space $H^0\left(\mathbb{P}^N_k, \mathcal{N}_X\right)$ can be canonically identified with the tangent space to $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$ at the point X. In particular X is obstructed in $\mathbb{P}^N_k$ if and only if $h^0\left(\mathbb{P}^N_k, \mathcal{N}_X\right)$ is greater than the local dimension of $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$ at the point X.

For all the other notations and results we refer to [25].

2. The locus $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$

In this section we summarize some facts about smoothability and obstructedness of schemes $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$.

We recall that the locus of reduced schemes $\mathcal{R} \subseteq \mathcal{H}ilb_d(\mathbb{P}^N_k)$ is birational to a suitable open subset of the d-th symmetric product of $\mathbb{P}^N_k$, thus it is irreducible of dimension dN (see [11]) and we will denote by $\mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k)$ its closure in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$.

Note that $\mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k)$ is necessarily an irreducible component of $\mathcal{H}ilb_d(\mathbb{P}^N_k)$. Indeed, in any case, we can always assume $\mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k) \subseteq \mathcal{H}$ for a suitable irreducible component $\mathcal{H}$ in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$. If the inclusion were proper then there would exist a flat family with special point in $\mathcal{R}$, hence reduced, and non–reduced general point, which is absurd. We conclude that $\mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k) = \mathcal{H}$.

Definition 2.1. A scheme X is said to be smoothable in $\mathbb{P}^N_k$ if $X \in \mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k)$.

Thus X is smoothable if and only if there exists an irreducible scheme B and a flat family $\mathcal{X} \subseteq \mathbb{P}^N_k \times B \to B$ with special fiber X and general fiber in $\mathcal{R}$, hence reduced.

The following result is well-known. Since we are not able to find a precise reference for it we will provide an explicit proof for it.

Lemma 2.2. *Let X be a scheme of dimension 0 and degree d and let $X \subseteq \mathbb{P}^N_k$ and $X \subseteq \mathbb{P}^{N'}_k$ be two embeddings. Then X is smoothable in $\mathbb{P}^N_k$ if and only if it is smoothable in $\mathbb{P}^{N'}_k$.*

Proof. Let $X = \bigcup_{i=1}^p X_i$ where the $X_1, \dots, X_p$ are irreducible and pairwise disjoint of respective degree $d_1, \dots, d_p$, with $d = \sum_{i=1}^p d_i$. It is clear that X is smoothable if and only if the same is true for all its connected component (which coincide with its irreducible components since X has dimension 0).

Fix one of such component and call it Y: from now on we will denote by δ its degree. Each such scheme is affine, say $Y \cong \operatorname{spec}(A)$ where A is an Artinian, Gorenstein k-algebra of degree δ, i.e. with $\dim_k(A) = \delta$, and maximal ideal $\mathfrak{M}$. In order to study our scheme Y, hence X, it is then natural to study A.

Let the embedding dimension $\operatorname{emdim}(A) := \dim_k(\mathfrak{M}/\mathfrak{M}^2)$ of A be $n \le N, N'$. Note that $\operatorname{emdim}(A)$ is, by definition, the dimension of the tangent space at the unique closed point $y \in Y$.

Then we have a surjective morphism from the symmetric k-algebra on $\mathfrak{M}/\mathfrak{M}^2$, which is $k[y_1, \dots, y_n]$, onto A. Hence an isomorphism $A \cong k[y_1, \dots, y_n]/I$, where $I \subseteq (y_1, \dots, y_n)^2$, i.e. an embedding $Y \subseteq \mathbb{A}^n_k \subseteq \mathbb{P}^n_k$ such that Y is supported on the origin. In order to prove the statement it suffices to prove that Y is smoothable in $\mathbb{P}^N_k$ if and only if it is smoothable in $\mathbb{P}^n_k$.

Assume that $Y \subseteq \mathbb{P}^N_k$ does not intersect the hyperplane $\{ x_0 = 0 \}$, so that we can assume $Y \subseteq \mathbb{A}^N_k$. Since the dimension of the tangent space of Y is n, there exists a subscheme $Q \subseteq \mathbb{A}^N_k$ of dimension n, containing Y and smooth around its support $y \in Q$.

Let Y be smoothable in $\mathbb{P}^n_k$. The embedding $Q \subseteq \mathbb{A}^N_k$ corresponds to an epimorphism $k[x_1, \dots, x_N] \twoheadrightarrow R$ where R is a suitable k-algebra. If the maximal ideal of y (the unique closed point in Y) in R is $\mathfrak{N}$, then $R_{\mathfrak{N}}$ is regular and $\dim_k(\mathfrak{N}/\mathfrak{N}^2) = n$, by definition of Q. Moreover the embedding $Y \subseteq Q$ corresponds to $\varphi_0 \colon R \twoheadrightarrow A$. Hence we obtain a morphism $k[y_1, \dots, y_n] \to R$, corresponding to $f \colon Q \cap \mathbb{A}^N_k \to \mathbb{A}^n_k$ and having, by definition of Q, invertible differential $\mathrm{d}f$ at y. It follows that $\mathrm{d}f$ is invertible in a neighborhood of y. Since $\dim(Q) = n$, we conclude that f is dominant, hence locally étale around y.

Let $B := \operatorname{spec}(S)$ be a smooth curve and let $\mathcal{Y} := \operatorname{spec}(S[y_1, \dots, y_n]/J) \subseteq \mathbb{A}^n_k \times B$ be a flat family with special fiber Y over $b = 0$ and smooth general fiber. We can embed, at least locally, B in $\mathbb{A}^n_k$ in such a way that $b = 0$ coincides with the origin and we set $\widetilde{B} := f^{-1}(B) \subseteq Q$. Since f is étale, $\widetilde{B}$ is smooth around $y \in Q$. Moreover f is flat, hence the family $\widetilde{\mathcal{Y}} := \mathcal{Y} \times_B \widetilde{B}$ is flat over $\widetilde{B}$. Finally $\widetilde{\mathcal{Y}}$ has special fiber Y over y and smooth general fiber. We now check that $\widetilde{\mathcal{Y}} \to \widetilde{B}$ factors through $\widetilde{\mathcal{Y}} \subseteq \mathbb{A}^N_k \times \widetilde{B}$.

The embeddings $Y \subseteq \mathbb{A}^N_k$ and $Y \subseteq \mathcal{Y}$ correspond to epimorphisms $\psi \colon k[x_1, \dots, x_N] \twoheadrightarrow A$ and $S[y_1, \dots, y_n]/J \twoheadrightarrow A$. By lifting ψ, we naturally obtain a morphism $k[x_1, \dots, x_N] \to S[y_1, \dots, y_n]/J$ which gives rise to a morphism $\mathcal{Y} \to \mathbb{A}^N_k$. By composition, we then obtain the morphism of schemes over $\widetilde{B}$,

$$\varphi \colon \widetilde{\mathcal{Y}} = \mathcal{Y} \times_B \widetilde{B} \to \mathcal{Y} \times \widetilde{B} \to \mathbb{A}^N_k \times \widetilde{B}.$$

Since the restriction of φ to the fiber over $y \in \widetilde{B}$ is exactly φ_0, which is surjective, it follows that φ is surjective around such fiber, thus it gives the desired factorization. We conclude that Y is smoothable in $\mathbb{P}^N_k$ too.

Conversely let Y be smoothable in $\mathbb{P}^N_k$ and let $\mathcal{Y} \to B$ be a flat family in $\mathbb{P}^N_k \times B$ with special fiber Y and reduced general fiber. By projecting the family from a general linear space of dimension $N - n - 1$ on the tangent space to Q at y we obtain a flat family with reduced general fiber and special fiber isomorphic to $Y \subseteq \mathbb{P}^n_k$ over a suitable open subset $B_0 \subseteq B$. □

Now we turn our attention to the singular locus of $\mathcal{H}ilb_d(\mathbb{P}^N_k)$. Recall that $X \subseteq \mathbb{P}^N_k$ is called obstructed if the corresponding point in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ is singular. Since $H^0\left(\mathbb{P}^N_k, \mathcal{N}_X\right)$ is canonically identified with the tangent space to $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ at the point X, it follows that X is obstructed if and only if $h^0\left(\mathbb{P}^N_k, \mathcal{N}_X\right)$ is greater than the local dimension of $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ at the point X. The next lemma allows us to relate such invariants.

Lemma 2.3. *Let X be a scheme of dimension 0 and degree d and let $X \subseteq \mathbb{P}^N_k$ and $X \subseteq \mathbb{P}^{N'}_k$ be two embeddings. Then*

$$h^0\left(X, \mathcal{N}_{X|\mathbb{P}^N_k}\right) - dN = h^0\left(X, \mathcal{N}_{X|\mathbb{P}^{N'}_k}\right) - dN'.$$

Proof. Again let $X = \bigcup_{i=1}^{p} X_i \in \mathcal{H}ilb_d(\mathbb{P}_k^N)$. Since $h^0(X, \mathcal{N}_X) = \bigoplus_{i=1}^{p} h^0\left(X_i, \mathcal{N}_{X_i}\right)$ and $h^0\left(X_i, \mathcal{N}_{X_i}\right) \geq d_i N$, it turns out that X is obstructed if and only if the same is true for at least one of the X_i.

Arguing as in the proof of Lemma 2.2, from now on, we will fix our attention on the above irreducible $Y \cong \operatorname{spec}(A) \in \mathcal{H}ilb_\delta^G(\mathbb{P}_k^N)$ since, in order to prove the statement, it suffices to prove that $h^0\left(Y, \mathcal{N}_{Y|\mathbb{P}_k^N}\right) - \delta N = h^0\left(Y, \mathcal{N}_{Y|\mathbb{A}_k^n}\right) - \delta n$.

Recall that Q denotes a subscheme in $\mathbb{A}_k^N \subseteq \mathbb{P}_k^N$ of dimension n, containing Y and smooth around the unique closed point $y \in Y$ (see the proof of the previous Lemma). Since Q is smooth around Y we have $\mathcal{N}_Q \otimes \mathcal{O}_Y \cong \mathcal{O}_Y^{\oplus N-n}$. Thus the embedding $\iota\colon Q \hookrightarrow \mathbb{A}_k^N$ yields the exact sequence

$$0 \longrightarrow \mathcal{T}_Q \otimes \mathcal{O}_Y \longrightarrow \mathcal{T}_{\mathbb{A}_k^N} \otimes \mathcal{O}_Y \longrightarrow \mathcal{O}_Y^{\oplus N-n} \longrightarrow 0,$$

The embedding ι also induces a morphism $\mathcal{O}_{\mathbb{A}_k^N} \to \mathcal{O}_Q$. By definition, its restriction to $\mathfrak{I}_Y$ maps onto $\mathfrak{I}_{Y|Q}$. Thus i induces a morphism

$$\iota'\colon \mathcal{N}_{Y|Q} := (\mathfrak{I}_{Y|Q}/\mathfrak{I}_{Y|Q}^2)\check{} \to \mathcal{N}_Y.$$

Finally ι induces an isomorphism on Y then its restricted differential $d(\iota_{|Y})$ is an isomorphism. Let T_Y^1 is the first cotangent sheaf of Y: for the same reason the induced morphism $\widetilde{\iota}\colon T_Y^1 \to T_Y^1$ is an isomorphism too.

The above discussion proves the existence of a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_Y & \longrightarrow & \mathcal{T}_Q \otimes \mathcal{O}_Y & \longrightarrow & \mathcal{N}_{Y|Q} & \longrightarrow T_Y^1 \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d\iota_{|Y}} & & \downarrow{\scriptstyle d\iota\otimes\mathrm{id}} & & \downarrow{\scriptstyle \iota'} & \downarrow{\scriptstyle \widetilde{\iota}} & \\
0 & \longrightarrow & \mathcal{T}_Y & \longrightarrow & \mathcal{T}_{\mathbb{A}_k^N} \otimes \mathcal{O}_Y & \longrightarrow & \mathcal{N}_Y & \longrightarrow T_Y^1 \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_Y^{\oplus N-n} & \xrightarrow{\sim} & \mathcal{O}_Y^{\oplus N-n} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Taking the cohomology of the third column we finally obtain

$$h^0(Y, \mathcal{N}_Y) = h^0\left(Y, \mathcal{N}_{Y|Q}\right) + (N-n)h^0(Y, \mathcal{O}_Y) = h^0\left(Y, \mathcal{N}_{Y|Q}\right) + (N-n)\delta.$$

Arguing as above with $f\colon Q \to \mathbb{A}_k^n$ instead of ι, since the differential of f (which is étale) induces an isomorphism of the restricted tangent sheaves $\mathcal{T}_Q \otimes \mathcal{O}_Y \cong \mathcal{T}_{\mathbb{A}_k^n} \otimes \mathcal{O}_Y$, we infer $\mathcal{N}_{Y|Q} \cong \mathcal{N}_{Y|\mathbb{A}_k^n}$ (the normal sheaf with respect to the canonical embedding $Y \subseteq \mathbb{A}_k^n$) and we finally obtain $h^0(Y, \mathcal{N}_Y) = h^0\left(Y, \mathcal{N}_{Y|\mathbb{A}_k^n}\right) + (N-n)\delta$. □

Now we restrict to $X \in \mathcal{H}ilb_d^G(\mathbb{P}_k^N) \subseteq \mathcal{H}ilb_d(\mathbb{P}_k^N)$ the Gorenstein locus, i.e. the locus of points in $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ representing Gorenstein subschemes of $\mathbb{P}_k^N$. The locus $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ is open, but is not necessarily dense, in $\mathcal{H}ilb_d(\mathbb{P}_k^N)$.

Trivially $\mathcal{R} \subseteq \mathcal{H}ilb_d^G(\mathbb{P}_k^N)$, i.e. reduced schemes represent points in $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$. It follows that the main component $\mathcal{H}ilb_d^{G,gen}(\mathbb{P}_k^N) := \mathcal{H}ilb_d^G(\mathbb{P}_k^N) \cap \mathcal{H}ilb_d^{gen}(\mathbb{P}_k^N)$ of $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ is irreducible of dimension dN and open in $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ since $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ is open in $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ (see the introduction).

As first step in the description of $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ we show that we have to restrict our attention to schemes $X \in \mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ having "big" tangent space at some point. More precisely we give the following

Definition 2.4. Let X be a scheme of dimension 0. We say that X is AS (almost solid) if the dimension of the tangent space at every point of X is at most three.

Such AS schemes well–behave with respect to smoothability and unobstructedness. Indeed we have the following

Proposition 2.5. *Let* $\operatorname{char}(k) \neq 2$. *If* $X \in \mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ *represents an AS scheme, then* $X \in \mathcal{H}ilb_d^{G,gen}(\mathbb{P}_k^N)$ *and it is unobstructed.*

Proof. In Corollary 4.3 of [26] and Proposition 2.2 and Remark 2.3 of [10] the smoothability and unobstructedness of each Gorenstein subscheme $X \subseteq \mathbb{P}_k^3$ are proved: in particular $h^0\left(X, \mathcal{N}_{X|\mathbb{P}_k^3}\right) = 3d$.

