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Common fixed point results for noncommuting mappings without continuity in cone metric spaces

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Abstract

The existence of coincidence points and common fixed points for mappings satisfying certain contractive conditions, without appealing to continuity, in a cone metric space is established. These results generalize several well-known comparable results in the literature.

Keywords: Weakly compatible maps; Common fixed point; Cone metric space

1. Introduction and preliminaries

The study of common fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity. In 1976, Jungck [4], proved a common fixed point theorem for commuting maps, generalizing the Banach contraction principle. This theorem has many applications but suffers from one drawback—the results require the continuity of one of the two maps involved. Sessa [11] introduced the notion of weakly commuting maps. Jungck [5] coined the term compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. Pant [9] defined R-weakly commuting maps and proved common fixed point theorems, assuming the continuity of at least one of the mapping. Kannan [12] proved the existence of a fixed point for a map that can have a discontinuity in a domain, however the maps involved in every case were continuous at the fixed point. Jungck [7,8] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts. For a survey of coincidence point theory, its applications, comparison of different contractive conditions and related results, we refer to [1,6,9,10] and references contained therein. Guang and Xian [3] generalized the concept of a metric space, replacing the set of

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real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. The aim of this paper is to present coincidence point results for two mappings which satisfy generalized contractive conditions. Common fixed point theorems for a pair of weakly compatible maps, which are more general than R-weakly commuting, and compatible mappings are obtained in the setting of cone metric spaces without exploiting the notion of continuity. These theorems generalize the results of Guang and Xian [3], Jungck [4], Kannan [12] and Pant [9].

Consistent with Guang and Xian [3], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a *cone* if and only if:

(a) P is closed, nonempty and $P \neq \{0\}$;
(b) $a, b \in R$, $a, b \geqslant 0$, $x, y \in P$ implies $ax + by \in P$;
(c) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leqslant x \leqslant y \quad \text{implies} \quad \|x\| \leqslant K\|y\|. \tag{1.1}$$

The least positive number satisfying the above inequality is called the *normal constant* of P, while $x \ll y$ stands for $y - x \in \operatorname{int} P$ (interior of P).

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leqslant d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a *cone metric space*. The concept of a cone metric space is more general than that of a metric space.

Definition 1.2. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(e) a *Cauchy* sequence if for every c in E with $c \gg 0$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;
(f) a *Convergent* sequence if for every c in E with $c \gg 0$, there is N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence is unique provided P is a normal cone with normal constant K (see Guang and Xian [3] and [2]).

Definition 1.3. Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g, and w is called a *point of coincidence* of f and g.

Proposition 1.4. *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Proof. Since $w = fx = gx$ and f and g are weakly compatible, we have $fw = fgx = gfx = gw$: i.e., $fw = gw$ is a point of coincidence of f and g. But w is the only point of coincidence of f and g, so $w = fw = gw$. Moreover if $z = fz = gz$, then z is a point of coincidence of f and g, and therefore $z = w$ by uniqueness. Thus w is a unique common fixed point of f and g. □

2. Common fixed point theorems

In this section we obtain several coincidence and common fixed point theorems for mappings defined on a cone metric space.

Theorem 2.1. *Let (X, d) be a cone metric space, and P a normal cone with normal constant K. Suppose mappings $f, g : X \to X$ satisfy*

$$d(fx, fy) \leqslant kd(gx, gy), \quad \text{for all } x, y \in X, \tag{2.1}$$

where $k \in [0, 1)$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the range of g contains the range of f. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \leqslant kd(gx_n, gx_{n-1}) \\ &\leqslant k^2 d(gx_{n-1}, gx_{n-2}) \leqslant \cdots \leqslant k^n d(gx_1, gx_0). \end{aligned}$$

Then, for $n > m$,

$$\begin{aligned} d(gx_n, gx_m) &\leqslant d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m+1}, gx_m) \\ &\leqslant (k^{n-1} + k^{n-2} + \cdots + k^m) d(gx_1, gx_0) \\ &\leqslant \frac{k^m}{1-k} d(gx_1, gx_0). \end{aligned}$$

From (1.1),

$$\left\| d(gx_n, gx_m) \right\| \leqslant \frac{k^m}{1-k} K \left\| d(gx_1, gx_0) \right\|,$$

which implies that $d(gx_n, gx_m) \to 0$ as $n, m \to \infty$. Hence $\{gx_n\}$ is a Cauchy sequence. Since $g(X)$ is complete, there exists a q in $g(X)$ such that $gx_n \to q$ as $n \to \infty$. Consequently, we can find p in X such that $g(p) = q$. Further,

$$d(gx_n, fp) = d(fx_{n-1}, fp) \leqslant kd(gx_{n-1}, gp),$$

which from (1.1) implies that

$$\left\| d(gx_n, fp) \right\| \leqslant Kk \left\| d(gx_{n-1}, gp) \right\| \to 0, \quad \text{as } n \to \infty.$$

Hence $d(gx_n, fp) \to 0$ as $n \to \infty$. Also, $d(gx_n, gp) \to 0$ as $n \to \infty$. The uniqueness of a limit in a cone metric space implies that $f(p) = g(p)$. Now we show that f and g have a unique point of coincidence. For this, assume that there exists another point q in X such that $fq = gq$. Now

$$d(gq, gp) = d(fq, fp) \leqslant kd(gq, gp),$$

which gives $\|d(gq, gp)\| = 0$ and $gq = gp$. From Proposition 1.4, f and g have a unique common fixed point. □

