The Richardson model in a random environment

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Let $\xi_t : \mathbb{Z}^d \to \{0, 2\}$ be such that all coordinates are independent and, in each coordinate, 2 changes to 0 at rate $\beta$ and 0 changes to 2 at rate 1. A family of particles (1's) move in the space occupied by 0's like Richardson model, i.e., a 0 becomes 1 at a rate proportional to 1-occupied neighbors. We prove the phase transition phenomena for the coexistence of 0's, 1's and 2's.

random environment * reversibility * phase transition * renormalization

1. Introduction

We consider Markov processes in which the state at time $t$ is $\xi_t : \mathbb{Z}^d \to \{0, 1, 2\}$. The evolution of this model is as follows: (i) 2's change to 0 at rate $\beta$; (ii) 0's change to 2 at rate 1; (iii) 1's change to 2 at rate 1; (iv) 1's give birth at rate $\alpha$; (v) if the birth occurs at $x$, the offspring is sent to a site chosen at random from $\{y : y - x \in \mathcal{N}\}$, $\mathcal{N}$ = the set of neighbors of 0; (vi) if $\xi_t(y) \geq \xi_t(x)$ then the birth is suppressed. Rules (i)-(iii) indicate a varying environment which, with (v) we can see, is independent of the evolution of 1's. Rules (iv)-(vi) indicate that 1's cannot live in the space occupied by 2's. More precisely, 1's run as a Richardson model in the random space where there are no 2's. The easy way to look at these processes is to give a random length to the death point in usual contact process. Also, it adds variation to the Epidemics With Recovery (see Durrett and Neuhauser, 1990) by allowing healthy changes to immune at rate 1. Here we concern about if 1's, initially with a finite number, can survive or if there is a nontrivial stationary distribution which with positive probability contains 1's. As below, we can see these two questions are equivalent.

Let $\xi_t : \mathbb{Z}^d \to \{0, 2\}$ be such that $\xi_t(x) = 2$ iff $\xi_t(x) = 2$. Then, $\{\xi_t(x) : x \in \mathbb{Z}^d\}$ are independent Markov chains. Furthermore, for each $x$, $\{\xi_t(x)\}$ is reversible with the reversible measure $\mu_\beta$, $\mu_\beta(\xi_t = 2) = 1/(1 + \beta) = 1 - \mu_\beta(\xi_t = 0)$. This reversibility makes it possible to build a selfdual for the system $\{\xi_t\}$. Let $\eta_t : \mathbb{Z}^d \to \{0, 1\}$ be such that $\eta_t(x) = 1$ iff $\xi_t(x) = 1$. We denote $\xi_t = (\eta_t, \xi_t)$. For any subset $A \subset \mathbb{Z}^d$, we denote...
the system starting with \( \xi_0 \) such that \( \xi_0 \) has the distribution \( \mu_\beta \) and \( \eta_0(x) - 1 \) iff \( x \in A \) and \( \xi_0(x) = 0 \).

From an argument in Durrett and Moller (1991), we can prove that \( \xi_t^{Z_{\alpha \mu \beta}} \) converges weakly to a limit \( \mu_{12} \) as \( t \to \infty \) (see Section 3). It is easy to see \( \mu_{12} \) is the largest stationary distribution. We are going to examine when \( \mu_{12} \) is not trivial.

From the above reversibility and a construction of the process, we can see there is a dual relation

\[
P(\eta_t^{A_{\mu \beta}} \cap B \neq \emptyset) = P(\eta_t^{B_{\mu \beta}} \cap A \neq \emptyset)
\]

where one of \( A \) and \( B \) is a finite set. Thus, we have

\[
\mu_{12}(\eta \cap B \neq \emptyset) = \lim_{t \to \infty} P(\eta_t^{Z_{\alpha \mu \beta}} \cap B \neq \emptyset) = P(\eta_t^{B_{\mu \beta}} \neq \emptyset, \forall t \geq 0).
\]

For any dimension \( d \geq 1 \) and \( \beta > 0 \), let

\[
\alpha_{c,d}(\beta) \equiv \inf\{\alpha > 0: \mu_{12}(\eta \neq \emptyset) > 0\}.
\]

Case \( d = 1 \) can be easily compared with a contact process. As above, 2 changes to a 0 at rate \( \beta \) and the birth rate of 2 is 1. Thus, 2's dominate an ordinary contact process with birth rate 1 death rate \( \beta \). In a one-dimensional case, 1's cannot be born in an interval consisting of 2's and/or 0's. Therefore, if \( 1/\beta > \lambda_{c,1} \), the critical value of a one-dimensional contact process, 2 and 0 will take over with probability 1, i.e., 1's will die out with probability 1. With the above notation, this can be explained as:

**Theorem 1.1.** \( \alpha_{c,1}(\beta) = \infty \) for any \( \beta < 1/\lambda_{c,1} \). \( \square \)

To find a condition guaranteeing \( \mu_{12} \) is not trivial, we first notice that the open sites (0 sites) of \( \{\xi_t\} \) dominate a usual contact process with birth rate \( \beta \) and then, applying the results in Durrett (1989a), we can show:

**Theorem 1.2.** For all \( \beta > \lambda_{c,d} \), the critical value for \( d \)-dimensional contact processes, we have \( 0 < \alpha_{c,d}(\beta) < \infty \).

As we can see, Theorem 1.2 says that, for certain \( \beta \) (i.e., certain random environments), there are nontrivial stationary distributions for large \( \alpha \) and there are only trivial stationary distribution for small \( \alpha \). But, from Theorem 1.1, we can see things are dramatically different for small \( \beta \). In that case, there are no nontrivial stationary distributions at all for any \( \alpha \). We will give further comment on this issue in Section 5. In Section 2, we give a construction of the model and prove the duality. In Section 3, we construct the largest stationary distribution. Theorem 1.2 is proved in Section 4.

2. Construction and duality

We construct the process from a graphical representation in three steps. First, we construct the standard Richardson model. Secondly, we construct the random
environment. Then, we use the random environment to delete points from the standard Richardson model.

Let $\alpha > 0$ and let $\{p(x, y): x, y \in \mathbb{Z}^d\}$ be the transition probability of the simple random walk on $\mathbb{Z}^d$, i.e., $p(x, y) = (1/(2d))1_{|y-x|=1}$. For each pair $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, let $\{T_n(x, y): n \geq 1\}$ be a rate $\alpha p(x, y)$ Poisson process. At $t = T_n(x, y)$, we draw an arrow from $x$ to $y$. For each $t > 0, x, y \in \mathbb{Z}^d$, we say there is a path from $(x, 0)$ to $(y, t)$ if there is a sequence of times $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t$ and spatial locations $x_0 = x, x_1, \ldots, x_n = y$ such that, for $i = 1, 2, \ldots, n$, there is an arrow from $x_{i-1}$ to $x_i$ at time $s_i$. Let $A \subset \mathbb{Z}^d$ be any subset. We define $r^A_t: \mathbb{Z}^d \to \{0, 1\}$ to be such that $r^A_t(y) = 1$ iff there is a path from $(x, 0)$ to $(y, t)$ for some $x \in A$. Then, the process $\{r^A_t: t \geq 0\}$ is the standard Richardson model starting at $A$.

The random environment is defined by a family of independent Markov chains $\{\xi_t(x): t \geq 0, x \in \mathbb{Z}^d\}$ where

$$P(\xi_{t+h}(x) = 2 | \xi_t(x) = 0) = h + o(h)$$

and

$$P(\xi_{t+h}(x) = 0 | \xi_t(x) = 2) = \beta h + o(h).$$

Here $\beta > 0$. For each $\beta$, let $\mu_\beta$ be the probability measure on $\{0, 2\}$ such that

$$\mu_\beta(\{0\}) = \frac{\beta}{1 + \beta} \quad \text{and} \quad \mu_\beta(\{2\}) = \frac{1}{1 + \beta}.$$ 

**Lemma 2.1.** $\mu_\beta$ is the unique stationary distribution of $\{\xi_t(x): t \geq 0\}$ for any $x$ and hence $\xi_t(x)$ converges weakly to $\mu_\beta$ as $t \to \infty$ for any initial distribution. Furthermore, $\mu_\beta$ is reversible.

**Proof.** This is a standard result.  \(\square\)

With the above independent Markov chains given, we let $\{\xi_t: t \geq 0\}$ be the $\{0, 2\}^{\mathbb{Z}^d}$-valued process such that each coordinate $\xi_t(x)$ is as above and with initial distribution $\mu_\beta$ and $\{\xi_t(x): x \in \mathbb{Z}^d\}$ are independent.

Now, we construct $\{\eta_s: 0 \leq s \leq t\}$. We start with the standard Richardson model $\{r^A_s: 0 \leq s \leq t\}$ and go from 0 to $t$ with the path of $\{\xi_s: 0 \leq s \leq t\}$. Whenever 2 in $\{\xi_s: 0 \leq s \leq t\}$ meets a 1 in $\{r^A_s: 0 \leq s \leq t\}$, we delete that 1 and all its pure descendents (i.e., those who are not the descendents of other 1's) from the graph. Then, the remaining part of $\{r^A_s: 0 \leq s \leq t\}$ will be the graph of $\{\eta_s: 0 \leq s \leq t\}$. We can see that $\{\eta_s\}$ is well defined.