Since each AS scheme X can be embedded in $\mathbb{P}_k^3$, it follows from Lemma 2.2 above that $X \in \mathcal{H}ilb_d^{G,gen}(\mathbb{P}_k^N)$. Moreover Lemma 2.3 implies

$$h^0\left(X, \mathcal{N}_{X|\mathbb{P}_k^N}\right) - dN = h^0\left(X, \mathcal{N}_{X|\mathbb{P}_k^3}\right) - 3d = 0,$$

thus X is unobstructed too. □

For the reader's benefit we recall the following

Corollary 2.6. *Let* $\mathrm{char}(k) \neq 2$. *If* $N \leq 3$ *then* $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ *is irreducible and smooth.*

Proof. Each point in $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ with $N \leq 3$ is an AS scheme. It follows that $\mathcal{H}ilb_d^G(\mathbb{P}_k^N) = \mathcal{H}ilb_d^{G,gen}(\mathbb{P}_k^N)$ which is then irreducible. □

It is then natural to ask if $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ is irreducible. Or, equivalently, are non-AS schemes smoothable?

The answer to this question is, in general, negative. As pointed out in the introduction, in Section 6.2 of [13] the authors states the existence of local Artinian, Gorenstein k-algebras of degree 14 and embedding dimension 6 whose deformations are all of the same type, using a method previously introduced in [12]: thus such kinds of algebras define an irreducible component in $\mathcal{H}ilb_{14}^G(\mathbb{P}_k^6)$ distinct from $\mathcal{H}ilb_{14}^{G,gen}(\mathbb{P}_k^6)$.

A second natural question is to ask if $\mathrm{Sing}(\mathcal{H}ilb_d^G(\mathbb{P}_k^N))$ coincides with the locus of non-AS schemes.

When d increases the answer to the above question is again negative. E.g. take $X := \mathrm{spec}(A) \in \mathcal{H}ilb_{16}^G(\mathbb{P}_k^4)$, where $A := k[x_1, x_2, x_3, x_4]/(x_1^2, x_2^2, x_3^2, x_4^2)$. Thus X is a complete intersection, hence it is trivially smoothable, thus it belongs to the component $\mathcal{H}ilb_{16}^{G,gen}(\mathbb{P}_k^4)$ which has dimension 64. Being a complete intersection, $\mathcal{N}_X$ is locally free, thus it is actually free, since X has dimension 0. This means that $\mathcal{N}_X \cong \mathcal{O}_X^{\oplus 4}$, hence $h^0(X, \mathcal{N}_X) = 4h^0(X, \mathcal{O}_X) = 64$ and we finally conclude that X is unobstructed.

The object of our paper is to prove the irreducibility of $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ and to characterize its singular locus, when $d \leq 9$. Due to Proposition 2.5 above it is clear that we have to focus our attention on non-AS schemes $X \in \mathcal{H}ilb_d^G(\mathbb{P}_k^N)$. To this purpose we first look at the intrinsic structure of local Artinian, Gorenstein k-algebras A of degree $d \leq 9$ and $\mathrm{emdim}(A) \geq 4$.

Let A be a local Artinian k-algebra of degree d with maximal ideal $\mathfrak{M}$. In general we have a filtration

$$A \supset \mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots \supset \mathfrak{M}^e \supset \mathfrak{M}^{e+1} = 0$$

for some integer $e \geq 1$, so that its associated graded algebra

$$\mathrm{gr}(A) := \bigoplus_{i=0}^{\infty} \mathfrak{M}^i/\mathfrak{M}^{i+1}$$

is a vector space over $k \cong A/\mathfrak{M}$ of finite dimension

$$d = \dim_k(A) = \dim_k(\mathrm{gr}(A)) = \sum_{i=0}^{e} \dim_k(\mathfrak{M}^i/\mathfrak{M}^{i+1}). \tag{2.7}$$

Definition 2.8. Let A be a local, Artinian k-algebra. If $\mathfrak{M}^e \neq 0$ and $\mathfrak{M}^{e+1} = 0$ we define the maximum socle degree of A as e and denote it by $\mathrm{msdeg}(A)$.

If $e = \mathrm{msdeg}(A)$ and $n_i := \dim_k(\mathfrak{M}^i/\mathfrak{M}^{i+1})$, $0 \leq i \leq e$, we define the Hilbert function of A as the vector $H(A) := (n_0, \ldots, n_e) \in \mathbb{N}^{e+1}$.

Some other authors (see e.g. [27]) prefer to use *level* instead of maximum socle degree. In any case $n_0 = 1$. Recall that the Gorenstein condition is equivalent to saying that the socle $\mathrm{Soc}(A) := 0 \colon \mathfrak{M}$ of A is a vector space over $k \cong A/\mathfrak{M}$ of dimension 1. If $e = \mathrm{msdeg}(A) \geq 1$ trivially $\mathfrak{M}^e \subseteq \mathrm{Soc}(A)$, hence if A is Gorenstein then equality must hold and $n_e = 1$. Thus if $\mathrm{emdim}(A) \geq 2$ we deduce that $\mathrm{msdeg}(A) \geq 2$ and $\deg(A) \geq \mathrm{emdim}(A) + 2$.

Taking into account of Section 5F of [28] (see also [29]), the list of all possible shapes of Hilbert functions of local, Artinian, Gorenstein k-algebras A with $\mathrm{emdim}(A) \geq 4$ of degree 7, 8, 9 is given in Table 1.

Table 1

Degree	Hilbert function
7	(1, 4, 1, 1), (1, 5, 1)
8	(1, 4, 1, 1, 1), (1, 5, 1, 1), (1, 6, 1), (1, 4, 2, 1)
9	(1, 4, 1, 1, 1, 1), (1, 5, 1, 1, 1), (1, 6, 1, 1), (1, 7, 1), (1, 4, 2, 1, 1), (1, 5, 2, 1), (1, 4, 3, 1)

The sequences in Table 1 can be divided into three different families according to $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3)$. We will show that they all actually occur as Hilbert functions of a local, Artinian, Gorenstein k-algebra.

When $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) = 1$ the above sequences completely characterize the algebra if $\mathrm{char}(k) \neq 2$. Indeed we have the following

Theorem 2.9. *Let* $n \geq 1$ *be an integral number. If* A *is a local, Artinian, Gorenstein,* k*-algebra with* $H(A) = (1, n, 1, \ldots, 1)$ *of degree* d *and* $\mathrm{char}(k) \neq 2$, *then* $A \cong A_{n,d} := k[x_1, \ldots, x_n]/I$ *where*

$$I := \begin{cases} (x_1^d) & \text{if } n = 1, \\ (x_i x_j, x_h^2 - x_1^{d-n})_{\substack{1 \leq i < j \leq n, \\ 2 \leq h \leq n}} & \text{if } n \geq 2. \end{cases}$$

Moreover $A_{n,d} \cong A_{n',d'}$ *if and only if* $n = n'$ *and* $d = d'$.

Proof. See [20]; another proof can be found in [17]. □

Following [20] the algebras $A_{n,d}$ are usually called *stretched*.

Proposition 2.10. *Let $X := \operatorname{spec}(A_{n,d}) \subseteq \mathbb{P}^N_k$, $N \geq n$. Then X is smoothable in $\mathbb{P}^N_k$.*

Proof. Due to Lemma 2.2 it suffices to check that $\operatorname{spec}(A_{n,d})$ is smoothable in $\mathbb{A}^n_k$. Such an assertion is trivial if $n = 1$ and we check it by induction on n, proving that $A_{n,d}$ is a flat specialization of the simpler algebra $A_{n,d-1} \oplus A_{0,1}$, for each $d \geq n+2 \geq 4$. Indeed in $k[b, x_1, \ldots, x_n]$ we have

$$\begin{aligned} J &:= (x_ix_j, x_h^2 - bx_1^{d-n-1} - x_1^{d-n}, x_1^{d-n+1})_{\substack{1\leq i<j\leq n,\\ 2\leq h\leq n}} \\ &= (x_1 + b, x_2, \ldots, x_n) \cap (x_ix_j, x_h^2 - bx_1^{d-n-1}, x_1^{d-n})_{\substack{1\leq i<j\leq n,\\ 2\leq h\leq n}}, \end{aligned}$$

for each $d \geq n+2 \geq 4$. In order to check the above equality we set $J_0 := (x_1+b, x_2, \ldots, x_n) \cap (x_ix_j, x_h^2 - bx_1^{d-n-1}, x_1^{d-n})_{\substack{1\leq i<j\leq n,\\ 2\leq h\leq n}}$. Trivially $J \subseteq J_0$. On the other hand an element in J_0 can be written as

$$y := \sum_{1\leq i<j\leq n} u_{i,j}x_ix_j + \sum_{2\leq h\leq n} v_h(x_h^2 - bx_1^{d-n-1}) + wx_1^{d-n}$$

with the obvious extra condition $b\sum_{2\leq h\leq n} v_hx_1^{d-n-1} - wx_1^{d-n} \in (x_1 + b, x_2, \ldots, x_n)$. Since $d \geq n + 1$ such last condition is equivalent to $b\sum_{2\leq h\leq n} v_h - wx_1 \in (x_1 + b, x_2, \ldots, x_n)$ i.e. $w = -\sum_{2\leq h\leq n} v_h$. We conclude that

$$y = \sum_{1\leq i<j\leq n} u_{i,j}x_ix_j + \sum_{2\leq h\leq n} v_h(x_h^2 - bx_1^{d-n-1} - x_1^{d-n})$$

hence $y \in J$. Thus $\mathcal{A}_{n,d} := k[b, x_1, \ldots, x_n]/J \to \mathbb{A}^1_k$ is a flat family having special fiber over $b = 0$ isomorphic to $A_{n,d}$ and general fiber isomorphic to $A_{n,d-1} \oplus A_{0,1}$. □

In the next two sections we will classify the two remaining cases. More precisely in Section 3 we deal with the case $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) = 2$, i.e. $H(A) = (1, n, 2, 1, \ldots, 1)$, $n \geq 2$. We obtain the same results proved in [23] and [24] under the restrictive hypothesis $\operatorname{char}(k) = 0$. Finally, in Section 4, we will examine the remaining case, namely $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) = 3$, i.e. $H(A) = (1, n, 3, 1)$, $n \geq 3$.

Our first remark is the following

Lemma 2.11. *Let $n \geq m$ be integral numbers. If A is a local, Artinian k-algebra with $H(A) = (1, n, m, \ldots)$ and $\operatorname{char}(k) \neq 2$, then there exists a minimal set of generators $a_1, \ldots, a_n$ of its maximal ideal $\mathfrak{M}$ such that $\mathfrak{M}^2 = (a_1^2, \ldots, a_m^2)$.*

Proof. Consider an arbitrary minimal set of generators $a_1, \ldots, a_n \in \mathfrak{M}$. In any case we have $\mathfrak{M}^2 = (a_ia_j)_{i,j=1,\ldots,n}$. If $a_1a_2 \in \mathfrak{M}^2 \setminus \mathfrak{M}^3$, since $(a_1 + a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2$, it follows that at least one among $(a_1 + a_2)^2$, a_1^2, a_2^2 is not in $\mathfrak{M}^3$. Thus, up to a linear change of the minimal generators of $\mathfrak{M}$, we can always take a_1^2 as minimal generator for $\mathfrak{M}^2$. Now we can repeat the above argument in $A/(a_1^2)$ and we prove the statement with an easy induction on m. □

3. k-Algebras with Hilbert function $(1, n, 2, 1, \ldots, 1)$

The aim of this section is to prove the following

Theorem 3.1. *Let $n \geq 2$ be an integral number. If A is a local, Artinian, Gorenstein, k-algebra with $H(A) = (1, n, 2, 1, \ldots, 1)$ of degree d and either $\operatorname{char}(k) = 0$ or $\operatorname{char}(k) \geq d - n - 1$, then $A \cong A^t_{n,2,d} := k[x_1, \ldots, x_n]/I_t$, $t = 1, 2$, where*

$$I_1 := \begin{cases} (x_1^2x_2 - x_1^3, x_2^2, x_ix_j, x_h^2 - x_1^3)_{\substack{1\leq i<j\leq n, 3\leq j\\ 3\leq h\leq n}} & \text{if } d = n + 4,\\ (x_1^2x_2, x_2^2 - x_1^{d-n-2}, x_ix_j, x_h^2 - x_1^{d-n-1})_{\substack{1\leq i<j\leq n,\ 3\leq j\\ 3\leq h\leq n}} & \text{if } d \geq n + 5, \end{cases}$$

$$I_2 := (x_1x_2, x_2^3 - x_1^{d-n-1}, x_ix_j, x_h^2 - x_1^{d-n-1})_{\substack{1\leq i<j\leq n, 3\leq j\\ 3\leq h\leq n}}.$$

Moreover $A^t_{n,2,d} \cong A^{t'}_{n',2,d'}$ if and only if $n = n'$, $d = d'$ and $t = t'$.

Proof. Let A be as in the statement. Its maximum socle degree, $e := \operatorname{msdeg}(A)$, is then equal to $d - n - 1 \geq 3$ and we will assume in this section that $\operatorname{char}(k) > e \geq 3$. Due to Lemma 2.11 we can assume

$$\mathfrak{M} = (a_1, \ldots, a_n), \qquad \mathfrak{M}^2 = (a_1^2, a_2^2),$$

for a suitable set of minimal generators of $\mathfrak{M}$.

It follows the existence of a non-trivial relation of the form

$$\alpha_1 a_1^2 + \alpha_2 a_2^2 + \overline{\alpha} a_1 a_2 \in \mathfrak{M}^3, \tag{3.1.1}$$

where $\alpha_1, \alpha_2, \overline{\alpha} \in k \subseteq A$. The first member of the above relation can be interpreted as the defining polynomial of a single quadric Q in the projective line $\mathbb{P}(V)$ associated with the subspace $V \subseteq \mathfrak{M}/\mathfrak{M}^2$ generated by the classes of a_1, a_2. Such a quadric has rank $\mathrm{rk}(Q)$ either 2 or 1.

Claim 3.1.2. *If* $\mathrm{rk}(Q) = t$ *then* $A \cong A^t_{n,2,d}$.

In order to complete the proof it suffices to check that $A^t_{n,2,d} \cong A^{t'}_{n',2,d'}$ if and only if $n = n'$, $d = d'$ and $t = t'$. The proof of the if part is trivial, thus it remains to check that $A^t_{n,2,d} \cong A^{t'}_{n',2,d'}$ only if $n = n'$, $d = d'$ and $t = t'$. The first two equalities are immediate hence we have only to show that $A^1_{n,2,d} \ncong A^2_{n,2,d}$.

To this purpose set H_t be the ideal generated by $\{u \in A^t_{n,2,d} | u^2 \in \mathfrak{M}^e\}$. Then $\sum_{i=1}^n \lambda_i \overline{x_i} \in H_t$ if and only if

$$\sum_{i,j=1}^n \lambda_i\lambda_j \bar{x}_i\bar{x}_j = \lambda_1^2\bar{x}_1^2 + \lambda_1\lambda_2\bar{x}_1\bar{x}_2 + \lambda_2^2\bar{x}_2^2 + \sum_{i=3}^n \lambda_i^2\bar{x}_i^2 \in \mathfrak{M}^e.$$

This is equivalent to $\lambda_i = 0$, $i \le t$. In particular $H_t = (\bar{x}_{h+1}, \dots, \bar{x}_n)$, hence $\dim_k(H_t \otimes k) = n - t$.

