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On the Hojman conservation quantities in Cosmology

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ABSTRACT

We discuss the application of the Hojman's Symmetry Approach for the determination of conservation laws in Cosmology, which has been recently applied by various authors in different cosmological models. We show that Hojman's method for regular Hamiltonian systems, where the Hamiltonian function is one of the involved equations of the system, is equivalent to the application of Noether's Theorem for generalized transformations. That means that for minimally-coupled scalar field cosmology or other modified theories which are conformally related with scalar-field cosmology, like $f(R)$ gravity, the application of Hojman's method provide us with the same results with that of Noether's Theorem. Moreover we study the special Ansatz. $\phi(t) = \phi(a(t))$, which has been introduced for a minimally-coupled scalar field, and we study the Lie and Noether point symmetries for the reduced equation. We show that under this Ansatz, the unknown function of the model cannot be constrained by the requirement of the existence of a conservation law and that the Hojman conservation quantity which arises for the reduced equation is nothing more than the functional form of Noetherian conservation laws for the free particle. On the other hand, for $f(T)$ teleparallel gravity, it is not the existence of Hojman's conservation laws which provide us with the special function form of $f(T)$ functions, but the requirement that the reduced second-order differential equation admits a Jacobi Last multiplier, while the new conservation law is nothing else that the Hamiltonian function of the reduced equation.

The source for the late-time cosmic acceleration [1–4] has been attributed to an unidentified type of matter–energy with a negative parameter in the equation of state, the dark energy. The cosmological constant, Λ, is the simplest candidate for dark energy with a parameter in the equation of state, $w_\Lambda = -1$. However, the cosmological constant suffers from two major problems. They are the fine tuning and the coincidence problems [5,6]. Consequently other dark energy candidates have been introduced, such as cosmologies with time-varying $\Lambda(t)$, quintessence, Chaplygin gas, matter creation, $f(R)$ gravity and many others. This status of art imposes a discrimination among the various cosmological models and the search for new approaches to find out exact solutions to be matched with data. In particular, one of the issue is that cosmological models should come out from some first principles in order to be related to some fundamental theory.

Below we restrict our consideration to a spatially flat Friedmann–Robertson–Walker (FRW) spacetime with line element

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{1}$$

Let us start our considerations from cosmological solutions derived from General Relativity (GR). Standard GR provides us with a set of second-order differential equations. Consider a Riemannian manifold $\mathcal{M}^4$, induced with a metric tensor g_{ij}, and GR with cosmological constant, Λ. The Action integral of the field equations is as follows:

$$S = \int dx^4 \sqrt{-g}\,(R - 2\Lambda), \tag{2}$$

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where R is the Ricci scalar of $\mathcal{M}^4$. For the spacetime, (1), the Ricci scalar is

$$R=6\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right], \tag{3}$$

and from the variational principle for the Action integral, (2), we have the set of differential equations,

$$-3a\dot{a}^2+2a^3\Lambda=0 \tag{4}$$

and

$$\ddot{a}+\frac{1}{2a}\dot{a}^2-a\Lambda=0, \tag{5}$$

where $\dot{a}=da/dt$. It is well known that the solution of the system (4), (5) is

$$a(t)=a_0\exp\left[\sqrt{\frac{2\Lambda}{3}}t\right] \tag{6}$$

which is the de Sitter solution. In terms of dynamics, the system (4), (5) is that of a one-dimensional hyperbolic oscillator for which equation (4) can be interpreted as the Hamiltonian constraint on the oscillator. There are two ways to observe this. The first way is to apply the "coordinate transformation" $a(t)=r(t)^{\frac{2}{3}}$. When we use this transformation, the system, (4), (5), reduces to the simplest form,

$$-\frac{1}{2}\dot{r}^2+\frac{\omega^2}{2}r^2=0\ ,\quad \ddot{r}-\omega^2 r=0 \text{ with } \omega^2=\frac{3}{2}\Lambda. \tag{7}$$

The second way is to study the point symmetries. We consider the Noether point symmetries of the Lagrangian of the field equations. The Lagrangian which follows from (2) is

$$L(a,\dot{a})=3a\dot{a}^2+2\Lambda a^3 \tag{8}$$

and we can easily see that this admits five Noether point symmetries. Moreover, because $L(a,\dot{a})$ admits the Noether algebra of maximal dimension, this indicates that $L(a,\dot{a})$ can describe two systems: the one-dimensional free particle and the one-dimensional linear equation[1] [7]. However, as the potential in $L(a,\dot{a})$ is not constant, we can say that the dynamical system is the one-dimensional linear system and, specifically, that of the hyperbolic oscillator, when Λ is a positive constant.

