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Vertex Ramsey Properties of Families of Graphs

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For graphs $F, G_1, \ldots, G_r$, we write $F \to (G_1, \ldots, G_r)$ if for every coloring of the vertices of F with r colors there exists i, $i = 1, 2, \ldots, r$, such that a copy of G_i is colored with the ith color. For two families of graphs $G_1, \ldots, G_r$ and $H_1, \ldots, H_s$, by $(G_1, \ldots, G_r) \to (H_1, \ldots, H_s)$ we denote the fact that $F \to (G_1, \ldots, G_r)$ implies $F \to (H_1, \ldots, H_s)$ for every graph F. In this paper, we give necessary and sufficient

problem.

1. RESULTS

For a natural number r and graphs $F, G_1, \ldots, G_r$, we write $F \to (G_1, \ldots, G_r)$ if for every coloring of the *vertices* of F with r colors there exists $i \in \{1, 2, \ldots, r\}$ and a copy of the graph G_i with all vertices colored by the ith color. Thus, every finite family of graphs $G_1, \ldots, G_r$ can be associated with a (vertex) Ramsey property. The aim of this paper is to study how this property depends on the structure of graphs $G_1, \ldots, G_r$. In order to do that, we introduce first a natural relation between Ramsey properties. Let $\mathbf{G} = G_1, \ldots, G_r$ and $\mathbf{H} = H_1, \ldots, H_s$ be two families of graphs, where we allow repetitions in each of the families. If for every graph F such that $F \to (\mathbf{G})$ we also have $F \to (\mathbf{H})$, we write $(\mathbf{G}) \to (\mathbf{H})$. Finally, if $(\mathbf{G}) \to (\mathbf{H})$ and $(\mathbf{H}) \to (\mathbf{G})$, we write $(\mathbf{G}) \leftrightarrow (\mathbf{H})$.

Note that the relation "$\to$" is transitive and $(G) \to (H)$ means that H is a subgraph of G, so that $(G) \to (\mathbf{H})$ if and only if $G \to (\mathbf{H})$. Consequently, the notation $(\mathbf{G}) \to (\mathbf{H})$ can be viewed as a generalization of the standard arrow notation used in Ramsey theory.

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We start with a few simple observations. Assume that $(G_1, \ldots, G_r) \to (H_1, \ldots, H_s)$. Then,

$$\sum_{i=1}^{r} (|G_i|-1) \geqslant \sum_{j=1}^{s} (|H_j|-1), \tag{1}$$

since otherwise, for the complete graph K_a on a vertices, where $a = \sum_i (|G_i|-1)+1$, we have, by the pigeonhole principle, $K_a \to (G_1, \ldots, G_r)$, while $K_a \nrightarrow (H_1, \ldots, H_s)$. Similarly,

$$\sum_{i=1}^{r} (\chi(G_i)-1) \geqslant \sum_{j=1}^{s} (\chi(H_j)-1), \tag{2}$$

since otherwise, for every complete t-partite graph F with $t = \sum_{i=1}^{r} (\chi(G_i)-1)+1$, in which each of the partition sets is large enough (say, larger than $\sum_{i=1}^{r} |G_i|$), we have $F \to (G_1, \ldots, G_r)$ and $F \nrightarrow (H_1, \ldots, H_s)$.

Are any of conditions (1) and (2) sufficient to imply $(G_1, \ldots, G_r) \to (H_1, \ldots, H_s)$? Certainly, (1) is, for graphs which contain no edges. Thus, in order to avoid this and similar pathological cases, throughout the paper we assume that each of the graphs $H_1, \ldots, H_s$ is connected and contains at least one edge (i.e., is not equal to K_1). For a moment let us strengthen this assumption slightly and suppose that each of the graphs $H_1, \ldots, H_s$ is 2-connected. (Throughout this paper, for convenience, the graph K_2 is declared to be 2-connected.) The following result was proved by Nešetřil and Rödl in [3].

Lemma 1. *If H is a 2-connected graph and none of the graphs $F_1, \ldots, F_r$ contains H, then there exists a graph K which does not contain H such that $K \to (F_1, \ldots, F_r)$.*

As an immediate consequence of Lemma 1 we get that if $(G_1, \ldots, G_r) \to (H_1, \ldots, H_s)$ and all $H_1, \ldots, H_s$ are 2-connected, then for every H_j, $1 \leqslant j \leqslant s$, there exists G_i, $1 \leqslant i \leqslant r$, such that $H_j \subseteq G_i$, i.e., $G_i \to H_j$. On the other hand, if for every i, $1 \leqslant i \leqslant r$, we have $G_i \to (H_i^1, \ldots, H_i^{s_i})$, then clearly

$$(G_1, \ldots, G_r) \to (H_1^1, \ldots, H_1^{s_1}, H_2^1, \ldots, H_2^{s_2}, \ldots, H_r^1, \ldots, H_r^{s_r}). \tag{3}$$

Our first result states that, basically, if we deal with 2-connected graphs, each relation $(G_1, \ldots, G_r) \to (H_1, \ldots, H_s)$ is of the type described by (3).