If $A^1_{n,2,d} \cong A^2_{n,2,d}$ then the ideals H_1 and H_2 would correspond each other in this isomorphism, hence $\dim_k(H_1 \otimes k)$ and $\dim_k(H_2 \otimes k)$ should coincide. □

In order to complete the proof of Theorem 3.1, we have to prove Claim 3.1.2. This proof is rather technical. First we will find relations among the generators of the maximal ideal. Then we will show that such a system of relations is complete, i.e. all the other relations are generated by these ones. We will examine separately the two cases $\mathrm{rk}(Q) = 2$ (the general case) and $\mathrm{rk}(Q) = 1$ (the special case).

3.2. *The case* $\mathrm{rk}(Q) = 2$

In the first case, via a suitable linear transformation in V we can assume $\alpha_1 = \alpha_2 = 0$, $\overline{\alpha} = 1$ in Relation (3.1.1), i.e. $a_1a_2 \in \mathfrak{M}^3$, thus $\mathfrak{M}^h = (a_1^h, a_2^h)$, for each $h \ge 2$. In particular, possibly interchanging a_1 and a_2, we can assume $a_1^e \neq 0$. Thus we obtain both $\mathfrak{M}^h = (a_1^h)$, $h \ge 3$ and the relations

$$a_ia_j = \alpha^1_{i,j}a_1^2 + \alpha^2_{i,j}a_2^2 + \alpha_{i,j}a_1^3, \quad i, j \ge 1,$$

where $\alpha_{i,j} = \sum_{h=0}^{e-4} \beta^h_{i,j}a_1^h + \beta_{i,j}a_1^{e-3}$, $\alpha^h_{i,j}, \beta^h_{i,j}, \beta_{i,j} \in k$, $\alpha^h_{i,j} = \alpha^h_{j,i}$, $\alpha_{i,j} = \alpha_{j,i}$, $\alpha^1_{1,1} = \alpha^2_{2,2} = 1$ and $\alpha^2_{1,1} = \alpha_{1,1} = \alpha^1_{1,2} = \alpha^2_{1,2} = \alpha^1_{2,2} = \alpha_{2,2} = 0$.

Via $a_2 \mapsto a_2 + \alpha_{1,2}a_1^2$ we can assume

$$a_1a_2 = 0, \tag{3.2.1}$$

i.e. $\alpha_{1,2} = 0$.

Again via $a_j \mapsto a_j + \alpha^1_{1,j}a_1 + \alpha^2_{2,j}a_2 + \alpha_{1,j}a_1^2$, $j \ge 3$, we can assume $\alpha^1_{1,j} = \alpha_{1,j} = 0$, $j \ge 2$, and $\alpha^2_{2,j} = 0$, $j \neq 2$. In particular $a_1^2a_j = a_2a_ia_j = 0$, $j \neq 3$. Explicitly $a_1a_j = \alpha^2_{1,j}a_2^2$, $j \ge 2$, and $a_2a_j = \alpha^2_{2,j}a_2^2 + \alpha_{2,j}a_1^3$, $j \ge 2$.

Moreover $a_2^3 = \sum_{i=3}^e \mu_ia_1^i$, $\mu_i \in k$, then $\sum_{i=3}^{e-1} \mu_ia_1^{i+1} = a_1a_2^3 = 0$, thus $\mu_i = 0$, $i = 3, \dots, e-1$, whence $a_2^3 = \mu_ea_1^e$. If $\mu_e = 0$ then $a_2^2 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^e$ that is a contradiction because A is Gorenstein. Up to multiplying a_1 times an e-root of μ_e, we can thus assume

$$a_2^3 - a_1^e = 0. \tag{3.2.2}$$

Let $n \ge 3$. Since $\alpha^2_{2,j} = 0$, $j \neq 2$, it follows $\alpha^2_{i,j}a_2^3 = a_2(a_ia_j) = a_2a_ia_j = (a_2a_j)a_i = 0$, $(i,j) \neq (2,2)$, whence $\alpha^2_{i,j} = 0$, $(i,j) \neq (2,2)$, thus

$$a_1a_j = 0, \quad j \neq 1. \tag{3.2.3}$$

Moreover $\alpha^1_{i,j}a_1^3 + \alpha_{i,j}a_1^4 = (a_ia_j)a_1 = a_1a_ia_j = (a_1a_j)a_i = 0$, thus $\alpha^1_{i,j} = 0$ and necessarily $\alpha_{i,j} = \beta_{i,j}a_1^{e-3}$. Recall that $\beta_{1,j} = \beta_{2,2} = 0$.

Let $y := y_0 + \sum_{i=1}^{n} y_i a_i + y_{n+1}a_1^2 + y_{n+2}a_2^2 + \sum_{h=3}^{e} y_{n+h}a_1^h \in \mathrm{Soc}(A)$, $y_h \in k$. Then the conditions $a_j y = 0$, $j = 1, \dots, n$, become

$$\begin{cases} y_0a_1 + y_1a_1^2 + y_{n+1}a_1^3 + \displaystyle\sum_{h=3}^{e-1} y_{n+h}a_1^{h+1} = 0, \\ y_0a_2 + y_2a_2^2 + \left(\displaystyle\sum_{i=3}^{n} y_i\beta_{2,i}\right) a_1^e + y_{n+2}a_1^e = 0, \\ y_0a_j + y_2\beta_{2,j}a_1^e + \left(\displaystyle\sum_{i=3}^{n} y_i\beta_{i,j}\right) a_1^e = 0, \quad j \geq 3. \end{cases}$$

It is clear that $y_0 = y_1 = y_2 = y_{n+1} = y_{n+3} = \cdots = y_{n+e-1} = 0$ and

$$\sum_{i=3}^{n} y_i\beta_{i,j} = 0, \quad j \geq 3.$$

In particular if the symmetric matrix $B := \left(\beta_{i,j}\right)_{i,j\geq 3}$ is singular then each non-zero solution $(y_3, \dots, y_n) \in k^{\oplus n-3}$ of the above linear system would yield $y := \sum_{i=3}^{n} y_i a_i \in \mathrm{Soc}(A) \setminus \mathfrak{M}^e$, that is again a contradiction since A is Gorenstein. We conclude that B is non-singular, hence we can make a linear change on $a_3, \dots, a_n$ in such a way that

$$a_i a_j = \delta_{i,j}a_1^e, \quad i, j \geq 3. \tag{3.2.4}$$

Now we finally have $a_2a_j = \gamma_j a_1^e$: via $a_2 \mapsto a_2 + \sum_{j=3}^{n} \gamma_j a_j$ we also obtain

$$a_2a_j = 0, \quad j \geq 3. \tag{3.2.5}$$

Combining Equalities (3.2.1), (3.2.2), (3.2.3), (3.2.4) and (3.2.5), we obtain an epimorphism $A^2_{n,2,d} \twoheadrightarrow A$ where $A^2_{n,2,d}$ has been defined in the statement of Theorem 3.1. Since in $\mathrm{gr}(A^2_{n,2,d})$ the relations

$$\bar{x}_1\bar{x}_2 = \bar{x}_2^3 = \bar{x}_i\bar{x}_j = \bar{x}_h^2 = \bar{x}_1^{d-n} = 0, \quad 1 \leq i < j \leq n, 3 \leq j, 3 \leq h \leq n$$

($\bar{x}$ denotes the class of the element $x \in A^2_{n,2,d}$ in $\mathrm{gr}(A^2_{n,2,d})$) hold true, thanks to Formula (2.7), we obtain $\dim_k(A^2_{n,2,d}) \leq d$, thus the above epimorphism is forced to be an isomorphism.

Notice that when $n = 2$ then $A^2_{2,2,d}$ is a complete intersection. When $n = 3$ the well-known structure theorem for Gorenstein local rings, proved in [30], guarantees that the ideal defining $A^2_{3,2,d}$ in $k[x_1, x_2, x_3]$ is minimally generated by the submaximal pfaffians of a suitable skew-symmetric matrix M. E.g. one may take

$$M := \begin{pmatrix} 0 & 0 & x_1 & -x_2 & 0 \\ 0 & 0 & 0 & x_2 & -x_3 \\ -x_1 & 0 & 0 & -x_3 & x_2^2 \\ x_2 & -x_2 & x_3 & 0 & -x_1^{d-5} \\ 0 & x_3 & -x_2^2 & x_1^{d-5} & 0 \end{pmatrix}.$$

3.3. *The case* $\mathrm{rk}(Q) = 1$

In this case, via a suitable linear transformation in V we can assume that Relation (3.1.1) is $a_2^2 \in \mathfrak{M}^3$, thus $\mathfrak{M}^h = (a_1^h, a_1^{h-1}a_2)$, $h \geq 2$. In particular $\mathfrak{M}^e = (a_1^e, a_1^{e-1}a_2)$ and $a_1^{e-t}a_2^t = 0$, $t \geq 2$. If $a_1^e = 0$ then $a_1^{e-1}a_2 \neq 0$, hence $(a_1 + a_2)^e = a_1^e + ea_1^{e-1}a_2 \neq 0$, and so the linear change $a_1 \mapsto a_1 + a_2$ allows us to assume both $\mathfrak{M}^h = (a_1^h)$, $h \geq 3$, and the relations

$$a_ia_j = \alpha^1_{i,j}a_1^2 + \alpha^2_{i,j}a_1a_2 + \alpha_{i,j}a_1^3, \quad i, j \geq 1,$$

where $\alpha_{i,j} = \sum_{h=0}^{e-4} \beta^h_{i,j}a_1^h + \beta_{i,j}a_1^{e-3}$, $\alpha^h_{i,j}, \beta^h_{i,j}, \beta_{i,j} \in k$, $\alpha^h_{i,j} = \alpha^h_{j,i}$, $\alpha_{i,j} = \alpha_{j,i}$, $\alpha^1_{1,1} = \alpha^2_{1,2} = 1$ and $\alpha^2_{1,1} = \alpha_{1,1} = \alpha^1_{1,2} = \alpha_{1,2} = \alpha^1_{2,2} = \alpha^2_{2,2} = 0$.

Via the transformation $a_j \mapsto a_j + \alpha^1_{1,j}a_1 + \alpha^2_{1,j}a_2 + \alpha_{1,j}a_1^2$ we can assume

$$a_1a_j = 0 \quad j \geq 3. \tag{3.3.1}$$

Moreover $a_1^2a_2 = \sum_{i=3}^{e} \mu_i a_1^i \in \mathfrak{M}^3$, where $\mu_i \in k$.

From now on we will assume $e \geq 4$ and we will come back to the case $e = 3$ later on. Since $e \geq 4$, it follows $a_1^{e-1}a_2 = (a_1^2a_2)a_1^{e-3} = \mu_3 a_1^e$. On the other hand $\mu_3^2 a_1^e = \mu_3 a_1^{e-1}a_2 = a_1^{e-4}(a_1^2a_2)a_2 = a_1^{e-2}a_2^2 = 0$, whence $\mu_3 = 0$. Via $a_2 \mapsto a_2 + \sum_{i=4}^{e} \mu_i a_1^{i-2}$ we obtain

$$a_1^2a_2 = 0. \tag{3.3.2}$$

Since $a_2^2 \in \mathfrak{M}^3$, we have $a_2^2 = \sum_{h=3}^{e} \gamma_h a_1^h$. Assume that $\gamma_h = 0$, $h < t$. Then $\gamma_t a_1^e = a_1^{e-t}a_2^2 = a_1^{e-t-2}(a_1^2a_2)a_2 = 0$. It follows that $\gamma_h = 0$, $h \leq e-2$, thus $a_2^2 = \gamma_{e-1}a_1^{e-1} + \gamma_e a_1^e$. If $\gamma_{e-1} = 0$ then $a_1a_2 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^e$, a contradiction, since A Gorenstein implies $\mathrm{Soc}(A) = \mathfrak{M}^e$. Thus $\gamma_{e-1} \neq 0$, hence we can find a square root u of $\gamma_{e-1} + \gamma_e a_1$ and via $a_2 \mapsto ua_2$ we finally obtain

$$a_2^2 - a_1^{e-1} = 0. \tag{3.3.3}$$

From now on let $n \geq 3$. Equalities (3.3.1) and (3.3.2) yield $\alpha_{i,j}^1 a_1^3 + \alpha_{i,j}a_1^4 = a_1(a_ia_j) = a_1a_ia_j(a_1a_i)a_j = 0$, $(i,j) \neq (1,1), (1,2), (2,2)$, thus $\alpha_{i,j}^1 = 0$ and $\alpha_{i,j} = \beta_{i,j}^h = 0$, $h = 0, \dots, e-4$. Moreover Equalities (3.3.1), (3.3.2) and (3.3.3) imply $\alpha_{i,j}^2 a_1^e = \alpha_{i,j}^2 a_1 a_2^2 = a_2(a_ia_j) = (a_2a_i)a_j = 0$, $(i,j) \neq (1,1), (1,2), (2,2)$, whence $\alpha_{i,j}^2 = 0$ too.

Let $y := y_0 + \sum_{i=1}^{n} y_ia_i + y_{n+1}a_1^2 + y_{n+2}a_1a_2 + \sum_{h=3}^{e} y_{n+h}a_1^h \in \mathrm{Soc}(A)$, $y_h \in k$. Then the conditions $a_jy = 0$, $j = 1, \dots, n$, become

$$\begin{cases} y_0a_1 + y_1a_1^2 + y_2a_1a_2 + y_{n+1}a_1^3 + \displaystyle\sum_{h=3}^{e-1} y_{n+h}a_1^{h+1} = 0, \\ y_0a_2 + y_1a_1a_2 + y_2a_1^{e-1} + \left(\displaystyle\sum_{i=3}^{n} y_i\beta_{2,i}\right) a_1^e + y_{n+2}a_1^e = 0, \\ y_0a_j + y_2\beta_{2,j}a_1^e + \left(\displaystyle\sum_{i=3}^{n} y_i\beta_{i,j}\right) a_1^e = 0, \quad j \geq 3. \end{cases}$$

It is clear that $y_0 = y_1 = y_2 = y_{n+1} = y_{n+3} = \cdots = y_{n+e-1} = 0$ and $\sum_{i=3}^{n} y_i\beta_{i,j} = 0$, $j \geq 3$. As in the previous case $\mathrm{rk}(Q) = 2$ the matrix $B := \left(\beta_{i,j}\right)_{i,j \geq 3}$ is non-singular, thus there exists $P \in \mathrm{GL}_{n-3}(k)$ such that ${}^tPBP = I_{n-3}$ is the identity, ${}^tP$ being the transpose of P. This matrix corresponds to a linear change of the generators $a_3, \dots, a_n$ which allows us to assume

$$a_ia_j = \delta_{i,j}a_1^e, \quad i,j \geq 3. \tag{3.3.4}$$

At this point we have $a_2a_j = \vartheta_j a_1^e$. Via $a_2 \mapsto a_2 + \sum_{i=3}^{n} \vartheta_i a_i/2$ we finally obtain $a_2a_j = 0$, $j \geq 3$, and $a_2^2 = a_1^{e-1} + \lambda a_1^e$ for a suitable $\lambda \in A$. Let v be a square root of $1 + \lambda a_1$. Then via $a_2 \mapsto va_2$ we finally obtain again Equality (3.3.3) and also

$$a_2a_j = 0, \quad j \geq 3. \tag{3.3.5}$$

If $e = 3$ then $a_1^2a_2 = \mu_3a_1^3$, $a_2^2 = \nu_3a_1^3$. If $\mu_3 = 0$ then $a_1a_2 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^e$, that is a contradiction since A is Gorenstein. Thus, up to multiplying a_2 by a suitable square root of μ_e, we can assume $\mu_e = 1$. Via $a_2 \mapsto a_2 + \beta_{2,2}a_1^2/2$ we finally obtain

$$a_1^2a_2 - a_1^3 = a_2^2 = 0. \tag{3.3.6}$$

If $n \geq 3$, we can repeat word by word the discussion above and we finally obtain Equalities (3.3.4) and (3.3.5) with $\lambda = 0$.