Consider now as a candidate for dark energy a quintessence scalar field with action

$$S_\phi=\int dx^4\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\phi^{;\mu}\phi^{;\nu}+V(\phi)\right). \tag{9}$$

Consequently, for a model with a scalar field, the Action integral of the field equations is

$$S=\int dx^4\sqrt{-g}R+S_\phi,$$

and for the FRW spacetime, (1), the field equations comprise the set of differential equations,

$$-3a\dot{a}^2+\frac{1}{2}a^3\dot{\phi}^2+V(\phi)=0, \tag{10}$$

and

$$\ddot{a}+\frac{1}{2a}\dot{a}^2+\frac{a}{2}\dot{\phi}^2-aV(\phi)=0, \tag{11}$$

where the scalar field, $\phi(t)$, satisfies the equation

$$\ddot{\phi}+\frac{3}{a}\dot{a}\dot{\phi}+V(\phi)_{,\phi}=0. \tag{12}$$

Similarly equation (10) can be viewed as the Hamiltonian constraint[2] of equations (11) and (12). In order that we can study the (Liouville) integrability of the Hamiltonian system (10)–(12), we have to study the existence of conservation laws. As the system has dimension two, being the configuration space $\mathbb{Q}\equiv\{a,\phi\}$, and (10) can be seen as a conservation law, we have to determine the exact form of the potential $V(\phi)$ in which the system admits additional conservation laws. We remark that Liouville integrability means that the field equations can be reduced to quadratures from which we can seek for determining the solution of the scalar factor, $a(t)$, in a closed form. The most common method to determine conservation laws for Hamiltonian systems is the well-known Noether Symmetry Approach. This has lead many researchers to the study of the Noether symmetries for dynamical system (10)–(12) (see for example [8–11] and reference therein).

Recently, in [12], it has been proposed that the unknown potential of the scalar field be constrained by the existence of Hojman conserved quantities. The results have been applied also to the cosmological scenario with a nonminimally coupled scalar-field model [13], in $f(R)$-gravity in the metric formalism [14] and in $f(T)$-gravity [15]. In what follows we follow the notation by [12, 13].

Hojman proved that, if a system of second-order differential equations,

$$\ddot{q}^i=F^i\left(t,q^k,\dot{q}^k\right), \tag{13}$$

is invariant under the transformation,

$$q'^i=q^i+X^i\left(t,q^k,\dot{q}^k\right), \tag{14}$$

then the quantity $Q\left(t,q^i,\dot{q}^i\right)$ with form

$$Q=\frac{\partial X^i}{\partial q^i}+\frac{\partial}{\partial\dot{q}^i}\left(\frac{dX^i}{dt}\right) \tag{15}$$

for

$$\frac{\partial F^i}{\partial\dot{q}^i}=0, \tag{16}$$

or

$$Q=\frac{1}{\gamma}\frac{\partial\left(\gamma X^i\right)}{\partial q^i}+\frac{\partial}{\partial\dot{q}^i}\left(\frac{dX^i}{dt}\right) \tag{17}$$

for

$$\frac{\partial F^i}{\partial\dot{q}^i}=-\frac{d}{dt}\ln\gamma, \tag{18}$$

is a conservation law of (13), i.e. $\frac{dQ}{dt}=0$, where $\gamma=\gamma\left(q^k\right)$ [16]. We remark that the differential equation (13) is invariant under the action of (14) and this means that the following condition holds

$$\ddot{X}^i-\frac{\partial F^i}{\partial q^j}X^j-\frac{\partial F^i}{\partial\dot{q}^j}\dot{X}^j=0. \tag{19}$$

[1] They are mathematically equivalent under a point transformation, but we prefer to maintain a physical distinction.

[2] We note that in a lapse time $dt=N(\tau)d\tau$ of (1), equations (4) and (10) arise from the variation of the variable $N(\tau)$ and are restricted to that form when we consider $N(\tau)=1$.

Furthermore, conditions (16) or (18) are necessary to hold in order the conservation law to exist.