Theorem 1. *Let $\mathbf{G} = G_1, \ldots, G_r$ and $\mathbf{H} = H_1, \ldots, H_s$ be families of graphs such that for every $j = 1, 2, \ldots, s$, H_j is 2-connected. Then the following two conditions are equivalent.*

(i) $(G_1, ..., G_r) \to (H_1, ..., H_s)$,

(ii) *there exists a partition* $A_1 \cup A_2 \cup \cdots \cup A_r = \{1, 2, ..., s\}$ *such that for every* $i \in \{1, 2, ..., r\}$ *either* $G_i \to (H_j\colon j \in A_i)$ *or* $A_i = \emptyset$.

Theorem 1 has a particularly simple form for complete graphs. For natural numbers $a_1, ..., a_r$, $b_1, ..., b_s$, all greater than 1, it asserts that $(K_{a_1}, ..., K_{a_r}) \to (K_{b_1}, ..., K_{b_s})$ if and only if there exists a partition $A_1 \cup A_2 \cup \cdots \cup A_r = \{1, 2, ..., s\}$ such that for every $i \in \{1, 2, ..., r\}$ either $a_i > \sum_{j \in A_i} (b_j - 1)$ or $A_i = \emptyset$. This special case of Theorem 1 was also proved by Andrzej Kurek (personal communication).

Note that Theorem 1 is best possible in the following sense. For every connected graph H which is not 2-connected, there exists a graph G such that $G \not\supseteq H$ but $(G, G) \to (H)$, while, obviously, condition (ii) of Theorem 1 is not satisfied. In fact, it was proved by Kierstead and Rödl [2] that as G one can take a graph obtained from H by splitting one of its cut vertices into two new vertices in such a way that G is disconnected.

We also remark that if the assumption of the 2-connectivity in Theorem 1 is omitted, then not only does the condition (i) not imply (ii) but one can have $(G_1, ..., G_r) \to (H_1, ..., H_s)$ for graphs $H_1, ..., H_s$ which are "larger" than $G_1, ..., G_r$. Indeed, consider the following example. Let T_1 and T_2 be two trees on r_1 and r_2 vertices, respectively. Denote by $\mathbf{G}$ the family of $r_1 + r_2 - 2$ graphs K_2. We show that $(\mathbf{G}) \to (T_1, T_2)$. In order to verify this claim, suppose that $F \to (\mathbf{G})$ and, consequently, $\chi(F) \geqslant r_1 + r_2 - 1$. Color F with two colors, red and blue. Then, either the red graph F_1 has the chromatic number at least r_1 or the graph F_2 spanned by the blue vertices has the chromatic number at least r_2. Without loss of generality let us assume that the former case holds. Then F_1 contains a subgraph F_1' of the minimum degree $\chi(F_1) - 1 \geqslant r_1 - 1$, and so F_1' contains a copy of every tree on r_1 vertices; in particular it contains T_1.

Our next result states that, in a way, the above example is generic for the case in which the graphs $G_1, ..., G_r$ are bipartite. Note that by (2), in this case, if $(G_1, ..., G_r) \to (H_1, ..., H_s)$ then $r \geqslant s$.

THEOREM 2. *Let* $\mathbf{G} = G_1, ..., G_r$ *and* $\mathbf{H} = H_1, ..., H_s$ *be sequences of graphs such that for every* $i = 1, 2, ..., r$, *the graph* G_i *is bipartite and all the graphs* $H_1, ..., H_s$ *are connected. Then the following two conditions are equivalent.*

(i) $(G_1, ..., G_r) \to (H_1, ..., H_s)$,

(ii) *there exists a partition* $B_1 \cup B_2 \cup \cdots \cup B_s = \{1, 2, ..., r\}$ *into nonempty sets such that for every* $j \in \{1, 2, ..., s\}$ *and for every graph* F *such that* $F \to (G_i\colon i \in B_j)$ *we also have* $F \supseteq H_j$.

Let us mention an interesting consequence of Theorems 1 and 2.