In particular, combining Equalities (3.3.1), (3.3.2), (3.3.3), (3.3.4), (3.3.5) and (3.3.6), as in the previous case $\mathrm{rk}(Q) = 2$, we obtain the isomorphism $A \cong A_{n,2,d}^1$, the k-algebra $A_{n,2,d}^1$ being defined in the statement of Theorem 3.1.

Again, when $n = 2$ then $A_{2,2,d}^1$ is a complete intersection. When $n = 3$, $A_{3,2,d}^1$ is defined in $k[x_1, x_2, x_3]$ by the submaximal pfaffians of either

$$M := \begin{pmatrix} 0 & 0 & x_2 & -x_1 & 0 \\ 0 & 0 & 0 & x_2 & -x_3 \\ -x_2 & 0 & 0 & x_3 & -x_1^2 \\ x_1 & -x_2 & -x_3 & 0 & x_1^2 \\ 0 & x_3 & x_1^2 & -x_1^2 & 0 \end{pmatrix}$$

if $d = 7$ or

$$M := \begin{pmatrix} 0 & 0 & x_2 & -x_1 & 0 \\ 0 & 0 & 0 & x_1^2 & -x_3 \\ -x_2 & 0 & 0 & x_3 & -x_1^{d-6} \\ x_1 & -x_1^2 & -x_3 & 0 & x_2 \\ 0 & x_3 & -x_1^{d-6} & -x_2 & 0 \end{pmatrix}$$

if $d \geq 8$.

It is natural to study the smoothability of schemes corresponding to the above described algebras. As is Section 2 we have the following

Proposition 3.4. *Let* $X := \operatorname{spec}(A^t_{n,2,d}) \subseteq \mathbb{P}^N_k$, $N \geq n$. *Then* X *is smoothable in* $\mathbb{P}^N_k$.

Proof. Due to Lemma 2.2 it suffices to check that $\operatorname{spec}(A^t_{n,2,d})$ is smoothable in $\mathbb{A}^n_k$. We check this by proving that the k-algebra $A^t_{n,2,d}$ is a flat specialization of easier algebras as in the proof of Proposition 2.10. Indeed take for $t = 2$

$$J := (x_1x_2, x_2^3 - bx_1^{d-n-2} - x_1^{d-n-1}, x_ix_j, x_h^2 - bx_1^{d-n-2} - x_1^{d-n-1}, x_1^{d-n})_{\substack{1\leq i\leq n,\\ 3\leq j\leq n, i\neq j\\ 3\leq h\leq n}}.$$

The same argument of Proposition 2.10 shows that

$$J = (x_1 + b, x_2, \dots, x_n) \cap (x_1x_2, x_2^3 - bx_1^{d-n-2}, x_ix_j, x_h^2 - bx_1^{d-n-2}, x_1^{d-n-1})_{\substack{1\leq i\leq n,\\ 3\leq j\leq n, i\neq j\\ 3\leq h\leq n}},$$

for $b \neq 0$. Then the family $\mathcal{A} := k[b, x_1, \dots, x_n]/J \to \mathbb{A}^1_k$ is flat with special fiber over $b = 0$ isomorphic to $A^2_{n,2,d}$ and general fiber isomorphic to $A^2_{n,2,d-1} \oplus A_{0,1}$ if $e \geq 4$ and $A_{n,n+3} \oplus A_{0,1}$ if $d - n - 1 = e = 3$.

Finally consider the case of the k-algebra $A^1_{n,2,d}$. Then let us consider the ideal J defined as

$$(bx_1x_2 + x_1^2, x_1^2x_2 + bx_2^3 - x_1^3, x_ix_j, x_h^2 - x_1^3, x_1^4)_{\substack{1\leq i\leq n,\\ 3\leq j\leq n, i\neq j\\ 3\leq h\leq n}}$$

if $d = n + 4$ (i.e. $e = 3$), and

$$(bx_1x_2 + x_2^2 - x_1^{d-n-2}, bx_2^3 - bx_1^{d-n-1} + x_1^2x_2, x_ix_j, x_h^2 - x_1^{d-n-1}, x_1^{d-n})_{\substack{1\leq i\leq n,\\ 3\leq j\leq n, i\neq j\\ 3\leq h\leq n}}$$

if $d \geq n + 5$ (i.e. $e \geq 4$). In this case $\mathcal{A} := k[b, x_1, \dots, x_n]/J \to \mathbb{A}^1_k$ is a flat family of local Artinian, Gorenstein k-algebras with constant Hilbert function $H(\mathcal{A}_b) = (1, n, 2, 1, \dots, 1)$. For $b = 0$ we have trivially $\mathcal{A}_0 \cong A^1_{n,2,d}$. For general $b \neq 0$ the algebra $\mathcal{A}_b$ is again local and Gorenstein, thus it must be either $A^1_{n,2,d}$ or $A^2_{n,2,d}$.

In any case we have the relations $bx_1x_2+x_1^2 \in \mathfrak{M}^3$ (if $d = n+4$) or $bx_1x_2+x_2^2 \in \mathfrak{M}^3$ (if $d \geq n+5$) in $\mathcal{A}_b$, thus, computing the invariant $\dim_k(H_t \otimes k)$ defined in the proof of Theorem 3.1 in these cases, we finally obtain $\mathcal{A}_b \cong A^2_{n,2,d}$ for general $b \neq 0$. □

4. *k*-Algebras with Hilbert functions $(1, n, 3, 1)$

In this section we will prove the following

Theorem 4.1. *Let* $n \geq 3$ *be an integral number. If* A *is a local, Artinian, Gorenstein,* k*-algebra with* $H(A) = (1, n, 3, 1)$ *and* $\operatorname{char}(k) \neq 2, 3$, *then* $A \cong A^{t,\alpha}_{n,3,n+5} := k[x_1, \dots, x_n]/I_{t,\alpha}$, $t = 1, \dots, 6$ *and* $\alpha \in k$ ($\alpha = 0$ *if* $t \geq 2$) *where*

$$I_{1,\alpha} := (x_1x_2 + x_3^2, x_1x_3, x_1^2 + x_2^2 - \alpha x_3^2, x_ix_j, x_j^2 - x_1^3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{2,0} := (x_1^2, x_2^2, x_3^2 + 2x_1x_2, x_ix_j, x_j^2 - x_1x_2x_3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{3,0} := (x_1^2, x_2^2, x_3^2, x_ix_j, x_j^2 - x_1x_2x_3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{4,0} := (x_1x_2, x_1x_3, x_2x_3, x_2^3 - x_1^3, x_3^3 - x_1^3, x_ix_j, x_j^2 - x_1^3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{5,0} := (x_1^2, x_1x_2, x_2x_3, x_2^3 - x_3^3, x_1x_3^2 - x_3^3, x_ix_j, x_j^2 - x_3^3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{6,0} := (x_1^2, x_1x_2, 2x_1x_3 + x_2^2, x_3^3, x_2x_3^2, x_ix_j, x_j^2 - x_1x_3^2)_{\substack{1\leq i<j\leq n\\ 4\leq j}}.$$

Moreover $A^{t,\alpha}_{n,3,n+5} \cong A^{t',\alpha'}_{n',3,n'+5}$ *if and only if* $n = n'$, $d = d'$, $t = t'$ *and* $\alpha^2 = \alpha'^2$.

Proof. Due to Lemma 2.11 one can always assume that

$$\mathfrak{M} = (a_1, a_2, a_3, \dots, a_n), \qquad \mathfrak{M}^2 = (a_1^2, a_2^2, a_3^2).$$

Thus we have three linearly independent relations of the form

$$\alpha_1a_1^2 + \alpha_2a_2^2 + \alpha_3a_3^2 + 2\overline{\alpha}_1a_2a_3 + 2\overline{\alpha}_2a_1a_3 + 2\overline{\alpha}_3a_1a_2 \in \mathfrak{M}^3, \tag{4.1.1}$$

where $\alpha_i, \overline{\alpha}_j \in k \subseteq A$, $i, j = 1, 2, 3$, and hence a net $\mathcal{N}$ of conics in the projective plane $\mathbb{P}(V)$, associated to the subspace $V \subseteq \mathfrak{M}/\mathfrak{M}^2$ generated by the classes of a_1, a_2, a_3.

Claim 4.1.2. *Let* $\operatorname{char}(k) \neq 2, 3$ *and let* $\mathcal{N}$ *be a net of conics in the projective plane* $\mathbb{P}^2_k$ *with coordinates* x_1, x_2, x_3. *Then, up to projectivities on* x_1, x_2, x_3, *we can choose as generators of the net* $\mathcal{N}$ *three polynomials as indicated in Table 2.*

Table 2

Wall symbol	Generators
ABB^*C	$x_1x_2+x_3^2, x_1x_3, x_2^2-6px_3^2+qx_1^2$
D	$x_1^2, x_2^2, x_3^2+2x_1x_2$
D^*	$x_1x_3, x_2x_3, x_3^2+2x_1x_2$
E	x_1^2, x_2^2, x_3^2
E^*	x_1x_3, x_2x_3, x_1x_2
F	$x_1x_3, x_2x_3, x_3^2+x_1^2$
F^*	$x_1^2, x_1x_2, x_2^2+x_3^2$
G	x_1^2, x_2^2, x_2x_3
G^*	x_1^2, x_1x_2, x_2x_3
H	$x_1^2, x_1x_2, 2x_1x_3+x_2^2$
I	x_1^2, x_1x_2, x_2^2
I^*	x_1x_3, x_2x_3, x_3^2

Thus we can make use of the above result in order to list all the possible relations for the algebras we are dealing with.

Claim 4.1.3. *If $\mathcal{N}$ is of type ABB^*C, D, E, E^*, G^*, H, then the k-algebra A is Gorenstein and it is respectively isomorphic to $A^{1,\alpha}_{n,3,n+5}$, $A^{2,0}_{n,3,n+5}$, $A^{3,0}_{n,3,n+5}$, $A^{4,0}_{n,3,n+5}$, $A^{5,0}_{n,3,n+5}$, $A^{6,0}_{n,3,n+5}$.*
If $\mathcal{N}$ is one of the remaining types D^*, F, F^*, G, I, I^*, then the corresponding algebra is not Gorenstein.

In order to complete the proof it suffices to check that $A^{t,\alpha}_{n,3,n+5}\cong A^{t',\alpha'}_{n',3,n'+5}$ if and only if $n=n'$, $d=d'$, $t=t'$ and $\alpha^2=\alpha'^2$. As in the proof of Theorem 3.1 it suffices to check that $A^{t,\alpha}_{n,3,n+5}\not\cong A^{t',\alpha'}_{n,3,n+5}$ if either $t\neq t'$ or $\alpha^2\neq {\alpha'}^2$.

For each $t=1,\dots,6$ we define H^α_t as the ideal in $\mathrm{gr}(A^{t,\alpha}_{n,3,n+5})$ generated by

$$\{\,u\in \mathrm{gr}(A^{h,\alpha}_{n,3,n+5})|u^2=0\,\}\cap\mathfrak{M}/\mathfrak{M}^2.$$

Using our representation for $A^{h,\alpha}_{n,3,n+5}$, one checks that $H^\alpha_1=(x_4,\dots,x_n)$ if $\alpha\neq\pm2$, $H^\alpha_1=(x_1\pm x_2,x_4,\dots,x_n)$ if $\alpha=\pm2$, $H^0_2=(x_1,x_2,x_4,\dots,x_n)$, $H^0_3=(x_1,\dots,x_n)$, $H^0_4=(x_4,\dots,x_n)$, $H^0_5=H^0_6=(x_1,x_4,\dots,x_n)$. If $\psi\colon A^{t,\alpha}_{n,3,n+5}\to A^{t',\alpha'}_{n,3,n+5}$ is an isomorphism then it would induce a graded isomorphism $\Psi\colon \mathrm{gr}(A^{t,\alpha}_{n,3,n+5})\to\mathrm{gr}(A^{t',\alpha'}_{n,3,n+5})$, hence an isomorphism $\overline{\Psi}\colon \mathrm{gr}(A^{t,\alpha}_{3,3,8})/H^\alpha_t\to\mathrm{gr}(A^{t',\alpha'}_{3,3,8})/H^{\alpha'}_{t'}$ since trivially $\Psi(H^\alpha_t)=H^{\alpha'}_{t'}$.

Thus it suffices to examine the existence of ψ in the four cases $t=t'=1,\alpha,\alpha'\neq\pm2$, $t=1,t'=4,\alpha\neq\pm2,\alpha'=0$, $t=1,t'=5,6,\alpha=\pm2,\alpha'=0$ and finally $t=5,t'=6,\alpha=\alpha'=0$.

Let us consider first the case $t=5,t'=6,\alpha=\alpha'=0$. In this case we have a graded isomorphism

$$\overline{\Psi}\colon k[x_2,x_3]/(x_2x_3,x_2^3,x_3^3)\longrightarrow k[x_2,x_3]/(x_2^2,x_3^3,x_2x_3^2).$$

In the first ring each non-zero element of degree 1 has a non-zero square. In the second we obviously have $x_2^2=0$. Thus $A^{5,0}_{n,3,n+5}\not\cong A^{6,0}_{n,3,n+5}$. In the case $t=1$, $t'=5,6$, $\alpha=\pm2$ the domain of $\overline{\Psi}$ is $k[x_2,x_3]/(x_2x_3,x_2^2\pm x_3^2)$ which is Gorenstein. Hence it cannot be isomorphic to either $k[x_2,x_3]/(x_2x_3,x_2^3,x_3^3)$ or $k[x_2,x_3]/(x_2^2,x_3^3,x_2x_3^2)$ which are not Gorenstein.

Now we examine the other cases $t=t'=1,\alpha,\alpha'\neq\pm2$ and $t=1,t'=4,\alpha\neq\pm2,\alpha'=0$. In this cases $\overline{\Psi}$ can be identified with a graded isomorphism $A^{1,\alpha}_{3,3,8}\to A^{1,\alpha'}_{3,3,8}$. We thus conclude that it suffices to prove the statement in the particular case $n=3$.

Let $A^{1,\alpha}_{3,3,8}\cong k[x_1,x_2,x_3]/I_{1,\alpha}$ and $A^{1,\alpha'}_{3,3,8}\cong k[x_1,x_2,x_3]/I_{1,\alpha'}$ be the standard representations defined above. Then ψ finally induces an automorphism ψ_0 of $k[x_1,x_2,x_3]$ such that $\psi_0(I_{1,\alpha})=I_{1,\alpha'}$, thus the corresponding nets of conics, which are generated by the generators of degree 2 of the ideals $I_{1,\alpha}$ and $I_{1,\alpha'}$, must be projectively equivalent, hence they have isomorphic discriminant curves. In particular the case $t=1,t'=4,\alpha\neq\pm2,\alpha'=0$ does not occur since the discriminant of the net associated to the domain is integral, while the one associated with the codomain splits as three non-concurrent lines.

Thus we have finally to examine the case $t=t'=1,\alpha,\alpha'\neq\pm2$. Since the discriminant curves associated with the domain and with the codomain of $\overline{\Psi}$ should be isomorphic, they should have the same j-invariant.