In order to simplify the problem the authors in [12] reduced the two-dimensional dynamical system $\{a, \phi\}$ of the field equations to a one-dimensional system in $\{a\}$ by selecting $\phi(t) = \phi(a(t))$. When this is used, equation (11), with the use of (10), becomes

$$\ddot{x} + f(x)\dot{x}^2 = 0, \tag{20}$$

where $x = \ln a$, and $f(x) = \frac{1}{2}\left(\frac{d\phi(x)}{dx}\right)^2$. Moreover, from equation (12) and (10), we have

$$(\ln V(\phi))_{,\phi} = \frac{f(x)\phi_{,x} - \phi_{,xx} - 3\phi_{,x}}{3 - \frac{1}{2}(\phi_{,x})^2}. \tag{21}$$

In [12] the authors substituted $F(t, x, \dot{x}) = -f(x)\dot{x}^2$ into (19) and constrained the function $f(x)$ in order that equation (20) admit Hojman conservation laws. Furthermore from (21) they determined the potential $V(\phi)$ and found closed-form cosmological solutions.

We continue by studying the Lie point symmetries of the second-order differential equation (20). We consider a point transformation in the space $\{t, x\}$ and we find that equation (20) is invariant under an eight-dimensional Lie algebra of point transformations, automatically $sl(3, R)$, for arbitrary functional form of $f(x)$. Therefore the Lie theorem holds, i.e., it means that there exists a "coordinate" transformation $\{t, x\} \to \{\tau, y\}$, whereby equation (20) is reduced to that of the free particle, $y'' = 0$ [17]. By using the Lie symmetries, we have that the transformation is

$$t = t \ , \ \ y = \int \exp\left(\int f(x)\,dx\right) dx \tag{22}$$

whence (21) becomes $\ddot{y} = 0$. Hence $y(t) = y_1 t + y_0$, $y_1, y_0 \in \mathbb{R}$, that is $\int \exp\left(\int f(x)\,dx\right) dx = y_1 t + y_0$ which is the solution for the scale factor $a(t) = \exp[x(t)]$ for an arbitrary function, $f(x)$. Moreover the potential is constrained by equation (21).

The function $L_f(y, \dot{y}) = \frac{1}{2}\dot{y}^2$, is one of the possible Lagrangians of the free particle. It is straightforward to see that L_f admits five Noetherian point symmetries, the vector fields

$$Z_1 = \partial_t \ , \ Z_2 = \partial_y \ , \ Z_3 = t\partial_y \tag{23}$$

$$Z_4 = 2t\partial_t + y\partial_y \ , \ Z_5 = t^2\partial_t + ty\partial_y \tag{24}$$

for which the corresponding Noetherian conservation laws are

$$Z_1 : \mathcal{H}_f = \frac{1}{2}\dot{y}^2 \ , \ Z_2 : I_p = \dot{y} \ , \ Z_3 : I'_p = t\dot{y} - y \tag{25}$$

$$Z_4 : I_s = 2t\mathcal{H}_f - y\dot{y} \ , \ Z_5 : I'_s = t^2\mathcal{H}_f - ty\dot{y} + \frac{1}{2}y^2. \tag{26}$$

We remark that the conservation laws, $\mathcal{H}_f$, and I_p, are the well-known conservation of energy and momentum, in particular it holds that $\mathcal{H}_f = \frac{1}{2}(I_p)^2$. Furthermore, under the coordinate transformation (22), the momentum, I_p, becomes $I_p = \exp\left(\int f(x)\,dx\right)\dot{x}$.

We continue with the determination of the Hojman conservation laws for equation (20). Without loss of generality, we study the Hojman conservation laws of $\ddot{y} = 0$. For this equation, condition (19) becomes $\ddot{X} = 0$, that is,

$$X_{,tt} + 2\dot{y}X_{,ty} + \dot{y}^2 X_{,yy} = 0. \tag{27}$$

Hence

$$X(x, y, \dot{y}) = X_1(t\dot{y} - y, \dot{y}) + tX_2(t\dot{y} - y, \dot{y}), \tag{28}$$

or

$$X(x, y, \dot{y}) = X_1\left(I_p, I'_p\right) + tX_2\left(I_p, I'_p\right).$$

It is easy to see that, when $\frac{\partial X}{\partial \dot{y}} = 0$, we have the Lie symmetries Z_2, Z_3 and $Y_L = y\partial_y$. Therefore from (15) we have the general Hojman conservation law

$$Q(t, y, \dot{y}) = Q(I'_p, I_p) = \frac{\partial}{\partial I'_p} X_1(I'_p, I_p), \tag{29}$$

