Lie Nilpotent and Solvable Associative Algebras

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First we consider nine properties of an associative algebra analogous to properties of nilpotent Lie algebras and connected nilpotent algebraic groups. We demonstrate the order of implication of these properties and that all nine properties are equivalent when the ground field is algebraically closed.

Next we consider eight properties of an associative algebra analogous to properties of solvable Lie algebras of characteristic zero and connected solvable algebraic groups of any characteristic. We demonstrate the order of implication of these properties and that all eight properties are equivalent when the ground field is algebraically closed and of characteristic different from 2.

INTRODUCTION

Unless specified to the contrary associative algebras are assumed to have unit, subalgebras have the same unit as the over algebra, algebra homomorphisms preserve unit, modules are unital, etc.

Let \( A \) be a finite dimensional associative algebra over the field \( k \). Let \( \{e_i\}_{i=1}^n \) be the minimal orthogonal central idempotents of \( A \) so that if \( A_i = A e_i \) then \( \{A_i\} \) are the indecomposable summands of \( A \). Let \( R \) denote the Jacobson radical of \( A \). For each \( i \), \( A_i \) is an ideal in \( A \), but \( A_i \) is also an associative algebra with unit \( e_i \). Let \( R_i \) be the Jacobson radical of \( A_i \), considered as an associative algebra with unit. Of course \( R_i \supseteq R_j \).

\( Z(A) \) denotes the center of \( A \). \( A^- \) denotes the underlying Lie algebra of \( A \) with respect to \( [a, b] = ab - ba \) for \( a, b \in A \).

NILPOTENCE

We consider the following nine statements about \( A \):

1. For each finite dimensional left \( A \)-module \( M \), \( M \) can be written as the direct sum of submodules \( M = \bigoplus_j M_j \), where for each \( a \in A \), the restriction of \( a \) to \( M_j \) is a scalar plus a nilpotent transformation.

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2. Each indecomposable summand of $A$ can be represented faithfully by matrices of the form
\[
\begin{pmatrix}
\lambda & \star \\
\vdots & \\
0 & \lambda
\end{pmatrix}
\]

3. $A$ has a Wedderburn factor which is central in $A$.

4. For each indecomposable summand of $A$, $A/R_i$ is a purely inseparable field extension of $k$.

5. $A = R + Z(A)$, the sum not necessarily direct.

6. There is a chain of ideals of $A$, $A = I_0 \cap I_1 \cap I_2 \cap \cdots \cap I_{m+1} = 0$, where the image of $I_i$ is central in $A/I_{j+1}$, for $i = 0, \ldots, m$.

7. $A$ is a nilpotent Lie algebra.

8. All idempotents in $A$ are central.

9. For each indecomposable summand of $A$, $A/R_i$ is a field.

**Theorem I.** (a) The following implications hold for the above nine statements:

(b) If $k$ is separably closed, then statements 4, 7, and 9 are equivalent.

(c) If $k$ is perfect, then statements 3, 5, 6, and 7 are equivalent.

(d) If $k$ is algebraically closed, then all nine statements are equivalent.

**Definition.** A finite dimensional associative algebra is called Lie nilpotent if it satisfies condition 7.

Notice that if $L$ is a field extension of $k$ then $A$ satisfies condition 7 if and only if $A \otimes_k L$ satisfies condition 7. This is true because for subspaces $V, W \subset A$,

$$[V, W] \otimes_k L = [V \otimes_k L, W \otimes_k L].$$

Condition 3 of course implies that the Wedderburn factor of $A$ is commutative so that $A/R$ is commutative. But also since separable Wedderburn factors are conjugate by inner automorphism condition 3 implies that $A$ has a unique Wedderburn factor if it is separable.