The net is generated by the three forms

$$x_1x_2+x_3^2,\qquad x_1x_3,\qquad x_2^2-4px_3^2+x_1^2+2px_1x_2,$$

(here we modified the last generator of the net in order to obtain a cubic in Weierstrass form as discriminant of the net, setting $\alpha=6p$, $p\in k$) thus its discriminant curve Δ has equation

$$\lambda_1^2\lambda_2=(\lambda_0^2+4p\lambda_0\lambda_2+4(p^2-1)\lambda_2^2)(4p\lambda_2-\lambda_0),$$

which is singular if and only if $p=\pm1/3$ and, in this case, it carries a node.

In all the remaining cases its j-invariant is

$$j(\Delta) = -\frac{27p^2(1-p^2)^2}{(1-9p^2)^2}.$$

We recall that, once we fix the discriminant curve Δ of the net, there are exactly three nets of conics with discriminant curve Δ (e.g. see [31], Chapter VI). They correspond to the three non-trivial theta-characteristics on Δ. In our case there are exactly six possible values of p corresponding to the same j-invariant for Δ. Notice that the transformation $(x_1, x_2, x_3) \mapsto (x_1, -x_2, ix_3)$ ($i^2 = -1$) allows us to identify the two cases $\pm p$. Thus we have exactly three possible values of p giving rise to possible non-isomorphic nets of conics for a fixed j-invariant. We conclude that such values actually correspond to non-isomorphic nets of conics, thus to non-isomorphic k-algebras. □

We have now to prove Claims 4.1.2 and 4.1.3.

Proof of Claim 4.1.2. The classification of nets of conics when $\mathrm{char}(k) = 0$ can be found in [21]. The hard part of that paper is the classification in the real case while in the complex case the arguments are, at least conceptually, easy. We will show, step by step, that such a classification still holds over each algebraically closed field k with $\mathrm{char}(k) \neq 2, 3$. So, from now on we will assume such a restriction on the base field k.

Let $\mathcal{N}$ be a net of conics in the projective plane $\mathbb{P}(V)$ with coordinates x_0, x_1, x_2 and let Δ be its discriminant in the plane $\mathbb{P}^2_k$ with coordinates $\lambda_0, \lambda_1, \lambda_2$. The discriminant Δ is either a cubic curve or it is identically zero.

We first assume that Δ is integral: following [21] we call this case ABB^*C. Since $\mathrm{char}(k) \neq 2$, up to a suitable projective transformation in $\mathbb{P}^2_k$, we can assume that Δ is defined by $\lambda_1^2\lambda_2 = f(\lambda_0, \lambda_2)$ where f is a suitable binary form of degree 3 (see Proposition IV.4.6 of [25]). Since $\mathrm{char}(k) \neq 3$ we can finally reduce the equation of Δ in the Weierstrass form $\lambda_1^2\lambda_2 = \lambda_0^3 + u\lambda_0\lambda_2^2 + v\lambda_2^3$ via an easy Tschirnhaus transformation, where $u, v \in k$. We recall that Δ has a flex M at the point $[0, 1, 0]$ with inflectional tangent line of equation $\lambda_2 = 0$. It follows that the pencil of conics $\mathcal{N}_2$ corresponding to $\lambda_2 = 0$, has discriminant Δ_2 of equation $\lambda_0^3 = 0$.

We recall in Table 3 the complete classification of pencils of conics due to B. Segre which is valid on each algebraically closed field k with $\mathrm{char}(k) \neq 2$.

Table 3

Segre symbol	Generators	Discriminant
[1 1 1]	$x_1^2+x_2^2, x_2^2+x_3^2$	$\lambda_0\lambda_1(\lambda_0+\lambda_1)$
[2 1]	$2x_1x_2, x_2^2+x_3^2$	$-\lambda_0^2\lambda_1$
[(1 1) 1]	$x_1^2+x_2^2, x_3^2$	$\lambda_0^2\lambda_1$
[3]	$x_2^2+2x_1x_3, 2x_2x_3$	$-\lambda_0^3$
[(2 1)]	$2x_1x_2+x_3^2, x_2^2$	$-\lambda_0^3$
[; 1 ;]	$2x_1x_2, 2x_2x_3$	0
[1 1 ; 1]	x_1^2, x_2^2	0
[2 ; ; 1]	$2x_1x_2, x_2^2$	0

Thus $\mathcal{N}_2$ is of type either [3] or [(2 1)]. Since Δ is smooth at M, such a flex corresponds to a conic of rank 2, hence $\mathcal{N}_2$ is of type [3]. In particular, up to a suitable projectivity in $\mathbb{P}(V)$ we can assume that two generators of $\mathcal{N}$ are $x_2^2 + 2x_1x_3, 2x_2x_3$ and the third one is $s(x_1, x_2, x_3) := s_{1,1}x_1^2 + s_{2,2}x_2^2 + s_{3,3}x_3^2 + 2s_{1,2}x_1x_2 + 2s_{1,3}x_1x_3 + 2s_{2,2}x_2x_3$. Computing the equation of Δ from this three generators one finally obtains that $s(x_1, x_2, x_3) = x_1^2 + 2s_{1,3}x_2^2 - s_{3,3}x_3^2 - 2s_{1,3}x_1x_3$.

Now we turn our attention to the case of nets with non-integral discriminant. Thus the polynomial defining Δ necessarily contains a linear factor, say λ_2. This factor corresponds to a pencil in $\mathcal{N}$ consisting entirely of degenerate conics, thus its discriminant must be identically zero. Examining Table 3 such a pencil is of type either [; 1 ;] or [1 1 ; 1] or [2 ; ; 1].

In the first case we can choose as third generator $s(x_1, x_2, x_3) := s_{1,1}x_1^2 + s_{2,2}x_2^2 + s_{3,3}x_3^2 + 2s_{1,3}x_1x_3$, where $s_{i,j} \in k$. Since $\mathrm{char}(k) \neq 2$, a suitable linear transformation on x_1 and x_3 allows us to restrict to one of the following cases for the third generator: $x_1^2 + x_2^2, x_1^2, 2x_1x_3 + x_2^2, x_1x_3, x_2^2$ (following [21] we denote such cases with the symbols F, G^*, D^*, E^*, I^*).

In the second case we can choose as third generator $s(x_1, x_2, x_3) := s_{3,3}x_3^2 + 2s_{1,2}x_1x_2 + 2s_{1,3}x_1x_3 + s_{2,3}x_2x_3$, where $s_{i,j} \in k$. If $s_{3,3} \neq 0$ we can complete the square obtaining either $x_3^2 + 2x_1x_2$ (case D) or x_3^2 (case E^*): notice that we only need $\mathrm{char}(k) \neq 2$. If $s_{3,3} = 0$ but at least one between $s_{1,3}$ and $s_{2,3}$ is non-zero via $x_3 \mapsto x_3 + \ell(x_1, x_2)$ (here ℓ is a suitable binary linear form) we obtain $s_{1,2} = 0$. Scaling the coordinates we finally obtain as third generator either x_1x_3 (case G) or $x_1x_3 + x_2x_3$: in this last case via $(x_1, x_2) \mapsto (x_1 + x_2, x_1 - x_2)$ (recall that $\mathrm{char}(k) \neq 2$) we finally obtain the three generators $x_1x_3, x_1x_2, x_1^2 + x_2^2$ (again case F).

Finally, in the third case, we can choose as third generator $s(x_1, x_2, x_3) := s_{1,1}x_1^2 + s_{3,3}x_3^2 + 2s_{1,3}x_1x_3 + s_{2,3}x_2x_3$, where $s_{i,j} \in k$. If $s_{3,3} \neq 0$ we can complete the square obtaining either $x_3^2 + x_1^2$ (case F^*) or x_3^2 (again case G). If $s_{3,3} = 0$, via $x_1 \mapsto \ell(x_1, x_2)$ (ℓ is a suitable binary linear form) we obtain as third generator either x_1^2 or $x_1^2 + 2x_2x_3$ or x_2x_3 or x_1x_3 (respectively cases I, H, I^*, G^* of [21])

Via suitable permutations and rescaling of x_1, x_2, x_3 we finally obtain Table 2. □

In order to complete the proof of Theorem 4.1 we have to prove Claim 4.1.3. As in the proof of Claim 3.1.2 we will first find relations among the generators of the maximal ideal and then we will show that such a system of relations is complete. Of course, we will use the classification of nets of conics summarized in Table 2, examining separately the different listed cases.

4.2. The case ABB^*C

Taking into account the results proved in [21], we obtain that Relations (4.1.1) above become $a_1a_2+a_3^2$, a_1a_3, $a_2^2-6pa_3^2+qa_1^2 \in \mathfrak{M}^3$, where $p, q \in k$. In particular

$$a_1^2a_2 = a_1^2a_3 = a_1a_2a_3 = a_1a_3^2 = a_2^2a_3 = a_3^3 = 0,$$
$$a_2a_3^2 = -a_1a_2^2 = qa_1^3, \qquad a_2^3 = -6pqa_1^3,$$

thus $\mathfrak{M}^2 = (a_1^2, a_3^2, a_2a_3)$ and $\mathfrak{M}^3 = (a_1^3)$. Relations (4.1.1) thus become

$$a_1a_2 = -a_3^2 + \beta_{1,2}a_1^3, \qquad a_1a_3 = \beta_{1,3}a_1^3, \qquad a_2^2 = \alpha_{2,2}^1a_1^2 + \alpha_{2,2}^3a_3^2 + \beta_{2,2}a_1^3,$$

where $\alpha_{i,j}^h, \beta_{i,j} \in k$, $\alpha_{2,2}^1 = -q$, $\alpha_{2,2}^3 = 6p$.

In general, we have relations of the form

$$a_ia_j = \alpha_{i,j}^1a_1^2 + \alpha_{i,j}^2a_2a_3 + \alpha_{i,j}^3a_3^2 + \beta_{i,j}a_1^3, \quad i \geq 1, j \geq 4, \tag{4.2.1}$$

where the $\alpha_{i,j}^h, \beta_{i,j} \in k$, $\alpha_{i,j}^h = \alpha_{j,i}^h$, $\beta_{i,j} = \beta_{j,i}$. Via $(a_2, a_3) \mapsto (a_2 + \beta_{1,2}a_1^2, a_3 + \beta_{1,3}a_1^2)$, we can assume $\beta_{1,2} = \beta_{1,3} = 0$.

If $\alpha_{2,2}^1 = 0$ then $a_2a_3 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^3$, a contradiction because A is Gorenstein. Let u be a fourth root of $-\beta_{2,2}a_1 - \alpha_{2,2}^1$. Via $(a_2, a_3) \mapsto (u^2a_2, ua_3)$ then we can assume $\alpha_{2,2}^1 = -1$ and $\beta_{2,2} = 0$, whence

$$a_1a_2 = -a_3^2, \qquad a_1a_3 = 0, \qquad a_2^2 = -a_1^2 + \alpha a_3^2, \tag{4.2.2}$$

where $\alpha := \alpha_{2,2}^3/u^2$. Since $-\beta_{2,2}a_1 - \alpha_{2,2}^1$ is in the subring A_1 generated by 1 and a_1, the same is true for u, hence for u^{-2}. Thus we can write $\alpha = \alpha' + \alpha''a_1$, $\alpha' \in k$ and $\alpha'' \in A_1$. Since $a_1a_3^2 = 0$ we can finally assume that $\alpha = \alpha' \in k$.

We will assume $n \geq 4$ from now on. Via $a_j \mapsto a_j + \alpha_{1,j}^1a_1 + \alpha_{2,j}^2a_3 - \alpha_{3,j}^2a_2 + \beta_{1,j}a_1^2 - \beta_{2,j}a_3^2 - \beta_{3,j}a_2a_3$, we can assume $\alpha_{1,j}^1 = \alpha_{2,j}^2 = \alpha_{3,j}^2 = \beta_{1,j} = \beta_{2,j} = \beta_{3,j} = 0, j \geq 4$.

Since $a_1a_3 = 0$, it follows $\alpha_{1,j}^2a_2a_3^2 = (a_1a_j)a_3 = a_1a_3a_j = (a_3a_j)a_1 = \alpha_{3,j}^1a_1^3$, thus $\alpha_{1,j}^2 = \alpha_{3,j}^1 = 0, j \geq 4$. Since $a_1a_2a_j = -a_3^2a_j = (a_3a_j)a_3 = 0$, we have $\alpha_{1,j}^3a_2a_3^2 = (a_1a_j)a_2 = a_1a_2a_j = (a_2a_j)a_1 = \alpha_{2,j}^1a_1^3$, thus $\alpha_{1,j}^3 = \alpha_{2,j}^1 = 0, j \geq 4$. Moreover $0 = (a_2a_j)a_3 = a_2a_3a_j = (a_3a_j)a_2 = \alpha_{3,j}^3a_2a_3^2$, thus $\alpha_{3,j}^3 = 0, j \geq 4$. Finally $0 = a_2^2a_j = (a_2a_j)a_2 = \alpha_{2,j}^3a_2^2a_3$, thus $\alpha_{2,j}^3 = 0, j \geq 4$. We conclude that $a_1a_j = a_2a_j = a_3a_j = 0, j \geq 4$.

It follows that $0 = (a_1a_i)a_j = (a_ia_j)a_1 = \alpha_{i,j}^1a_1^3$, $0 = (a_2a_i)a_j = (a_ia_j)a_2 = \alpha_{i,j}^3a_2a_3^2$, $0 = (a_3a_i)a_j = (a_ia_j)a_3 = \alpha_{i,j}^2a_2a_3^2$, thus $a_ia_j = \beta_{i,j}a_1^3, i, j \geq 4$.

Let $y := y_0 + \sum_{i=1}^n y_ia_i + y_{n+1}a_1^2 + y_{n+2}a_2a_3 + y_{n+3}a_3^2 + y_{n+4}a_1^3 \in \mathrm{Soc}(A)$, $y_h \in k$. Then the conditions $a_jy = 0, j = 1, \dots, n$, become

$$\begin{cases} y_0a_1 + y_1a_1^2 - y_2a_3^2 + y_{n+1}a_1^3 = 0, \\ y_0a_2 - y_1a_3^2 + y_2\alpha a_3^2 + y_2a_1^{e-1} + y_3a_2a_3 + y_{n+3}a_1^3 = 0, \\ y_0a_3 + y_2a_2a_3 + y_3a_3^2 + y_{n+2}a_1^3 = 0, \\ y_0a_j + \left(\sum_{i=4}^n y_i\beta_{i,j}\right)a_1^3 = 0, \quad j \geq 4. \end{cases}$$

It is clear that $y_0 = y_1 = y_2 = y_3 = y_{n+1} = y_{n+3} = 0$ and $\sum_{i=4}^n y_i\beta_{i,j} = 0, j \geq 4$. If the symmetric matrix $B := \left(\beta_{i,j}\right)_{i,j\geq 4}$ would be singular again $\mathrm{Soc}(A) \neq \mathfrak{M}^e$ and A would not be Gorenstein (see the argument for the analogous assertion in Section 3.2). We conclude that we can make a linear change on $a_4, \dots, a_n$ in such a way that

$$a_ia_j = \delta_{i,j}a_1^3, \quad i, j \geq 4. \tag{4.2.3}$$

Combining Equalities (4.2.1), (4.2.2) and (4.2.3) we obtain as in Section 3.2 that $A \cong A_{n,3,n+5}^{1,\alpha}$.