Finally, the Richardson model in the random environment, $\{\xi_t^{A, \mu_\beta}: t \geq 0\}$, is defined as

$$\xi_t^{A, \mu_\beta}(x) = \max\{\eta_s(x), \xi_t(x)\}.$$ 

We can see that $\{\eta_s\}, \{\xi_t\}$ can be obtained from $\{\xi_t^{A, \mu_\beta}\}$ as in the Introduction.
To construct the dual process of \( \{ \xi_t^{A, \mu} \} \), we first construct the dual of \( \{ r_t^A ; t \geq 0 \} \), which is given in Durrett (1988). We reverse the direction of arrows in the graphical representation for the standard Richardson model and construct the process \( \{ r_t^B, 0 \leq s \leq t \} \) starting with \( B \subset \mathbb{Z}^d \) as above. Then, we have

\[
\{ \omega : r_t^A \cap B = \emptyset \} = \{ \omega : r_t^B \cap A \neq \emptyset \}.
\]

(2.1)

Let \( \{ \hat{\xi}_s : 0 \leq s \leq t \} \) be the \( \{0, 2\}^d \)-valued process such that \( \hat{\xi}_s = \xi_{t-s}, 0 \leq s \leq t \). We construct \( \{ \hat{\eta}_t \} \) from \( \{ \hat{r}_t^B, 0 \leq s \leq t \} \) and \( \{ \hat{\xi}_s : 0 \leq s \leq t \} \) as we did for \( \{ \eta_t \} \). Then, from (2.1), we have

\[
\{ \omega : \eta_t^{A, \mu} \cap B \neq \emptyset \} = \{ \omega : \hat{\eta}_t^{B, \mu} \cap A \neq \emptyset \}.
\]

(2.2)

Lemma 2.2. For the process defined above, we have

\[
P(\eta_t^{A, \mu} \cap B \neq \emptyset) = P(\eta_t^{B, \mu} \cap A \neq \emptyset)
\]

where either \( A \) or \( B \) is a finite set.

Proof. Note that \( \{ \hat{r}_t^B, 0 \leq s \leq t \} \) and \( \{ r_t^B, 0 \leq s \leq t \} \) have the same distribution. From Lemma 2.1, \( \{ \hat{\xi}_s : 0 \leq s \leq t \} \) and \( \{ \xi_s : 0 \leq s \leq t \} \) have the same finite-dimensional distributions. Thus, from (2.2), we have

\[
P(\eta_t^{A, \mu} \cap B \neq \emptyset) = P(\hat{\eta}_t^{B, \mu} \cap A \neq \emptyset) = P(\eta_t^{B, \mu} \cap A \neq \emptyset).
\]

(2.2)

3. The largest stationary distribution

In this section, we are going to use an idea in Durrett and Möller (1991) to give the largest stationary distribution.

Lemma 3.1. The distribution of \( \xi_t^{Z^d, \mu} \) converges weakly to a limit \( \mu_{12} \).

Proof. As in Durrett and Möller (1991), it suffices to show \( P(\eta_t^{Z^d, \mu} \cap B_1 \neq \emptyset, \xi_t^{Z^d, \mu} \cap B_2 = \emptyset) \) is decreasing in \( t \) for any finite sets \( B_1 \) and \( B_2 \). From Markov property of \( \xi_t^{Z^d, \mu} \), we have, for \( t > s \),

\[
P(\eta_t^{Z^d, \mu} \cap B_1 \neq \emptyset, \xi_t^{Z^d, \mu} \cap B_2 = \emptyset) = EE \left( 1_{(\eta_t^{Z^d, \mu} \cap B_1 \neq \emptyset, \xi_t^{Z^d, \mu} \cap B_2 = \emptyset)} \mid \mathcal{F}_{t-s} \right)
\]

\[
= EE \xi_t^{Z^d, \mu} 1_{(\eta_t \cap B_1 \neq \emptyset, \xi_t \cap B_2 = \emptyset)}
\]

\[
\leq EE \xi_t^{Z^d, \mu} 1_{(\eta_t \cap B_1 \neq \emptyset, \xi_t \cap B_2 = \emptyset)}
\]

\[
= P(\eta_t^{Z^d, \mu} \cap B_1 \neq \emptyset, \xi_t^{Z^d, \mu} \cap B_2 = \emptyset)
\]

where \((Z^d, \xi_{t-s})\) is the probability distribution of \( \xi \) and

\[
\xi_t(x) = \begin{cases} 
\xi_{t-s}(x) & \text{if } \xi_{t-s}(x) = 2, \\
1 & \text{if } \xi_{t-s}(x) = 0.
\end{cases}
\]