Condition 6 is the condition of nilpotency in the category of associative algebras.
One should compare the properties of nilpotent associative algebras with the properties of nilpotent Lie algebras and connected nilpotent algebraic groups. Condition 1 shows that they have similar module theoretic properties. Condition 5 corresponds to the property of connected nilpotent algebraic groups being the direct product of a unipotent group by a torus. To get the actual connection between associative algebras, Lie algebras, and algebraic groups one must dualize the notion of “nilpotent associative algebra” to “conilpotent coalgebra.” It then turns out that if \( \mathcal{C} \) is the coalgebra of representative functions of a finite dimensional nilpotent Lie algebra or \( \mathcal{C} \) is the coalgebra of polynomial functions on a connected nilpotent affine algebraic group, then \( \mathcal{C} \) is a conilpotent coalgebra. The details of this dualization are left to the interested reader.

**Solvability**

We consider the following eight statements about \( A \):

1. \( A \) can be faithfully represented by upper triangular matrices.
2. All simple \( A \)-modules are one dimensional.
3. \( A \) contains a commutative subalgebra \( B \) with \( A = R + B \), the sum not necessarily direct.
4. \( R \supseteq [A, A] \) or \( A/R \) is a commutative algebra.
5. \( A/\mathcal{H} \) is commutative hence a field for each maximal ideal \( \mathcal{H} \) of \( A \).
6. There is a chain of ideals in \( A \)
   \[ A = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{m-1} = 0, \]
   where the image of \( I_j \) in \( A/I_{j+1} \) is commutative, \( j = 0 \cdots m \).
7. There is a chain of subspaces
   \[ A = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{m+1} = 0, \]
   where \( I_jI_j \subseteq I_j \), \( I_jI_{j+1} \subseteq I_{j+1} \supseteq I_jI_{j+1} \) (so that \( I_{j+1} \) is an ideal in the associative algebra not necessarily with unit \( I_j \)) and \( I_j/I_{j+1} \) is commutative, for \( j = 0, \ldots, m \).
8. \( A^- \) is a solvable Lie algebra.

**Theorem II.** (a) The following implications holds for the above eight statements:
   \[ 1 \iff 2 \iff 3 \iff 4 \iff 5 \iff 6 \iff 7 \iff 8. \]

(b) If \( k \) is algebraically closed, then statements 1 and 7 are equivalent.
(c) If \( k \) is perfect, then statements 3–7 are equivalent.

(d) If \( k \) has characteristic different from 2, then statements 4–8 are equivalent.

(e) If \( k \) has characteristic 2, and \( A \otimes_k k \) has no simple modules which are two dimensional over \( k \) (or \( A \otimes_k k \) has no quotient algebra of the form \( M(2, k) \)) then conditions 4–8 are equivalent. (\( k \) denotes the algebraic closure of \( k \)).

**Definition.** A finite dimensional associative algebra is called solvable if it satisfies the equivalent conditions 4–7.

Notice that if \( L \) is a field extension of \( k \) then \( A \) is a solvable \( k \) algebra if and only if \( A \otimes_k L \) is a solvable \( L \) algebra. This is easily seen by means of condition 7. If \( A = I_0 \supset I_1 \supset \cdots \supset I_{n-1} = 0 \) is a chain of \( k \) subspaces satisfying condition 7 then \( A \otimes_k L = I_0 \otimes_k L \supset I_1 \otimes_k L \supset \cdots \supset I_{n-1} \otimes_k L = 0 \) is a chain of \( L \) subspaces in \( A \otimes_k L \) satisfying condition 7. Conversely, if \( A \otimes_k L = I_0 \supset I_1 \supset \cdots \supset I_{n+1} = 0 \) is a chain of \( L \) subspaces in \( A \otimes_k L \) satisfying condition 7 let \( I = A \cap \bar{I} \) where we identify \( A \) with \( A \otimes_k k \subset A \otimes_k L \), then \( A = I_0 \supset I_1 \supset \cdots \supset I_{n+1} = 0 \) is a chain of \( k \) subspaces in \( A \) satisfying condition 7.

Condition 7 is the condition of solvability in the category of associative algebras not necessarily having unit.