Note that when $n = 3$ the k-algebra $A_{3,3,8}^{1,\alpha}$ is a complete intersection.

4.3. The cases D and E

Relations (4.1.1) become $a_1^2, a_2^2, a_3^2 + 2pa_1a_2 \in \mathfrak{M}^3$ where $p = 1$ in case D and $p = 0$ in case E. In particular

$$a_1^2a_2 = a_1a_2^2 = a_1^2a_3 = a_1a_3^2 = a_2^2a_3 = a_2a_3^2 = 0, \tag{4.3.1}$$

thus $\mathfrak{M}^2 = (a_1a_2, a_1a_3, a_2a_3)$ and $\mathfrak{M}^3 = (a_1a_2a_3)$ and Relations (4.1.1) become

$$a_1^2 = \beta_{1,1}a_1a_2a_3, \qquad a_2^2 = \beta_{2,2}a_1a_2a_3, \qquad a_3^2 = -2pa_1a_2 + \beta_{2,2}a_1a_2a_3,$$

where $\beta_{i,j} \in k$.

Via $(a_1, a_2, a_3) \mapsto (a_1+\beta_{1,1}a_2a_3/2, a_2+\beta_{2,2}a_1a_3/2, a_3+\beta_{3,3}a_1a_2/2)$ we can assume $\beta_{1,1} = \beta_{2,2} = \beta_{3,3} = 0$. In general, we have relations of the form

$$a_ia_j = \alpha_{i,j}^1a_1a_2 + \alpha_{i,j}^2a_1a_3 + \alpha_{i,j}^3a_2a_3 + \beta_{i,j}a_1a_2a_3, \quad i \geq 1, j \geq 4,$$

where $\alpha_{i,j}^h = \alpha_{j,i}^h \in k$, $\beta_{i,j} = \beta_{j,i} \in k$. Via $a_j \mapsto a_j + \alpha_{1,j}^1a_2 + \alpha_{1,j}^2a_3 + \alpha_{2,j}^1a_1 + \beta_{1,j}a_2a_3 + \beta_{2,j}a_1a_3 + \beta_{3,j}a_1a_2$ we can assume also that $\alpha_{1,j}^1 = \alpha_{1,j}^2 = \alpha_{2,j}^1 = \beta_{1,j} = \beta_{2,j} = \beta_{3,j} = 0, j \geq 4$.

Since $a_1^2 = 0$, we have $0 = (a_1^2)a_j = (a_1a_j)a_1 = \alpha_{1,j}^3a_1a_2a_3$, hence $\alpha_{1,j}^3 = 0, j \geq 4$. Similarly, since $a_2^2 = 0$ we also obtain $\alpha_{2,j}^2 = 0, j \geq 4$. Since $a_1a_j = 0$, we have $0 = (a_1a_j)a_h = (a_ha_j)a_1 = \alpha_{h,j}^3a_1a_2a_3$, $h = 2, 3$, hence $\alpha_{2,j}^3 = \alpha_{3,j}^3 = 0$, $j \geq 4$. Similarly, looking at $a_2a_3a_j$, we also infer $\alpha_{3,j}^2 = 0$. Finally $0 = -2(a_2a_j)a_1 = (-2pa_1a_2)a_j = a_3^2a_j = \alpha_{3,j}^1a_1a_2a_3$, thus $\alpha_{3,j}^1 = 0, j \geq 4$. We conclude that $a_1a_j = a_2a_j = a_3a_j = 0, j \geq 4$.

As in the previous case, it follows that $0 = (a_1a_i)a_j = (a_ia_j)a_1 = \alpha_{i,j}^3a_1a_2a_3$, $0 = (a_2a_i)a_j = (a_ia_j)a_2 = \alpha_{i,j}^2a_1a_2a_3$, $0 = (a_3a_i)a_j = (a_ia_j)a_3 = \alpha_{i,j}^1a_1a_2a_3$, thus $a_ia_j = \beta_{i,j}a_1a_2a_3$, $i, j \geq 4$.

Let $y := y_0 + \sum_{i=1}^n y_ia_i + y_{n+1}a_1a_2 + y_{n+2}a_1a_3 + y_{n+3}a_2a_3 + y_{n+4}a_1a_2a_3 \in \mathrm{Soc}(A)$, $y_h \in k$. Then the conditions $a_jy = 0$, $j = 1, \dots, n$, become

$$\begin{cases} y_0a_1 + y_2a_1a_2 + y_3a_1a_3 + y_{n+3}a_1a_2a_3 = 0, \\ y_0a_2 + y_1a_1a_2 + y_3a_2a_3 + y_{n+2}a_1a_2a_3 = 0, \\ y_0a_3 + y_1a_1a_3 + y_2a_2a_3 - y_3pa_1a_2 + y_{n+1}a_1a_2a_3 = 0, \\ y_0a_j + \left(\displaystyle\sum_{i=4}^n y_i\beta_{i,j}\right) a_1a_2a_3 = 0, \quad j \geq 4. \end{cases}$$

It is clear that $y_0 = y_1 = y_2 = y_3 = y_{n+3} = 0$ and $\sum_{i=4}^n y_i\beta_{i,j} = 0, j \geq 4$. Again the standard argument and Equality (4.3.1) yield $A \cong A_{n,3,n+5}^{3-p,0}$.

Also in these cases, when $n = 3$, the k-algebras $A_{3,3,8}^{3-p,0}$, $p = 1, 2$, are complete intersections.

4.4. The case E^*

In this case our Relations (4.1.1) become $a_1a_3, a_2a_3, a_1a_2 \in \mathfrak{M}^3$. In particular $\mathfrak{M}^2 = (a_1^2, a_2^2, a_3^2)$ and we have

$$a_1^2a_3 = a_1a_2a_3 = a_1a_3^2 = a_2^2a_3 = a_2a_3^2 = a_1^2a_2 = a_1a_2^2 = 0,$$

thus we can always assume $\mathfrak{M}^3 = (a_1^3)$, whence

$$a_1a_3 = \beta_{1,3}a_1^3, \qquad a_2a_3 = \beta_{2,3}a_1^3, \qquad a_1a_2 = \beta_{1,2}a_1^3,$$

where $\beta_{i,j} \in k$.

In general, we have relations of the form

$$a_ia_j = \alpha_{i,j}^1a_1^2 + \alpha_{i,j}^2a_2^2 + \alpha_{i,j}^3a_3^2 + \beta_{i,j}a_1^3, \quad i \geq 1, j \geq 4,$$

where $\alpha_{i,j}^h = \alpha_{j,i}^h \in k$, $\beta_{i,j} = \beta_{j,i} \in k$.

Let $a_h^3 = \mu_ha_1^3$ for some $\mu_h \in k$, $h = 2, 3$. If $\mu_2 = 0$, then $a_2^2 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^e$ and similarly if $\mu_3 = 0$: of course, them both are contradictions because A is Gorenstein. It follows that we can always assume $\mu_2 = \mu_3 = 1$. Thus via $a_3 \mapsto a_3 + \beta_{2,3}a_2^2$ we also have $\beta_{2,3} = 0$. Hence we have

$$a_1a_2 = 0, \qquad a_1a_3 = 0, \qquad a_2a_3 = 0, \qquad a_2^3 = a_1^3, \qquad a_3^3 = a_1^3. \tag{4.4.1}$$

Since $0 = (a_1a_h)a_j = (a_ha_j)a_1 = \alpha_{h,j}^1a_1^3$, $h = 2, 3$, we obtain $\alpha_{2,j}^1 = \alpha_{3,j}^1 = 0$. Thus via $a_j \mapsto a_j + \alpha_{1,j}^1a_1 + \alpha_{2,j}^2a_2 + \alpha_{3,j}^3a_3 + \beta_{1,j}a_1^2 + \beta_{2,j}a_2^2 + \beta_{3,j}a_3^2$ we can assume $\alpha_{1,j}^1 = \alpha_{2,j}^2 = \alpha_{3,j}^3 = \beta_{1,j} = \beta_{2,j} = \beta_{3,j} = 0$.

Since $0=(a_1a_h)a_j=(a_1a_j)a_h=\alpha^h_{1,j}a_1^3$, we have $\alpha^h_{1,j}=0$, $h=2,3$. Similarly, since $a_2a_3a_j=0$, one also obtains $\alpha^3_{2,j}=\alpha^2_{3,j}=0$, thus $a_ia_j=0$, $i=1,2,3$, $j\geq 4$. It follows that $0=(a_1a_i)a_j=(a_ia_j)a_1=\alpha^1_{i,j}a_1^3$, $0=(a_ha_i)a_j=(a_ia_j)a_h=\alpha^h_{i,j}a_1^3$, $h=2,3$, thus $a_ia_j=\beta_{i,j}a_1^3$, $i,j\geq 4$.

Let $y:=y_0+\sum_{i=1}^n y_ia_i+y_{n+1}a_1^2+y_{n+2}a_2^2+y_{n+3}a_3^2+y_{n+4}a_1^3\in \mathrm{Soc}(A)$, $y_h\in k$. Then the conditions $a_jy=0$, $j=1,\dots,n$, become

$$\begin{cases} y_0a_1+y_1a_1^2+y_{n+1}a_1^3=0,\\ y_0a_2+y_2a_2^2+y_{n+2}a_1^3=0,\\ y_0a_3+y_3a_3^2+y_{n+3}a_1^3=0,\\ y_0a_j+\left(\sum_{i=4}^n y_i\beta_{i,j}\right)a_1^3=0, \quad j\geq 4.\end{cases}$$

By using the same argument as in the previous cases and Equality (4.4.1) we deduce that $A\cong A^{4,0}_{n,3,n+5}$.

When $n=3$, the ideal defining $A^{4,0}_{3,3,8}$ in $[x_1,x_2,x_3]$ is minimally generated by the 4×4 pfaffians of a suitable 5×5 skew-symmetric matrix M. E.g. one may take

$$M:=\begin{pmatrix} 0 & 0 & 0 & x_1 & x_2\\ 0 & 0 & x_3 & -x_1 & 0\\ 0 & -x_3 & 0 & x_2^2 & x_1^2\\ -x_1 & x_1 & -x_2^2 & 0 & -x_3^2\\ -x_2 & 0 & -x_1^2 & x_3^2 & 0\end{pmatrix}.$$

4.5. The case G^*

In this case our Relations (4.1.1) become $a_1^2, a_1a_2, a_2a_3\in\mathfrak{M}^3$. In particular $\mathfrak{M}^2=(a_3^2,a_2^2,a_1a_3)$ and we have

$$a_1^3=a_1^2a_2=a_1^2a_3=a_1a_2a_3=a_2^2a_3=a_2a_3^2=0.$$

We also have relations

$$a_ia_j=\alpha^1_{i,j}a_3^2+\alpha^2_{i,j}a_2^2+\alpha^3_{i,j}a_1a_3+\gamma_{i,j},\quad i\geq 1, j\geq 4,$$

where $\alpha^h_{i,j}=\alpha^h_{j,i}\in k$, $\gamma_{i,j}\in\mathfrak{M}^3$.

We can always assume $\mathfrak{M}^3=(a_3^3)$. Indeed if $a_1a_3^2=a_3^3=0$ then $a_3^2\in \mathrm{Soc}(A)\setminus\mathfrak{M}^3$ that is not the case because A is Gorenstein. If $a_1a_3^2\neq 0$ but $a_3^3=0$, then we can make the transformation $a_3\mapsto a_3+\lambda a_1$. Thus $\gamma_{i,j}=\beta_{i,j}a_3^3$, where $\beta_{i,j}\in k$. In particular

$$a_1^2=\beta_{1,1}a_3^3,\qquad a_1a_2=\beta_{1,2}a_3^3,\qquad a_2a_3=\beta_{2,3}a_3^3.$$

Let $a_2^3=\mu_1a_3^3$, $a_1a_3^2=\mu_2a_3^3$. If $\mu_1=0$, then $a_2^2\in\mathrm{Soc}(A)\setminus\mathfrak{M}^3$, while if $\mu_2=0$, then $a_1a_3\in\mathrm{Soc}(A)\setminus\mathfrak{M}^3$. Both cases are not allowed because A is Gorenstein. Thus we can assume $\mu_1=\mu_2=1$. It follows that, via $(a_1,a_2)\mapsto(a_1+\beta_{1,1}a_3^2/2+\beta_{1,2}a_2^2/2, a_2+\beta_{2,3}a_3^2)$, we obtain $\beta_{1,1}=\beta_{1,2}=\beta_{2,3}=0$.

Via $a_j\mapsto a_j+\alpha^3_{1,j}a_3+\beta_{1,j}a_2^2+\alpha^2_{2,j}a_2+\beta_{2,j}a_2^2+\alpha^3_{3,j}a_1+(\beta_{3,j}-\beta_{1,j})a_1a_3$, we obtain $\alpha^3_{1,j}=\alpha^2_{2,j}=\alpha^3_{3,j}=\beta_{1,j}=\beta_{2,j}=\beta_{3,j}=0$. Since $0=(a_1a_h)a_j=(a_1a_j)a_h=\alpha^h_{1,j}a_3^3$, it follows $\alpha^h_{1,j}=0$, $h=1,2$. Similarly $0=(a_1a_2)a_j=(a_2a_j)a_1=\alpha^1_{2,j}a_3^3$, then $\alpha^1_{2,j}=0$. Since $a_2a_3a_j=0$, one also obtains $\alpha^3_{2,j}=\alpha^2_{3,j}=0$. Finally $\alpha^3_{1,j}a_3^3=(a_1a_j)a_3=(a_3a_j)a_1=\alpha^1_{3j}a_3^3$, whence $\alpha^1_{3,j}=\alpha^3_{1,j}=0$. Thus $a_ha_i=0$, $h=1,2,3$, $i\geq 4$. It follows that $0=(a_ha_i)a_j=(a_ia_j)a_h=\alpha^h_{i,j}a_3^3$, whence $a_ia_j=\beta_{i,j}a_3^3$, $i,j\geq 4$.

Let $y:=y_0+\sum_{i=1}^n y_ia_i+y_{n+1}a_3^2+y_{n+2}a_2^2+y_{n+3}a_1a_3+y_{n+4}a_3^3\in\mathrm{Soc}(A)$, $y_h\in k$. Then the conditions $a_jy=0$, $j=1,\dots,n$, become

$$\begin{cases} y_0a_1+y_3a_1a_3+y_{n+1}a_3^3=0,\\ y_0a_2+y_2a_2^2+y_{n+2}a_3^3=0,\\ y_0a_3+y_1a_1a_3+y_3a_3^2+y_{n+1}a_3^3+y_{n+3}a_3^3=0,\\ y_0a_j+\left(\sum_{i=4}^n y_i\beta_{i,j}\right)a_3^3=0, \quad j\geq 4.\end{cases}$$

and we deduce as in the previous cases that $A\cong A^{5,0}_{n,3,n+5}$.