Example 2.2. Let $E = R^2$, $P = \{(x, y) \in E\colon\ x, y \geqslant 0\} \subset R^2$, $d : R \times R \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha > 0$ is a constant. Define

$$fx = \begin{cases} \frac{\alpha}{\beta+1} x, & x \neq 0, \\ \gamma, & x = 0, \end{cases}$$

and

$$gx = \begin{cases} \alpha x, & x \neq 0, \\ \gamma, & x = 0, \end{cases}$$

where $\beta \geqslant 1$, and $\gamma \neq 0$. It may be verified that $d(fx, fy) \leqslant kd(gx, gy)$, for all $x, y \in X$, where $k = \frac{1}{\beta} \in (0, 1]$. Moreover f and g have a coincidence point X.

In above example, f and g do not commute at the coincidence point 0, and therefore are not weakly compatible. And f and g do not have common fixed point. Thus, this example demonstrates the crucial role of weak compatibility in our results.

Theorem 2.3. *Let (X, d) be a cone metric space and P a normal cone with normal constant K. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \leqslant k\big(d(fx, gx) + d(fy, gy)\big), \quad \textit{for all } x, y \in X, \tag{2.2}$$

where $k \in [0, \frac{1}{2})$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done since the range of g contains the range of f. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1}) &\leqslant k\big(d(fx_n, gx_n) + d(fx_{n-1}, gx_{n-1})\big) \\ &= k\big(d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})\big). \end{aligned}$$

So

$$d(gx_{n+1}, gx_n) \leqslant hd(gx_n, gx_{n-1}),$$

where $h = \frac{k}{1-k}$. For $n > m$,

$$\begin{aligned} d(gx_n, gx_m) &\leqslant d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m+1}, gx_m) \\ &\leqslant \big(h^{n-1} + h^{n-2} + \cdots + h^m\big)d(gx_1, gx_0) \\ &\leqslant \frac{h^m}{1-h} d(gx_1, gx_0), \end{aligned}$$

which from (1.1) implies that $\|d(gx_n, gx_m\| \leqslant \frac{k^m}{1-k} K\|d(gx_1, gx_0)\|$. Then $d(gx_n, gx_m) \to 0$ as $n, m \to \infty$, and $\{gx_n\}$ is a Cauchy sequence. Since $g(X)$ is a complete subspace of X, there exists q in $g(X)$ such that $gx_n \to q$, as $n \to \infty$. Consequently we can find p in X such that $g(p) = q$. Thus,

$$d(gx_n, fp) = d(fx_{n-1}, fp) \leqslant kd(gx_{n-1}, gp),$$

which implies that

$$\big\|d(gx_n, fp)\big\| \leqslant Kk\big\|d(gx_{n-1}, gp)\big\| = 0, \quad \text{as } n \to \infty.$$

Hence $d(gx_n, fp) \to 0$ as $n \to \infty$. Also, $d(gx_n, gp) \to 0$ as $n \to \infty$. The uniqueness of a limit in a cone metric space implies that $f(p) = g(p)$. Now we show that f and g have a unique point of coincidence. For this, assume that there exists another point q in X such that $fq = gq$. Now

$$\begin{aligned} d(gq, gp) &= d(fq, fp) \\ &\leqslant k\big(d(fq, gq) + d(fp, gp)\big), \end{aligned}$$

which gives $\|d(gq, gp)\| = 0$ and $gq = gp$. From Proposition 1.4, f and g have a unique common fixed point. □

Theorem 2.4. *Let (X, d) be a cone metric space, and P a normal cone with normal constant K. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) < k\big(d(fx, gy) + d(fy, gx)\big), \quad \textit{for all } x, y \in X, \tag{2.3}$$

where $k \in [0, \frac{1}{2})$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the range of g contains the range of f. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \leqslant k\big(d(fx_n, gx_{n-1}) + d(fx_{n-1}, gx_n)\big) \\ &\leqslant k\big(d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})\big). \end{aligned}$$

So

$$d(gx_{n+1}, gx_n) \leqslant hd(gx_n, gx_{n-1}),$$

where $h - \frac{k}{1-k}$. For $n > m$,

$$\begin{aligned} d(gx_n, gx_m) &\leqslant d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + \cdots + d(gx_{m+1}, gx_m) \\ &\leqslant \big(h^{n-1} + h^{n-2} + \cdots + h^m\big) d(gx_1, gx_0) \\ &\leqslant \frac{h^m}{1-h} d(gx_1, gx_0). \end{aligned}$$

Following an argument similar to that given in Theorem 2.3, we obtain a point of coincidence of f and g. Now we show that f and g have a unique point of coincidence. For this, assume that there exist p and q in X such that $fp = gp$ and $fq = gq$. Now

$$\begin{aligned} d(gq, gp) &= d(fq, fp) \\ &\leqslant k\big(d(fq, gp) + d(fp, gq)\big) = 2kd(gq, gp) \end{aligned}$$

which implies that $d(gq, gp) = 0$ and $gq = gp$. From Proposition 1.4, the result follows. □

The above theorem generalizes Theorem 4 of [3], which itself is a generalization of a result of [12].

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