Condition 4 implies that if \( I \) is an ideal of \( A \) then \( I \) is a maximal left ideal if and only if \( I \) is a maximal two-sided ideal if and only if \( I \) is a maximal right ideal.

One should compare the properties of solvable associative algebras with the properties of solvable Lie algebras of characteristic zero and connected solvable algebraic groups. Conditions 1 and 2 show that they have similar module-representation properties. Over an algebraically closed field \( k \) a solvable associative algebra has a Wedderburn factor of the form \( k \oplus \cdots \oplus k \).

This corresponds to the property of a connected solvable affine algebraic group being the semidirect product of a unipotent group by a torus. Again to get the actual connection between associative algebras, Lie algebras and algebraic groups one must dualize the notion of “solvable associative algebra” to “cosolvable coalgebra.” From condition 4 one deduces that a coalgebra is a cosolvable if and only if its coradical is cocommutative. It then turns out that if \( C \) is the coalgebra of representative functions of a finite dimensional Lie algebra of characteristic zero or \( C \) is the coalgebra of polynomial functions on a connected solvable affine algebraic group then \( C \) is a cosolvable coalgebra.

Recently Sullivan has shown that over an algebraically closed field if \( G \) is any affine group scheme represented by a Hopf algebra which is cosolvable as a coalgebra then \( G \) is the semidirect product of a unipotent affine group scheme by a toroidal affine group scheme. No assumption that \( G \) be connected reduced or algebraic is necessary [3].
Proof of Theorem I

1. Let $M$ be a faithful finite dimensional $A$-module. Then $V = e_i M$ is a faithful (unital) finite dimensional $A_i$-module. Let $\rho: A_i \to \text{End} V$ be the representation of $A_i$ in $\text{End} V$. If statement 1 holds then $\rho(A_i) = k \oplus \rho(R_i)$. Since $\rho$ is injective this implies that $A_i = ke_i \oplus R_i$. This implies 2.

Conversely, the condition 2 implies that $A_i = ke_i \oplus R_i$. If $M$ is any $A$-module $M = \bigoplus_i e_i M$ and the $e_i M$ have the desired property in 1.

2. As observed in the preceding paragraph, condition 2 is equivalent to $A_i = ke_i \oplus R_i$. Thus, $T = \bigoplus_i ke_i$ is a central Wedderburn factor of $A_i$.

3. Again by 2 $A_i = ke_i \oplus R_i$ so that $A_i/R_i \cong k$ as a field. This is surely a purely inseparable extension of $k$.

4. If $T$ is the central Wedderburn factor of $A_i$ then $T \subseteq Z(A_i)$. Since $A_i \subseteq T \oplus R_i$ it follows that $A_i \subseteq Z(A_i) \subseteq R_i$.

5. By 5 any $a \in A_i$ can be written $a = z + r$ with $z \in Z(A_i)$, $r \in R_i$. Let $\pi: A_i \to A_i / I_i$ be the canonical map. Clearly $\pi(z)$ commutes with $\pi(z)$ and $\pi(r)$.

6. 7 is clear since associative ideals in $A_i$ are Lie ideals in $A_i$.

7. 8 follows from a trick. For $e$ an idempotent in $A_i$, $a \in A_i$,

$$[e, a] = ea - ae,$$
$$[e, [e, a]] = ea - 2ae + ae,$$
$$[e, [e, [e, a]]] = ea - ae.$$

Thus, $([e, -] \cdot e) \cdot e = (ad e)^i$ for odd $i$. By Engel's theorem [2, p. 12, Corollary 2] $ad e$ must be nilpotent if $A_i$ is a nilpotent Lie algebra. Thus, $ad e = 0$ and $e$ is central.