When $n = 3$, the ideal defining $A^{5,0}_{n,3,8}$ is generated by the 4×4 pfaffians of

$$M := \begin{pmatrix} 0 & 0 & x_2 & -x_3 & x_1 \\ 0 & 0 & -x_2 & x_1 & 0 \\ -x_2 & x_2 & 0 & x_3^2 & -x_3^2 \\ x_3 & -x_1 & -x_3^2 & 0 & x_2^2 \\ -x_1 & 0 & x_3^2 & -x_2^2 & 0 \end{pmatrix}.$$

4.6. The case H

In this case our Relations (4.1.1) become $a_1^2, a_1a_2, 2a_1a_3 + a_2^2 \in \mathfrak{M}^3$. In particular $\mathfrak{M}^2 = (a_2^2, a_3^2, a_2a_3)$ and we have

$$a_1^3 = a_1^2a_2 = a_1^2a_3 = a_1a_2^2 = a_1a_2a_3 = a_2^3 = 0.$$

Moreover $a_2^2a_3 = -2a_1a_3^2$ and

$$a_ia_j = \alpha^1_{i,j}a_2^2 + \alpha^2_{i,j}a_3^2 + \alpha^3_{i,j}a_2a_3 + \gamma_{i,j}, \quad i \geq 1, j \geq 4,$$

where $\alpha^h_{i,j} = \alpha^h_{j,i} \in k$ and $\gamma_{i,j} \in \mathfrak{M}^3$.

If $a_1a_3^2 = 0$, then $a_1a_3 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^3$ and this contradicts the hypothesis that A is Gorenstein. Thus we can assume $\mathfrak{M}^3 = (a_1a_3^2)$, whence $\gamma_{i,j} = \beta_{i,j}a_1a_3^2$ and

$$a_1^2 = \beta_{1,1}a_1a_3^2, \qquad a_1a_2 = \beta_{1,2}a_1a_3^2, \qquad a_1a_3 = -a_2^2/2 + \beta_{1,3}a_1a_3^2, \tag{4.6.1}$$

where $\beta_{i,j} \in k$. Via $(a_1, a_2, a_3) \mapsto (a_1 + \beta_{1,1}a_3^2/2, a_2 + \beta_{1,2}a_3^2, a_3 + \beta_{1,3}a_3^2)$, we obtain $\beta_{1,1} = \beta_{1,2} = \beta_{1,3} = 0$ in Relations (4.6.1).

Let $a_2a_3^2 = \mu_1a_1a_3^2$, $a_3^3 = \mu_2a_1a_3^2$. The transformation $(a_2, a_3) \mapsto (a_2 + \mu_1a_1, a_3 + \mu_2a_1/3)$ does not affect Relations (4.6.1), thus we can assume $\mu_1 = \mu_2 = 0$. Finally, via $a_j \mapsto a_j + \alpha^3_{2,j}a_3 - 2\alpha^1_{3,j}a_1 + \alpha^3_{3,j}a_2 + \beta_{1,j}a_3^2 - \beta_{2,j}a_2a_3/2 + \beta_{3,j}a_1a_3$, we can assume $\alpha^3_{2,j} = \alpha^1_{3,j} = \alpha^3_{3,j} = \beta_{1,j} = \beta_{2,j} = \beta_{3,j} = 0$.

Since $0 = (a_1a_h)a_j = (a_ha_j)a_1 = \alpha^2_{h,j}a_1a_3^2$, $h = 1, 2$, we obtain $\alpha^2_{1,j} = \alpha^2_{2,j} = 0$. Since $0 = (a_1a_2)a_j = (a_1a_j)a_2 = -2\alpha^3_{1,j}a_1a_3^2$, it follows $\alpha^3_{1,j} = 0$. Since $0 = (a_2a_3)a_j = (a_3a_j)a_2 = -2\alpha^1_{2,j}a_1a_3^2$, we obtain $\alpha^1_{2,j} = 0$, thus $a_2a_j = 0$ whence $(a_1a_3)a_j = -2a_2^2a_j = -2(a_2a_j)a_2 = 0$. It follows both that $0 = (a_1a_3)a_j = (a_1a_j)a_3 = -\alpha^1_{1,j}a_1a_3^2/2$ and $0 = (a_1a_3)a_j = (a_3a_j)a_1 = \alpha^2_{3,j}a_1a_3^2$: in particular $\alpha^1_{1,j} = \alpha^2_{3,j} = 0$ and we finally obtain $a_ia_j = 0$, $i = 1, 2, 3, j \geq 4$.

It follows that $0 = (a_1a_i)a_j = (a_ia_j)a_1 = \alpha^2_{i,j}a_1a_3^2$, $0 = (a_2a_i)a_j = (a_ia_j)a_2 = -2\alpha^3_{i,j}a_1a_3^2$, $0 = (a_3a_i)a_j = (a_ia_j)a_3 = -2\alpha^1_{i,j}a_1a_3^2$, thus $a_ia_j = \beta_{i,j}a_1a_3^2$, $i, j \geq 4$.

Again let $y := y_0 + \sum_{i=1}^n y_ia_i + y_{n+1}a_2^2 + y_{n+2}a_3^2 + y_{n+3}a_2a_3 + y_{n+4}a_1a_3^2 \in \mathrm{Soc}(A)$, $y_h \in k$. Then the conditions $a_jy = 0$, $j = 1, \ldots, n$, become

$$\begin{cases} y_0a_1 - y_3a_2^2/2 + y_{n+2}a_1a_3^2 = 0, \\ y_0a_2 + y_2a_2^2 + y_3a_2a_3 - 2y_{n+3}a_1a_3^2 = 0, \\ y_0a_3 - 2y_1a_2^2/2 + y_2a_2a_3 + y_3a_3^2 - 2y_{n+1}a_1a_3^2 = 0, \\ y_0a_j + \left(\displaystyle\sum_{i=4}^n y_i\beta_{i,j}\right) a_1a_3^2 = 0, \quad j \geq 4 \end{cases}$$

and we deduce as in the previous cases that $A \cong A^{6,0}_{n,3,n+5}$.

When $n = 3$, the ideal defining $A^{6,0}_{n,3,8}$ is generated by the 4×4 pfaffians of

$$M := \begin{pmatrix} 0 & 0 & -2x_3 & x_1 & -x_2 \\ 0 & 0 & -x_2 & 0 & x_1 \\ 2x_3 & x_2 & 0 & 0 & 0 \\ -x_1 & 0 & 0 & 0 & x_3^2 \\ x_2 & -x_1 & 0 & -x_3^2 & 0 \end{pmatrix}.$$

4.7. The cases D^*, F, F^*, G, I, I^*

The cases corresponding to the nets D^* and F cannot occur. Indeed in these cases our Relations (4.1.1) become $a_1a_3, a_2a_3, a_3^2 + 2pa_1a_2 + qa_1^2 \in \mathfrak{M}^3$ where $(p, q) = (1, 0), (0, 1)$ in cases D^* and F respectively. In particular $\mathfrak{M}^2 = (a_1^2, a_2^2, a_1a_2)$. Thus

$$a_1^2a_3 = a_1a_3^2 = a_2^2a_3 = a_2a_3^2 = a_3^3 = 0.$$

Since also

$$a_3a_j = \alpha^1_{3,j}a_1^2 + \alpha^2_{3,j}a_2^2 + \alpha^3_{3,j}a_1a_2 + \gamma_{3,j}, \quad j \geq 4,$$

where $\gamma_{3,j} \in \mathfrak{M}^3$, $\alpha^h_{3,j} \in k$, we conclude that $a_3^2 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^3$, and so A would not be Gorenstein.

The case corresponding to the net F^* cannot occur. Indeed in this case our Relations (4.1.1) become $a_1^2, a_1a_2, a_2^2+a_3^2 \in \mathfrak{M}^3$. In particular $\mathfrak{M}^2 = (a_2^2, a_2a_3, a_1a_3)$. Thus

$$a_1^2a_3 = a_1a_2^2 = a_1a_2a_3 = a_1a_2^2 + a_1a_3^2 = 0.$$

Since also

$$a_3a_j = \alpha^1_{3,j}a_2^2 + \alpha^2_{3,j}a_2a_3 + \alpha^3_{3,j}a_1a_3 + \gamma_{3,j}, \quad j \geq 4,$$

where $\gamma_{3,j} \in \mathfrak{M}^3$, $\alpha^h_{3,j} \in k$, we conclude that $a_1a_3 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^3$, a contradiction since A is Gorenstein.

The case corresponding to the net G cannot occur. Indeed in this case Relations (4.1.1) become $a_1^2, a_2^2, a_2a_3 \in \mathfrak{M}^3$. In particular $\mathfrak{M}^2 = (a_3^2, a_1a_3, a_1a_2)$. Thus

$$a_1^2a_2 = a_1a_2^2 = a_1a_2a_3 = a_2a_3^2 = 0.$$

Since we have

$$a_1a_j = \alpha^1_{1,j}a_3^2 + \alpha^2_{1,j}a_1a_3 + \alpha^3_{1,j}a_1a_2 + \gamma_{1,j}, \quad j \geq 4,$$

where $\gamma_{1,j} \in \mathfrak{M}^3$, $\alpha^h_{1,j} \in k$, we infer $a_1a_2 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^3$, a contradiction since A is Gorenstein.

The case corresponding to the net I cannot occur. Indeed in this case Relations (4.1.1) become $a_1^2, a_1a_2, a_2^2 \in \mathfrak{M}^3$. In particular $\mathfrak{M}^2 = (a_3^2, a_1a_3, a_2a_3)$. Thus

$$a_1^3 = a_1^2a_2 = a_1^2a_3 = a_1a_2^2 = a_1a_2a_3 = a_2^3 = a_2^2a_3 = 0.$$

Changing possibly a_3 in $a_3 + ua_2 + va_1$ we can always assume that $\mathfrak{M}^3 = (a_3^3)$. Let $a_ha_3^2 = \lambda_h a_3^3$, $h = 1, 2$. We have

$$a_3a_j = \alpha^1_{3,j}a_1a_3 + \alpha^2_{3,j}a_2a_3 + \alpha^3_{3,j}a_3^2 + \beta_{3,j}a_3^3, \quad j \geq 4.$$

where $\beta_{3,j}, \alpha^h_{3,j} \in k$. Via $a_j \mapsto a_j + \alpha^1_{3,j}a_1 + \alpha^2_{3,j}a_2 + \alpha^3_{3,j}a_3 + \beta_{3,j}a_3^2$, we can also assume $a_3a_j = 0$. Thus if $(\lambda_1, \lambda_2) \neq (0, 0)$, then $\lambda_1a_2a_3 - \lambda_2a_1a_3 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^3$. If $\lambda_1 = \lambda_2 = 0$, then $a_1a_3 \in \mathrm{Soc}(A) \setminus \mathfrak{M}^3$. In both cases A would not be a Gorenstein k-algebra.

We conclude by proving that also the last case I^*, corresponding to a net of conics with a fixed line, can be either reduced to one of the cases described above or it does not occur. In this case Relations (4.1.1) become $a_3^2, a_1a_3, a_2a_3 \in \mathfrak{M}^3$. In particular $\mathfrak{M}^2 = (a_1^2, a_1a_2, a_2^2)$. Moreover via $a_1 \mapsto a_1 + \lambda a_2$ we can always assume $a_1^3 \neq 0$ so that $\mathfrak{M}^3 = (a_1^3)$ and $a_1^2a_2 = \mu_1a_1^3$, $a_1a_2^2 = \mu_2a_1^3$, $a_2^3 = \mu_3a_1^3$. As usual, for each $j \geq 3$, we write

$$a_ia_j = \alpha^1_{i,j}a_1^2 + \alpha^2_{i,j}a_1a_2 + \alpha^3_{i,j}a_2^2 + \beta_{i,j}a_1^3, \quad i \geq 1, j \geq 3,$$

where $\beta_{i,j}, \alpha^h_{i,j} \in k$, $\beta_{i,j} = \beta_{j,i} \in k$, $\alpha^h_{i,j} = \alpha^h_{j,i}$. Via $a_j \mapsto a_j + \alpha^1_{1,j}a_1 + \alpha^2_{1,j}a_2 + \beta_{1,j}a_1^2$ we can assume $\alpha^1_{1,j} = \alpha^2_{1,j} = \beta_{1,j} = 0$.

Since $\mathfrak{M}^2 \subseteq (a_1, a_2, a_j)^2$, $j \geq 3$, we have a net of conics $\mathcal{N}_j$ corresponding to the relations above for a_1a_j, a_2a_j, a_j^2, in the projective plane $\mathbb{P}(V_j)$, associated with the subspace $V_j \subseteq \mathfrak{M}/\mathfrak{M}^2$ generated by the classes of a_1, a_2, a_j. If we find $j \geq 3$ such that $\mathcal{N}_j$ is not of type I^*, then we are in one of the cases examined above, so we have only to discuss the case when all the nets $\mathcal{N}_j$ are of type I^* i.e. each $\mathcal{N}_j$ has a fixed component.

Since we have modified a_j in order to have $a_1a_j = \alpha^3_{1,j}a_2^2$, the degeneracy condition yields $\alpha^3_{1,j} = 0$. In particular the fixed component of $\mathcal{N}_j$ corresponds to either a_1 or a_j. Looking at the relations for a_2a_j one then deduces that the fixed component corresponds to a_j, whence $\alpha^h_{2,j} = 0$, $h = 1, 2, 3$. Thus, for each $j \geq 3$, we have $a_2a_j \in \mathfrak{M}^3$.

Let $y := y_1a_1^2 + y_2a_1a_2 + y_3a_2^2$, $y_i \in k$. Since $ya_j = 0$, $j \geq 3$, due to the above discussion, it follows that $y \in \mathrm{Soc}(A)$ if and only if $ya_1 = ya_2 = 0$. Such conditions are equivalent to say that (y_1, y_2, y_3) is a solution of the system

$$\begin{cases} y_1 + \mu_1y_2 + \mu_2y_3 = 0 \\ \mu_1y_1 + \mu_2y_2 + \mu_3y_3 = 0. \end{cases}$$

Since the above system always has non-trivial solutions, we deduce that $\mathrm{Soc}(A) \neq \mathfrak{M}^3$, a contradiction because A is assumed to be Gorenstein.

We conclude the section with the following result about the smoothability of schemes corresponding to the above algebras.