Consequently, for equation (20), the Hojman conservation law has the following form

$$Q(t, x, \dot{x}) = Q\left(I_p(t, x, \dot{x}), I'_p(t, x, \dot{x})\right), \tag{30}$$

where now

$$I_p = \exp\left(\int f(x)\,dx\right)\dot{x} \tag{31}$$

$$I'_p = t\exp\left(\int f(x)\,dx\right)\dot{x} - \int \exp\left(\int f(x)\,dx\right)dx. \tag{32}$$

We have proved that equation (20) admits conservation laws for an arbitrary function $f(x)$, and that the Hojman conservation law $Q\left(I_p\right)$ for that case is a functional form of Noetherian conservation laws for the free particle.

We conclude that, under the Ansatz. $\phi(t) = \phi(a(t))$, the reduced field equation (20) is integrable for arbitrary function $f(x)$, i.e. Eq. (20) admits conservation laws for any $f(x)$. On the other hand, the Ansatz. $\phi = \phi(a)$, reduces the field equation in a one-dimensional autonomous dynamical system, which is integrable. Moreover the Hojman conservation quantities for equation (20) are functions of the Noetherian conservation laws of the free particle. Therefore there are no difference among the two methods at that level, that is, the claim in [12] that the class of scalar field potentials $V(\phi)$ is constrained by condition (19) for equation, (20) means that Hojman conservation laws exists when Noetherian conservation laws exists.

Furthermore we remark that the solutions which have been found in [12] are special solutions and are not the general solutions of the field equations in the sense that they hold when $\phi = \phi(a)$ and they are restricted to the case for which there exists an inverse function $t = a^{-1}(t)$. This can be seen easily for the closed-form solution are given for the exponential potential $V(\phi) = V_0 e^{\lambda\phi}$ in [12]. The solution found there is a power law. However, the solution for that model is more general and can be found in [19].

As far as concerns the application of Hojman's conservation quantities in nonminimally coupled scalar field cosmology [13], or in $f(R)$ gravity in the metric formalism [14] the situation is similar with above. Note that $f(R)$ gravity in the metric formalism can be seen as a Brans–Dicke-like model with vanishing Brans–Dicke parameter, i.e. the O' Hanlon gravity [20]. These theories are related under conformal transformations and are equivalent with the minimally coupled scalar field [21]. Moreover when there is no other matter source, then the solution of the field equations holds either in the Einstein or in the Jordan frame [22].

Because of that, the same Ansatz for the application of Hojman's method in scalar tensor theories can be used, i.e. the field "ϕ" to be a function of the scale factor. Hence the field equations reduced to a second-order ordinary differential equation of the form of (20). Therefore the above analysis and comments hold.

In the case of $f(T)$ gravity the situation is different since the field equations are a singular one-dimensional dynamical system.

Following [15], we take into account a spatially flat FRW spacetime (1) and the basis $e^{\alpha}\left(x^{\mu}\right)=h_{\mu}^{\alpha}dx^{\mu}$, such as, $g_{\mu\nu}=\eta_{\alpha\beta}e^{\alpha}e^{\beta}=\eta_{a\beta}h_{\mu}^{\alpha}h_{\mu}^{\beta}$, where

$$h_{\mu}^{i}=diag\left(1,e^{x(t)},e^{x(t)},e^{x(t)}\right). \tag{33}$$

$f(T)$ gravity is a straightforward extension of Teleparallel GR (TEGR), such as $f(R)$ gravity of GR. The Action integral in $f(T)$ gravity with a matter term is

$$A_{T}=\int dx^{4}\left|e\right|f(T)+\int dx^{4}\left|e\right|L_{m}, \tag{34}$$

where L_m is the Lagrangian of the matter term, and T is a function of the Torsion scalar (for details see [23,24]). For the basis (33) from the Action integral (34), for a perfect fluid with constant equation of state parameter $p=w\rho$, we have that $T=-6H^{2}$, $H=\dot{x}(t)$, the field equations are:

$$12f_{,T}H+f=\rho \tag{35}$$

$$48H^{2}f_{,TT}\dot{H}-f_{,T}\left(12H^{2}+4\dot{H}\right)-f=p \tag{36}$$

and the conservation law for the fluid, $\dot{\rho}+3(1+w)\rho H=0$, which gives $\rho=\rho_{m0}e^{-3(1+w)x}$. The field equations (35), (36) form a singular one-dimensional Hamiltonian system since $\det\left|\frac{\partial L}{\partial x^{i}}\right|=0$. Furthermore equation (36) is a second-order differential equation of $x(t)$, while (35) is a first order equation of $x(t)$. Typically, the system is integrable however it is not always possible to find the solution in closed-form. A classical analogue is the closed form solution of the one-dimensional Newtonian system, $E=\frac{1}{2}\dot{y}^{2}+V(y)$, which admits closed-form solutions for functions $V(x)$, where the equations of motion admit Lie symmetries. For the field equations (35), (36), the unknown function is defined by the Lagrange multiplier $T=-6H^{2}$, where, again, the existence of point symmetries provides closed-form solutions. With the use of (35) and the Lagrange multiplier T, equation (36) can be written in the following form,

$$\ddot{x}=F(\dot{x}), \tag{37}$$

where $F(\dot{x})=F(T)=F(f(T))$. In order to exist a Hojman conservation law for the second-order differential equation (37), condition (16) or (18) have to hold. In other words, $F(\dot{x})=-F_{1}\dot{x}^{2}-F_{0}$. This means that equation (37) is

$$\ddot{x}+F_{1}\dot{x}^{2}+F_{0}=0 \tag{38}$$

where the general solution is

$$x(t)=\frac{1}{2F_{1}}\ln\left[\frac{F_{1}}{F_{0}}\left(x_{1}\cos\left(\sqrt{F_{1}F_{0}}t\right)-x_{2}\sin\left(\sqrt{F_{1}F_{0}}t\right)\right)^{2}\right]. \tag{39}$$

However, equation (38) admits eight Lie point symmetries, i.e. the $sl(3,R)$, and according to the Lie theorem, it is equivalent to the free particle, $y''=0$. Moreover it is easy to see that equation (38) follows from the Lagrangian

$$L(x,\dot{x})=\frac{1}{2}e^{2F_{1}x}\dot{x}^{2}-F_{0}e^{2F_{1}x}, \tag{40}$$

and, since the latter is autonomous, admits as Noetherian conservation law the Hamiltonian function. In the case where $F_0=0$, (in the notation of [15], $c=0$), Lagrangian (40) is that of the free particle and also the momentum is another time-independent Noetherian conservation law. That is the conservations laws in [15] are not conservation laws which follows from the Hojman's formula but from Noetherian symmetries in the same way we discussed above.

However condition (18) for equation (37) is equivalent with the existence of a Jacobi Last multiplier for equation (37). The existence of a Jacobi Last multiplier for one-dimensional second order differential equations of the form of (13) is equivalent to the existence of a Lagrangian [25,26], and, in the simplest case that the dynamical system is autonomous, Hojman conservation laws are equivalent to the Noetherian conservation laws. Finally, the functional forms of $f(T)$, which follow from the solution of the system

$$F(\dot{x})=-F_{1}\dot{x}^{2}-F_{0}, \tag{41}$$

are not arising from the existent of a Hojman conservation law, but from the existence of a Jacobi Last multiplier for the reduced equation (37) and the conservation laws are Noetherian conservation laws, the well known conservation laws of the Hamiltonian or of the momentum for the "oscillator" or the free particle.

In conclusion, for regular dynamical systems, such as scalar-field cosmology, we consider the derivation of Hojman conservation laws for the field equations (11), (12) without the Ansatz. $\phi(t)=\phi(a(t))$. Hence we have to determine the group of invariant transformations of the system (11), (12). Moreover, at the same time the Hamiltonian function $\mathcal{H}=0$, i.e. equation (11), should be also invariant, which means that the following condition

$$X^{[1]}(\mathcal{H})=\lambda\mathcal{H}, \tag{42}$$

has to hold, where $X^{[1]}$ is the first prolongation of X and λ is an arbitrary function. However, equation (42) is nothing else than the Noether condition in Hamiltonian formalism for generalized symmetries (see [18,27]). In the case of point and contact transformations, this is a well known result [28,29]. Therefore the determination of Hojman conservation quantities in cosmological models, which arise from a Lagrange function, it is equivalent with the application of Noether's Theorem. That holds for all the physical systems where the Hamiltonian function is one of the equations involved.

When the field equations are a singular dynamical system, as in the case of $f(T)$ gravity, the functional forms of $f(T)$, in which the field equations admit a closed-form solution, follows from the existence of a Jacobi Last multiplier, and the conservation laws are again Noetherian.

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