Since \( \mu_\beta \) is stationary, it is equal to \( \xi_0^{Z^d, \mu} \) in distribution. So, the lemma is true.
Suppose \( \mu \) is a stationary distribution for the system \( \{ \xi_t \} \). Then, from Lemma 2.1, we know the projection of \( \mu \) of \( \xi \) will be the same as that of \( \mu_{12} \) or as the product measure of \( \mu_\beta \)'s. A similar argument in Lemma 3.1 will show that \( \mu \) does not give more mass on 1's than \( \mu_{12} \), i.e., we have

**Lemma 3.2.** Let \( \mu \) and \( \mu_{12} \) be as above. For any finite set \( A \subset \mathbb{Z}^d \), we have

\[
\mu(\eta \cap A \neq \emptyset) \leq \mu_{12}(\eta \cap A \neq \emptyset).
\]

**Proof.** As in Lemma 3.1, we have

\[
\mu(\eta \cap A \neq \emptyset) = P_\mu(\eta_t \cap A \neq \emptyset) \leq P_{(\mathbb{Z}^d,\mu_\beta)}(\eta_t \cap A \neq \emptyset) \rightarrow \mu_{12}(\eta \cap A \neq \emptyset)
\]

as \( t \rightarrow \infty \). So, the lemma is true. \( \square \)

It is easy to see that \( (0, \mu_\beta) \) and \( \mu_{12} \) are two stationary distribution for the system \( \{ \zeta_t \} \). From Lemma 3.2, we see the system \( \{ \zeta_t \} \) has a nontrivial stationary distribution iff \( \mu_{12} \neq (0, \mu_\beta) \).

4. Phase transition property

In this section, we will prove Theorem 1.2.

**Lemma 4.1.** For any \( \beta > 0 \), \( \alpha_\beta(\beta) \geq 1 \).

**Proof.** Suppose \( \alpha < 1 \). Then, the birth rate is less than the death rate in the system. A comparison with the result of branching process will show \( \mathbb{P}(\eta_t(0) = 0 \, \forall t \geq 0) = 0 \). Thus, from (1.1) or Lemma 2.2, \( \mu_{12}(\eta(0) = 1) = 0 \). Since \( \mu_{12} \) is translation invariant, we know \( \mu_{12} \) is trivial. \( \square \)

Now, we are going to show the more difficult part \( \alpha_\beta(\beta) < \infty \). The idea is as follows: 1's live in the vacant sites of \( \{ \xi_t \} \) and those vacant sites dominate a contact process. Applying result in Durrett (1989a), we have thick paths of vacant sites percolating. Here, percolation by a special type of sites means there is a path of that type of sites connecting two sites as in Durrett (1989a). By letting \( \alpha \) be large enough, we can form paths of 1's inside paths of vacant sites making percolation still possible. Let us compare the standard contact process \( \{ \nu_t \} \) with birth rate \( \beta \) and the above environment process \( \{ \xi_t \} \). In \( \{ \xi_t \} \), we know the rate at which 2 changes to 0 is simply \( \beta \) and is greater than or equal to the flip rate that a vacant site becomes occupied in \( \{ \nu_t \} \). Both of them have the same death rate 1. A simple coupling shows that if \( \xi_t \approx \nu_0 \) the two processes can be constructed on the same spaces so that \( \xi_t = \nu_t \) for all \( t \geq 0 \), where \( \xi_t \) denote the set of 0's, i.e., vacant sites, and \( \nu_t \) denote the set of 1's as usual. Thus, we can look at \( \{ \nu_t \} \) to get space for \( \{ \eta_t \} \) to go.
As in Durrett (1989a), let \( \mathcal{L} = \{(m, n) \in \mathbb{Z}^2: m + n \) is even\). Let \( B_0 = [-2L, 2L]^d \times [0, T] \) and \( B_{m,n} = (4Lm, 50Tn) + B_0 \) for \((m, n) \in \mathcal{L}\). Let \( \tilde{\nu}^{4L} \) be the process in which births outside \((4Lm, 50Tn) + (-4L, 4L)^d \) are not allowed. Let \( I = (-J, J)^d \) and call a site \((m, n) \in \mathcal{L}\) wet if \( \tilde{\nu}^{4L}_t \supset x + (-J, J)^d \) for some \((x, t) \in B_{m,n}\).