Proposition 4.8. *Let $X := \mathrm{spec}(A^{t,\alpha}_{n,3,n+5}) \subseteq \mathbb{P}^N_k$, $N \geq n$. Then X is smoothable in $\mathbb{P}^N_k$.*

Proof. As in the previous cases (see Propositions 2.10 and 3.4) it suffices to check that $\mathrm{spec}(A^{t,\alpha}_{n,3,n+5})$ is smoothable in $\mathbb{A}^n_k$ and again we look for the deformations of the k-algebras $A^{t,\alpha}_{n,3,n+5}$. We define the following ideals in $k[b, x_1, \dots, x_n]$:

$$J^{1,\alpha}_n := (x_1x_2 + x_3^2, x_1x_3, x_2^2 - \alpha x_3^2 + x_1^2, x_ix_j, x_h^2 - x_1^3, x_n^2 - bx_n - x_1^3)_{\substack{1\leq i<j\leq n, 4\leq j\\ 4\leq h\leq n-1}},$$
$$J^{2,0}_n := (x_1^2, x_2^2, x_3^2 + 2x_1x_2, x_ix_j, x_h^2 - x_1x_2x_3, x_n^2 - bx_n - x_1x_2x_3)_{\substack{1\leq i<j\leq n, 4\leq j\\ 4\leq h\leq n-1}},$$
$$J^{3,0}_n := (x_1^2, x_2^2, x_3^2, x_ix_j, x_h^2 - x_1x_2x_3, x_n^2 - bx_n - x_1x_2x_3)_{\substack{1\leq i<j\leq n, 4\leq j\\ 4\leq h\leq n-1}},$$
$$J^{4,0}_n := (x_2^3 - x_1^3, x_3^3 - x_1^3, x_ix_j, x_h^2 - x_1^3, x_n^2 - bx_n - x_1^3)_{\substack{1\leq i<j\leq n\\ 4\leq h\leq n}},$$
$$J^{5,0}_n := (x_1^2, x_1x_2, x_2x_3, x_2^3 - x_3^3, x_1x_3^2 - x_3^3, x_ix_j, x_h^2 - x_3^3, x_n^2 - bx_n - x_3^3)_{\substack{1\leq i<j\leq n, 4\leq j\\ 4\leq h\leq n-1}},$$
$$J^{6,0}_n := (x_1^2, x_1x_2, 2x_1x_3 + x_2^2, x_3^3, x_2x_3^2, x_ix_j, x_h^2 - x_1x_3^2, x_n^2 - bx_n - x_1x_3^2)_{\substack{1\leq i<j\leq n, 4\leq j\\ 4\leq h\leq n-1}}.$$

In $k[b, x_1, \dots, x_n]$ we have

$$J^{t,\alpha}_n = (x_1, \dots, x_{n-1}, x_n - b) \cap (J^{t,\alpha}_n + (x_n^2))$$

(the proof of this fact is easy: see Proposition 2.10 for the argument). Note that in $J^{t,\alpha} + (x_n^2)$ there always is a polynomial which is

$$bx_n + \text{cubic polynomial in } x_1, \dots, x_{n-1}.$$

With this in mind it is easy to check, case by case, that for a fixed $b \neq 0$ in k, we have

$$k[x_1, \dots, x_n]/(J^{t,\alpha}_n + (x_n^2)) \cong k[x_1, \dots, x_{n-1}]/J^{t,\alpha}_{n-1} \cong A^{t,\alpha}_{n-1,3,n+2}.$$

We thus conclude that the family $\mathcal{A}^{t,\alpha} := k[b, x_1, \dots, x_n]/J^{t,\alpha}_n \to \mathbb{A}^1_k$ is flat, it has special fiber over $b = 0$ isomorphic to $A^{t,\alpha}_{n,3,n+5}$ and general fiber isomorphic to $A^{t,\alpha}_{n-1,3,n+4} \oplus A_{0,1}$. □

5. The locus $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ for $d \leq 9$

Taking into account the results of the previous sections, it is now possible to study the irreducibility of $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ and its singular locus when $d \leq 9$. The first result is the following

Proposition 5.1. *Let $\mathrm{char}(k) \neq 2, 3$. If $d \leq 9$, then $\mathcal{H}ilb^G_d(\mathbb{P}^N_k) = \mathcal{H}ilb^{G,gen}_d(\mathbb{P}^N_k)$. In particular $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is irreducible.*

Proof. If $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is AS, then $X \in \mathcal{H}ilb^{G,gen}_d(\mathbb{P}^N_k)$ by Proposition 2.5. We can complete the proof of the above statement if we examine the case of non-AS irreducible schemes.

It suffices to prove that each such $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is a specialization of a flat family of schemes in $\mathcal{H}ilb^{G,gen}_d(\mathbb{P}^N_k)$. But these schemes are of the form $X \cong \mathrm{spec}(A)$ where A is either $A_{n,d}$, with $n = 4, 5, 6, 7$ and $6 \leq n+2 \leq d \leq 9$ or $A^t_{n,2,d}$ with $t = 1, 2$ and $8 \leq n+4 \leq d \leq 9$ or $A^{t,\alpha}_{4,3,9}$ with $t = 1, \dots, 6$.

For example, $A_{n,d}$ is in the flat family $\mathcal{A}_{n,d}$ (see the proof of Proposition 2.10). Its general member is $A_{n,d-1} \oplus A_{0,1}$ (when $\mathrm{char}(k) > 2$) which is in $\mathcal{H}ilb^{G,gen}_d(\mathbb{P}^N_k)$ due to the argument above. The same argument for $A^t_{n,2,d}$ with families $\mathcal{A}^t_{n,2,d}$, $t = 1, 2$, defined in the proof of Proposition 3.4 (when $\mathrm{char}(k) > 4 \geq \mathrm{msdeg}(A^t_{n,2,d}) = d - n - 1$) and for $A^t_{4,3,9}$ with families $\mathcal{A}^{t,\alpha}_{4,3,9}$, $t = 1, \dots, 6$, defined in the proof of Proposition 4.8 (when $\mathrm{char}(k) > 3$), completes the proof. □

Remark 5.2. When $d \leq 7$ the above result is classically known. Indeed, in this case, $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ is irreducible (see e.g. [16]: see also [14]), hence the same is true for the open dense subset $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$.

It is proved in [14] that $\mathcal{H}ilb_8(\mathbb{P}^N_k)$ is again irreducible if $N \leq 3$ and it consists of two distinct components if $N \geq 4$. In this case, beside the component $\mathcal{H}ilb^{gen}_8(\mathbb{P}^N_k)$, containing all the points representing smooth schemes, there is another component. Its points represent irreducible schemes $X = \mathrm{spec}(A)$, where A is a local and Artinian k-algebra with $H(A) = (1, 4, 3)$, thus $X \notin \mathcal{H}ilb^G_8(\mathbb{P}^N_k)$ which again turns out to be irreducible.

Now we examine $\mathrm{Sing}(\mathcal{H}ilb^G_d(\mathbb{P}^N_k))$. To this purpose let $X = \bigcup_{i=1}^p X_i$ where X_i is irreducible of degree d_i. We already checked that if X is AS, then it is unobstructed by Proposition 2.5.

Now assume that X contains a component $Y \cong \mathrm{spec}(A)$ of degree δ where A is either $A_{n,\delta}$, with $n = 4, 5, 6, 7$ and $6 \leq n+2 \leq \delta \leq 9$ or $A^t_{n,2,\delta}$ with $t = 1, 2$ and $8 \leq n+4 \leq 9$.

In the first case Y can be deformed in $\mathcal{H}ilb_\delta(\mathbb{P}^n_k)$ to $\widehat{Y} := \mathrm{spec}(A_{n,n+2} \oplus A_{0,1}^{\oplus\delta-n-2})$ (when $\mathrm{char}(k) > 2$). Due to Theorem 3.5 of [17], we have

$$h^0(Y, \mathcal{N}_Y) \geq \frac{(n+2)^3 - 7(n+2)}{6} + n(\delta - n - 2) \tag{5.3}$$

hence $h^0(Y, \mathcal{N}_Y) > \delta n$ for each $n \geq 4$. The same argument for $A^t_{n,2,\delta}$ with the families $\mathcal{A}^t_{n,2,\delta}$, $t = 1, 2$, yields the obstructedness of Y in the second case.

The above discussion proves the following

Proposition 5.4. *Let* $\mathrm{char}(k) \neq 2, 3$. *If* $d \leq 8$, *then* $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ *is obstructed if and only if it represents a non-AS scheme.* □

Now we analyze the case $d = 9$. In this case we have to examine the schemes $X := \mathrm{spec}(A)$ where $A \cong A^{t,\alpha}_{4,3,9}$ with $t = 1, \dots, 6$. Such algebras are flat specialization of $A^{t,\alpha}_{3,3,8}$ with $t = 1, \dots, 6$ which are unobstructed being AS. Thus we cannot prove (or disprove) that they are obstructed following the above method.

In order to solve our problem we will apply the following proposition to the case we are dealing with. The following result is well-known to the experts but for the sake of completeness we include its proof.

Proposition 5.5. *Let* $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ *and let* X_2 *be its first infinitesimal neighborhood in* $\mathbb{P}^N_k$. *Then*

$$h^0(X, \mathcal{N}_X) = \deg(X_2) - \deg(X).$$

Proof. We can assume that $X := \mathrm{spec}(A) \subseteq \mathbb{A}^N_k = \mathbb{P}^N_k \setminus \{x_0 = 0\}$ and that the embedding corresponds to a fixed quotient

$$k[x_1, \dots, x_N] \twoheadrightarrow k[x_1, \dots, x_N]/I \cong A.$$

Then (see [25], exercise III.2.3)

$$H^0(X, \mathcal{N}_X) \cong H^0\left(X, \mathcal{N}_{X|\mathbb{A}^N_k}\right) \cong H^0\left(X, \widetilde{(I/I^2)^\check{}}\right) \cong \mathrm{Hom}_A\left(I/I^2, A\right).$$

Since A is an Artinian, Gorenstein algebra, then $\mathrm{Hom}_A\left(I/I^2, A\right)$ and I/I^2 have the same length (see [32], Chapter 5, Theorem 21), thus $h^0\left(\mathbb{P}^N_k, (\mathfrak{I}/\mathfrak{I}^2)^\check{}\right) = \dim_k(I/I^2)$.

From the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow k[x_1, \dots, x_N]/I^2 \longrightarrow k[x_1, \dots, x_N]/I \longrightarrow 0,$$

since $\dim_k(k[x_1, \dots, x_N]/I) = \deg(X)$ and $\dim_k(k[x_1, \dots, x_N]/I^2) = \deg(X_2)$, the formula for $h^0\left(\mathbb{P}^N_k, \mathcal{N}_X\right)$ follows. □

Remark 5.6. The Gorenstein hypothesis in the above proposition cannot be skipped. E.g., take the scheme $X \subseteq \mathbb{A}^3_k$ defined by the ideal $I := (x_1, x_2, x_3)^2 \subseteq k[x_1, x_2, x_3]$, then $I^2 = (x_1, x_2, x_3)^4$.

We have $\deg(X) = \dim_k(k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2) = 4$ and $I^2 = (x_1, x_2, x_3)^4$, thus $\deg(X_2) = \dim_k(k[x_1, x_2, x_3]/(x_1, x_2, x_3)^4) = 20$. We conclude that

$$\deg(X_2) - \deg(X) = 16 < 18 = h^0(X, \mathcal{N}_X).$$

(see [3] for the second equality).

We are now able to deal with the singular locus of $\mathcal{H}ilb^G_9(\mathbb{P}^N_k)$. Obstructed schemes $X \in \mathcal{H}ilb^G_9(\mathbb{P}^N_k) = \mathcal{H}ilb^{G,gen}_9(\mathbb{P}^N_k)$ are necessarily non-AS due to Proposition 2.5, thus they must contain a component isomorphic to either $\mathrm{spec}(A_{n,\delta})$ or $\mathrm{spec}(A^t_{n,2,\delta})$ or $\mathrm{spec}(A^{t,\alpha}_{4,3,9})$ with $n \geq 4$.

The same argument already used to prove the obstructedness of non-AS schemes when $d \leq 8$ can be repeated word by word in the first two cases. We examine now the third case, i.e. $X \cong \mathrm{spec}(A^{t,\alpha}_{4,3,9})$. Since, due to Lemma 2.3, the obstructedness of X does not depend on the embedding, we can consider the canonical embedding $X \hookrightarrow \mathbb{P}^4_k$ induced by the quotient

$$k[x_1, x_2, x_3, x_4] \twoheadrightarrow k[x_1, x_2, x_3, x_4]/I_{t,\alpha} \cong A^{t,\alpha}_{4,3,9}$$

(see the notations introduced in the statement of Theorem 4.1).

With this idea, using any computer software for symbolic calculations, since

$$\deg(X_2) = \begin{cases} 50 & \text{if } t = 4, 5, 6, \\ 45 & \text{if } t = 2, 3. \end{cases}$$

Proposition 5.5 finally yields

$$h^0(X, \mathcal{N}_X) = \begin{cases} 41 & \text{if } t = 4, 5, 6, \\ 36 & \text{if } t = 2, 3. \end{cases}$$

In the case $t = 1$ we have an infinite family depending on the parameter $\alpha \in k$, thus we need some more care in order to compute $h^0(X, \mathcal{N}_X)$, since it could depend on the parameter.

We handle this case again by using a computer algebra software as follows. In the polynomial ring $k[x_1, x_2, x_3, x_4]$, we choose the lexicographic term-order associated to $x_4 > x_1 > x_2 > x_3$. It is easy to check that the leading terms of the polynomials in a minimal Gröbner basis of the ideal $I^2_{1,\alpha} \subseteq k[x_1, x_2, x_3, x_4]$ do not depend on $\alpha \in k$. Thus $\deg(X_2)$ does not depend on $\alpha \in k$, too. Hence we can fix any α, e.g. $\alpha = 0$. Since, in this case, one obtains $\deg(X_2) = 45$, it follows that $h^0(X, \mathcal{N}_X) = 36$ again by Proposition 5.5. We summarize the previous computations in the following

Proposition 5.7. *Let* $\mathrm{char}(k) \neq 2, 3$ *and* $X \in \mathcal{H}ilb^G_9(\mathbb{P}^N_k)$. *Then X is obstructed if and only if it contains an irreducible component isomorphic to either* $\mathrm{spec}(A_{n,\delta})$ *or* $\mathrm{spec}(A^t_{n,2,\delta})$, *where* $n \geq 4$, *or* $\mathrm{spec}(A^{t,\alpha}_{4,3,9})$, *where* $t = 4, 5, 6$. □

The proof of Proposition 5.4 can be easily generalized to prove that the closure in $\mathcal{H}ilb^G_d(\mathbb{P}^n_k)$ of the locus $H_{n,d}$ of schemes isomorphic to $\mathrm{spec}(A_{n,n+2} \oplus A^{\oplus d-n-2}_{0,1})$ is always contained in $\mathrm{Sing}(\mathcal{H}ilb^G_d(\mathbb{P}^N_k))$ for $n \geq 4$.

On one hand, this means also that the schemes $\mathrm{spec}(A^{t,\alpha}_{4,3,9})$, $h \leq 3$ are not in $H_{4,9}$. On the other hand, let us take $X \in H_{4,9}$. Equality (5.3) easily yields $h^0(X, \mathcal{N}_X) \geq 41$, thus we cannot exclude that $\mathrm{spec}(A^{t,\alpha}_{4,3,9}) \in H_{4,9}$, $t = 4, 5, 6$.

Indeed this is actually the case for $t = 4$. To prove this consider the ideal $J \subseteq k[b, x_1, \dots, x_n]$

$$\begin{aligned} J := &\ (x_2^3 - bx_1^2 - x_1^3, x_1^3 - bx_3^2 - x_1^3, x_ix_j, x_h^2 - bx_1^2 - x_1^3)_{\substack{1\leq i<j\leq n\\ 4\leq h\leq n}} \\ = &\ (x_1 + b, x_2, \dots, x_n) \cap (x_2^3 - bx_1^2, x_1^3 - bx_3^2, x_ix_j, x_h^2 - bx_1^2)_{\substack{1\leq i<j\leq n\\ 4\leq h\leq n}}. \end{aligned}$$

Thus $\mathcal{A} := k[b, x_1, \dots, x_n]/J \to \mathbb{A}^1_k$ is a flat family having special fiber over $b = 0$ isomorphic to $A^{4,0}_{n,3,n+5}$ and general fibre isomorphic to $A^1_{n,2,n+4} \oplus A_{0,1}$. Thus, for each $n \geq 3$, we have $X := \mathrm{spec}(A^{4,0}_{n,3,n+5}) \in H_{4,n+5}$.

It is then natural to state the following

Conjecture 5.8. $\mathrm{Sing}(\mathcal{H}ilb^{G,gen}_d(\mathbb{P}^N_k)) = H_{4,d}$ *for each d.*

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