COROLLARY. *Let* $\mathbf{G} = G_1, \ldots, G_r$, *and* $\mathbf{H} = H_1, \ldots, H_s$ *be two families of connected graphs of at least two vertices each such that* $(\mathbf{G}) \leftrightarrow (\mathbf{H})$. *Assume also that either*

(i) *each of the graphs* $H_1, \ldots, H_s$ *is 2-connected,*

or

(ii) *each of the graphs* $G_1, \ldots, G_r$ *is bipartite.*

Then $\mathbf{G} = \mathbf{H}$, *i.e.,* $r = s$ *and there exists a permutation* σ *of the set* $\{1, 2, \ldots, r\}$ *such that* $G_i = H_{\sigma(i)}$ *for* $i = 1, 2, \ldots, r$.

Finally, let $F \overset{\text{ind}}{\rightarrow} (G_1, \ldots, G_r)$ mean that for each coloring of F with r colors there is a color i, $i = 1, \ldots, r$, and an *induced* copy of G_i in F with all vertices colored by the ith color. For two families of graphs $\mathbf{G}$ and $\mathbf{H}$ define the relations $(\mathbf{G}) \overset{\text{ind}}{\rightarrow} (\mathbf{H})$ and $(\mathbf{G}) \overset{\text{ind}}{\leftrightarrow} (\mathbf{H})$ accordingly. Then, the results analogous to Theorems 1 and 2 and the corollary hold. We state and prove only the induced counterpart of Theorem 1; one can use a similar argument to verify that the induced versions of Theorem 2 and the corollary also hold.

THEOREM 1*. *Let* $\mathbf{G} = G_1, \ldots, G_r$ *and* $\mathbf{H} = H_1, \ldots, H_s$ *be families of graphs such that for every* $j = 1, 2, \ldots, s$, H_j *is* 2-*connected. Then the following two conditions are equivalent.*

(i) $(G_1, \ldots, G_r) \overset{\text{ind}}{\rightarrow} (H_1, \ldots, H_s)$,

(ii) *There exists a partition* $A_1 \cup A_2 \cup \cdots \cup A_r = \{1, 2, \ldots, s\}$ *such that for every* $i \in \{1, 2, \ldots, r\}$ *either* $G_i \overset{\text{ind}}{\rightarrow} (H_j\colon j \in A_i)$ *or* $A_i = \varnothing$.

2. PROOFS

Proof of Theorem 1. Clearly (ii) implies (i). Thus, let us assume that (ii) does not hold; i.e., for every partition $A_1 \cup \cdots \cup A_r = \{1, 2, \ldots, s\}$ there exists $i \in [r]$ such that $G_i \nrightarrow (H_j\colon j \in A_i)$. We construct a graph K such that $K \rightarrow (G_1, \ldots, G_r)$ and $K \nrightarrow (H_1, \ldots, H_s)$. The existence of such a K implies $\mathbf{G} \nrightarrow \mathbf{H}$, and so the assertion will follow.

For $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$, let $\mathscr{F}_i^j$ denote the family of all maximal, induced subgraphs of G_i which do not contain H_j. Denote by F_i^j the vertex-disjoint union of all graphs from the family $\mathscr{F}_i^j$; i.e., $F_i^j = \bigcup_{F \in \mathscr{F}_i^j} F$.

For every $j = 1, \ldots, s$, let K^j be a graph with the properties $H_j \not\subset K^j$ and $K^j \rightarrow (F_1^j, F_2^j, \ldots, F_r^j)$. The existence of K^j is guaranteed by Lemma 1.

Finally, let $K = K^1 \oplus K^2 \oplus \cdots \oplus K^s$; i.e., K is obtained by taking vertex-disjoint copies of each K^j (with $V_1, \ldots, V_s$ standing for their vertex sets) and connecting each pair of them by the complete bipartite graph. Since $H_j \not\subset K^j$, by coloring all vertices of V_j by color j, $j = 1, \ldots, s$, we see that $K \nrightarrow (H_1, \ldots, H_s)$.

To prove that $K \to (G_1, \ldots, G_r)$, consider now an arbitrary r-coloring $V(K) = C_1 \cup \cdots \cup C_r$. From the definition of K^j it follows that for every $j = 1, \ldots, s$, there exists $i \in \{1, \ldots, r\}$ such that $K^j[V^j \cap C_i] \supset F_i^j$. Now set $A_i = \{j : K^j[V^j \cap C_i] \supset F_i^j\}$ for every $i = 1, 2, \ldots, r$. By our assumption, there exists $i \in \{1, \ldots, r\}$ such that $G_i \nrightarrow (H_j : j \in A_i)$. Thus, for such i there exists a partition $V(G_i) = \bigcup_{j \in A_i} D_j$ with $H_j \not\subset G_i[D_j]$. Hence, by the definition of F_i^j, $G_i[D_j] \subset F_i^j$ and, consequently, $G_i \subset \bigoplus_{j \in A_i} F_i^j \subset K[C_i]$. ∎

Proof of Theorem 1*. The proof follows the lines of the proof of Theorem 1, but with an additional probabilistic ingredient.