7. Since $A = \bigoplus A_i$, if $A_i$ is Lie nilpotent, then $A_i$ is Lie nilpotent, and $(A_i/R_i)$ is Lie nilpotent. Thus, $A_i/R_i$ satisfies 7 and since it has been shown that 7 implies 8 it follows that all idempotents in $A_i/R_i$ are central. By Wedderburn theory $A_i/R_i$ is the tensor product over $k$ of a division algebra and a matrix algebra. Since all idempotents in $A_i/R_i$ are central the matrix algebra must be $k_i(1 \times 1$ matrices). Thus, $A_i/R_i$ is a division algebra. Let $K$ be the center of $A_i/R_i$ and let $L$ be a splitting field for $A_i/R_i$ over $K_i$, thus,

$$(A_i/R_i) \otimes_K L \cong M(t, L)$$

for some $t$. There is a natural associative algebra surjection

$$(A_i/R_i) \otimes_K L \twoheadrightarrow (A_i/R_i) \otimes_K L \cong M(t, L).$$
Since \((A_i/R_i)\) is Lie nilpotent so is \((A_i/R_i) \otimes_k L\) and hence \(M(tL)\) must be Lie nilpotent. This implies that \(t = 1\). Thus (by counting dimension say) \(K = A_i/R_i\), which gives statement 9.

4 \(\Rightarrow\) 7. Since \(A = \bigoplus A_i\) to show that \(A\) is Lie nilpotent it suffices to prove that each \(A_i\) is Lie nilpotent. Thus we may assume that \(A\) is indecomposable and \(M \twoheadrightarrow A/R\) is a purely inseparable field extension of \(k\).

The natural map \(M \otimes_k M \to M\) is surjective with nilpotent kernel. Thus the radical of \(A \otimes_k M\) (as an \(M\) algebra) has \(k \otimes_k M\) as a complement and \(A \otimes_k M\) satisfies condition 3. By what we have already shown we know that 3 \(\Rightarrow\) 7. Thus, \((A \otimes_k M)^-\) is Lie nilpotent which implies that \(A\) is Lie nilpotent.

This completes the proof of part (a). When \(k\) is separably closed 9 and 4 are the same statements so that part (b) follows from part (a).

Suppose \(k\) is perfect. Then \(A\) admits a Wedderburn factor \(W\), where \(A \twoheadrightarrow W \subset R\). Moreover, \(R \otimes_k k\) is the radical of \(A \otimes_k k\) and \(W \otimes_k k\) is a Wedderburn factor. If \(A^\cdot\) is Lie nilpotent then so is \(A \otimes_k k\) and by 7 \(\Rightarrow\) 8 it follows that all idempotents in \(A \otimes_k k\) are central. Since \(W \otimes_k k\) is a semisimple \(k\) algebra it is the direct sum of matrix algebras over \(k\). The condition of idempotents being central implies that all the matrix algebras must be isomorphic to \(k\). Thus,

\[ W \otimes_k k \cong k \oplus \cdots \oplus k, \]

and \(W \otimes_k k\) is spanned by idempotents. Again using that idempotents of \(A \otimes_k k\) are central it follows that \(W \otimes_k k\) is central in \(A \otimes_k k\). This implies that \(W\) is central in \(A\), and condition 3 is satisfied. Now part (c) follows from part (a).

Finally suppose that \(k\) is algebraically closed. Let \(W\) be a Wedderburn factor in \(A\) and suppose that \(A\) satisfies condition 8. As shown just above in the proof of part (c), (for \(W \otimes_k k\)) it follows that \(W \cong k \oplus \cdots \oplus k\) and \(W\) is central in \(A\). Let \(e\) be an idempotent in \(A\). Then \(k \cdot 1 \oplus k \cdot e\) is a separable subalgebra of \(A\), and there is an invertible element \(u \in A\), where \(u(k \cdot 1 \oplus k \cdot e)u^{-1} \subset W\). Thus, \(ueu^{-1} \in W\) and \(e = u^{-1}(ueu^{-1})u \in u^{-1}Wu\).