We quote his result as:

**Lemma 4.2.** Suppose \( \beta > \lambda_c \), the critical value for contact processes. For any \( \varepsilon > 0 \), we can choose \( J, L \) and \( T \) so that if \((0, 0)\) is wet then, with probability \( > 1 - \varepsilon \), \((1, 1)\) and \((-1, 1)\) will also be wet. \( \square \)

Suppose a particle at \( x \) gives a birth to \( y \) at time \( t \), we let \( \delta_{x,y,t} \) be the holding time in the sense that \( t + \delta_{x,y,t} \) is the first time there will be another change at \( x \) or \( y \), i.e.,

\[
\delta_{x,y,t} = \inf\{t' - t: \text{there is a death or birth at } x \text{ or } y \text{ at time } t'\}.
\]

For the random graph \( \{\nu^4_t: 50T(n-1) \leq t \leq 50Tn\} \), we let \( \delta \) be the minimum of all \( \delta_{x,y,t} \) with \((x, y, t)\) in the graph. Let \( N \) denote the number of deaths and births happened in the graph. It is easy to see

\[
\lim_{\delta_0 \to 0} P(\delta > \delta_0) = 1, \tag{4.1}
\]

\[
\lim_{N_0 \to \infty} P(N \leq N_0) = 1. \tag{4.2}
\]

Now, we define a new oriented percolation structure from the above construction by using \( \{\eta_t\} \) instead of \( \{\nu_t\} \) and prove a similar result. Here, we consider the contact process embedded in the process \( \{\xi_t\} \).

**Lemma 4.3.** Suppose \( \beta > \lambda_c \), the critical value for contact processes. For any \( \varepsilon > 0 \), we can choose \( J, L \) and \( T \) and \( \alpha_0 > 0 \) such that, for all \( \alpha > \alpha_0 \), if \((0, 0)\) is wet then, with probability \( > 1 - \varepsilon \), \((1, 1)\) and \((-1, 1)\) will also be wet.

**Proof.** We first choose \( J, L \) and \( T \) so that Lemma 4.2 holds. Then, from (4.1) and (4.2), we choose \( \delta_0 \) and \( N_0 \) so that \( P(\delta > \delta_0) > 1 - \varepsilon \) and \( P(N \leq N_0) > 1 - \varepsilon \). Conditional on \( \{\delta > \delta_0\}, \{N \leq N_0\} \) and the event that percolation succeeds in contact process, we can choose \( \alpha \) large enough so that with probability \( 1 - \varepsilon \) there will be a corresponding birth during every holding time; i.e., we can require, with probability \( 1 - \varepsilon \), \( N_0 \) independent rate \( \alpha \) exponential processes happen in intervals with length at least \( \delta_0 \). Therefore, with probability at least \( 1 - 5\varepsilon \), \( \{\eta_t\} \) will contain all \( x + (-J, J)^d \) as the embedded contact process, which completes the proof of the lemma. \( \square \)

**Proof of Theorem 1.2.** With Lemma 4.3 established, it follows from the induction argument in Durrett (1989a) that when viewed on suitable length and times scales the process \( \{\eta_t^{(0)}, \mu_n\} \) dominates a supercritical oriented percolation if we choose \( \alpha \) large enough. From (1.1), \( \mu_{12} \) is not trivial. From the discussion at the beginning of this section, we know the theorem is true. \( \square \)
5. Further discussion

As we can see in the Introduction, $\alpha_{c,d}(\beta) \in (0, \infty]$ is well defined for all $\beta > 0$. It is interesting to look at it as a curve in $(\beta, \alpha)$ plane. We can easily see the curve is decreasing in $\beta$. By controlling the 2-occupied length and using above renormalization argument, we can prove

$$\lim_{\beta \to \infty} \alpha_{c,d}(\beta) = \lambda_{c,d}.$$

Let $\beta_{c,d} = \inf\{\beta > 0: \alpha_{c,d}(\beta) < \infty\}$. Since the open sites connected to the origin in $\{\xi_t\}$ may not percolate at all for small $\beta$, it is reasonable, at least in low dimensions, to expect $\beta_{c,d} > 0$. Even though we did show this in the case of one dimension, we have not been able to prove it in higher dimensions or to decide if $\lim_{\beta \to \beta_{c,d}} \alpha_{c,d}(\beta) = \infty$ or not.

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