Since \(W\) is central in \(A\) it follows that \(e \in W\). Let \(\{f_i\}\) be the minimal orthogonal idempotents of \(W\). Then \(\{f_i\}\) are the minimal orthogonal central idempotents of \(A\) and \(\{f_i\} = \{e_i\}\). Thus from

\[ A = (\bigoplus ke_i) \oplus R\]

it follows that \(A_i = Ae_i = ke_i + Re_i\). As observed earlier \(A_i = ke_i \oplus R_i\) implies condition 2. Now 8 \(\Rightarrow\) 2 together with parts (a), (b), and (c) gives part (d).
Proof of Theorem II

1 \Rightarrow 2. Let \( M \) be a faithful finite dimensional \( A \)-module. By condition 2, \( M \) has a chain of submodules

\[ 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t, \]

where \( M_{i+1}/M_i \) is one dimensional for \( i = 0, \ldots, t-1 \). Choose \( m_i \in M_i \) where \( m_i \notin M_{i-1} \) for \( i = 1, \ldots, t \). Then \( \{m_{i-1}^{t+1}\} \) is a basis for \( M \) and the representation of \( A \) by matrices arising from the basis \( \{m_{i-1}^{t+1}\} \) is upper triangular.

Conversely let \( \nabla(t, k) \) denote the algebra of upper triangular \( t \times t \) matrices. Let \( \nabla(t, k) \) denote the strictly upper triangular \( t \times t \) matrices, (zeros along the diagonal). Then \( \nabla(t, k) \) is the radical of \( \nabla(t, k) \) and \( \nabla(t, k) \) \( \nabla(t, k) \cong k \oplus \cdots \oplus k \) (\( t \) times). Let \( \pi: \nabla(t, k) \rightarrow k \oplus \cdots \oplus k \) (\( t \) times) be a surjective algebra homomorphism with kernel \( \nabla(t, k) \).

If \( A \) is a subalgebra of \( \nabla(t, k) \) then \( A \cap \nabla(t, k) \) lies in the radical of \( A \). Since \( \pi(A) \) is a subalgebra of \( k \oplus \cdots \oplus k \) (\( t \)-times) it follows that \( \pi(A) \) is of the form \( k \oplus \cdots \oplus k \) (a number of times).

Thus, \( \pi(A) \) is semisimple and \( A \cap \nabla(t, k) \) is the radical of \( A \). Thus, \( A/R \cong k \oplus \cdots \oplus k \) (a number of times) and all simple \( A \)-modules are one dimensional.

1 \Rightarrow 3. As shown immediately above, condition 1 implies that \( A/R \cong k \oplus \cdots \oplus k \) a separable \( k \)-algebra. Thus, \( A \) has a Wedderburn factor \( W \cong k \oplus \cdots \oplus k \) and \( W \) is commutative. This given condition 3.

3 \Rightarrow 4. Let \( \pi \) be the natural map from \( A \rightarrow A/R \). By condition 3, \( \pi(B) = A/R \) and so condition 4 holds.

4 \Rightarrow 5. Since

\[ R = \bigcap_{\mathcal{M} \leq A} \mathcal{M} \]

4 \Rightarrow 6. Follows simply letting \( A = I_0 \) and \( R^j = I_j \) for \( j > 0 \).

6 \Rightarrow 7. 7 \Rightarrow 8. Both of these implications are clear.

7 \Rightarrow 5. In this demonstration of 7 \Rightarrow 5 the term "algebra" is used to mean associative algebra not necessarily with unit and so no unit condition is assumed about subalgebras. Assume that \( A \) satisfies statement 7 and that \( \mathcal{H} \) is a maximal two-sided ideal of \( A \). Choose \( t \) where

\[ I_t \varsubsetneq \mathcal{H} \supset I_{t-1}. \]

Let \( \mathcal{H}_t = I_t \cap \mathcal{H} \) a two-sided ideal in \( I_t \), (\( I_t \) is a subalgebra of \( A \)).

Suppose \( t = 0 \). Then \( \mathcal{H} \supset I_1 \) and \( A/\mathcal{H} \) is commutative since \( A/I_1 \) is.
Suppose \( t > 0 \). Then \( \mathcal{M} \not
subseteq I_1 \) and by maximality of \( \mathcal{M} \) it follows that
\[ A \cap \mathcal{M} = I_1. \]
Thus,
\[ A \cap \mathcal{M} = \frac{\mathcal{M} + I_1}{\mathcal{M}} \cong \frac{I_1}{I_1 \cap \mathcal{M}} = \frac{I_1}{I_1}. \]
If \( t = 1 \) then \( \mathcal{M}_1 \subseteq I_2 \) and \( A \cap \mathcal{M} \cong I_1 \cap \mathcal{M} \) is commutative since \( I_1 \cap I_2 \) is commutative. If \( t > 1 \) then \( \mathcal{M}_1 \not
subseteq I_2 \).

Since \( \mathcal{M}_1 \cap \mathcal{M} \cong A \cap \mathcal{M} \), it follows that \( \mathcal{M}_1 \) is a maximal ideal in \( I_1 \) and so
\[ \mathcal{M}_1 = I_1. \]
Thus
\[ A \cap \mathcal{M} \cong \frac{I_1}{\mathcal{M}_1} \cong \frac{I_1}{\mathcal{M}_2} \cong \ldots \cong \frac{I_t}{\mathcal{M}_t}, \]
Continuing we end up with
\[ A \cap \mathcal{M} \cong \frac{I_1}{\mathcal{M}_1} \cong \frac{I_2}{\mathcal{M}_2} \cong \ldots \cong \frac{I_t}{\mathcal{M}_t}, \]
and \( I_t \cap \mathcal{M}_t \) is commutative since \( \mathcal{M}_t \subseteq I_{t+1} \). This shows that \( 7 \Rightarrow 5 \).

We have proved part (a).

Suppose \( k \) is algebraically closed. Then condition 5 implies that
\[ A \cap \mathcal{M} \cong k \]
for each maximal ideal \( \mathcal{M} \) of \( A \).

This certainly implies condition 2, which together with part (a) gives part (b).

Suppose \( k \) is perfect then \( A \) has a Wedderburn factor \( W \) and \( W \cong A/\mathcal{R} \). Thus \( W \) is commutative if condition 4 holds and hence condition 3 holds. Together with part (a) this gives part (c).

Finally suppose condition 8 holds for \( \mathcal{M} \). Let \( k \) be the algebraic closure of \( k \).

Then condition 8 holds for \( A \otimes_k k \). Let \( \mathcal{R} \) denote the radical of \( A \otimes_k k \).

Since \( k \) is algebraically closed \( (A \otimes_k k)/\mathcal{R} \) is the direct sum of matrix algebras over \( k \) and since \( (A \otimes_k k) \) is Lie solvable, the matrix algebras must be Lie solvable.

What are the Lie solvable matrix algebras over \( k \)? Only \( k \) when the characteristic of \( k \) is different from 2. When the characteristic of \( k \) is 2 the Lie solvable matrix algebras over \( k \) are \( k \) and \( M(2, k) \). These results are easily verified.

Suppose the characteristic of \( k \) is different from 2. Then \( (A \otimes_k k)/\mathcal{R} \cong k \oplus \cdots \oplus k \) (a number of times). Thus, \( A \otimes_k k \) satisfies 4. We have already shown that condition 4 implies condition 7 and that if condition 7 holds for \( A \otimes_k k \) then it holds for \( A \). Thus with part (a) we have part (d).

Suppose \( k \) has characteristic 2 and \( A \otimes_k k \) has no two dimensional simple modules then \( M(2, k) \) cannot occur among the matrix algebras over \( k \) of which \( (A \otimes_k k)/\mathcal{R} \) is the direct sum. Thus \( (A \otimes_k k)/\mathcal{R} \cong k \oplus \cdots \oplus k \) (a number of times). As in the preceding paragraph this shows that \( A \) satisfies condition 7. With part (a) this proves part (e).
REFERENCES