As before, for every $j = 1, \ldots, s$, let K^j be a graph with the properties $H_j \not\subset K^j$ and $K^j \overset{\text{ind}}{\to} (F_1^j, F_2^j, \ldots, F_r^j)$. The existence of such graph K^j follows from the induced version of Lemma 1, which can also be found in [3].

For every natural N, let A_N be a complete s-partite graph A with vertex set $\bigcup_{j=1}^s W_j$, $|W_1| = \cdots = |W_s| = N$ and with random weights. The weight of an edge joining W_j and W_l is drawn uniformly from the set $\{1, \ldots, 2^{|K^j|\,|K^l|}\}$. (The weights can be viewed as all possible bipartite graphs between copies K^j and K^l.)

For every s-tuple of vertices $w_1 \in W_1, \ldots, w_s \in W_s$, we define its *type* as the sequence of $\binom{s}{2}$ weights on the edges between them.

We now present a probabilistic lemma concerning the existence of s-tuples of vertices of a given type in large subgraphs of the graph A_N.

LEMMA 2. *For every sequence* $c_1, \ldots, c_s$, $0 < c_j \leqslant 1$, *with probability approaching* 1 *as* $N \to \infty$, *for every type* τ, *and for every choice of subsets* $U_j \subset W_j$, $|U_j| \geqslant c_j N$, $j = 1, \ldots, s$, *there exists an* s-*tuple of vertices* $w_1 \in U_1, \ldots, w_s \in U_s$ *of type* τ.

Proof. There are at most 2^{sN} choices of U_j's. Since the graphs K^j have size independent of N, there are $O(1)$ different types, where here and below the implicit constant in $O(1)$ may depend on s, $c_1, \ldots, c_s$, and $\max\{|K_j| : 1 \leqslant j \leqslant s\}$. Given $U_1, \ldots, U_s$ and a type τ, our event reduces to the existence of at least one copy of the complete graph K_s in the random s-partite graph with edges between U_j and U_l appearing with constant probability $2^{-|K^j|\,|K^l|}$, $1 \leqslant j < l \leqslant s$. Let $X = X(U_1, \ldots, U_s; \tau)$ be the number of such copies. We apply inequality (ii) of Theorem 2.18 in [1]. For the expectation λ of X, we have $\lambda = \Theta(N^s)$ and $\Delta = \Theta(N^{2s-2})$. Hence, the probability that $X = 0$ is smaller than $\exp\{-\Omega(N^2)\}$, and consequently, the probability that there exists a choice of U_j's and τ with no s-tuple of type τ is bounded from above by $O(1)\, 2^{sN} \exp\{-\Omega(N^2)\} = o(1)$. ∎

Let $A = A_N$ be a weighted graph whose existence is guaranteed by Lemma 2. Let K be the graph obtained by blowing up every vertex of W_j to a copy of K^j, $j \in [s]$, and interconnecting these graphs by inserting between them the bipartite graphs determined by the weights of the corresponding edges of A. It is easy to note that $K \overset{\text{ind}}{\nrightarrow} (H_1, ..., H_s)$.

To prove that $K \overset{\text{ind}}{\rightarrow} (G_1, ..., G_r)$, consider an arbitrary r-coloring $V(K) = C_1 \cup \cdots \cup C_r$. There are $r^{|K^j|} = 1/c_j$ ways of coloring the vertices of K^j. Hence, for each j, there are at least $N_j = \lceil c_j N \rceil$ copies of K^j in K which have the same (i.e., preserved by isomorphism) coloring of their vertices. Suppose that these copies are $K^j_{l_j}$, where $j = 1, ..., s$ and $l_j = 1, ..., N_j$.

Let us now consider for a moment a graph K' consisting of disjoint copies, one of each K^j, say, $K' = \bigcup_{j=1}^{s} K^j_1$, and a coloring $V(K') = C'_1 \cup \cdots \cup C'_r$ which is the restriction of $V(K) = C_1 \cup \cdots \cup C_r$ to the chosen copies. As in the proof of Theorem 1, one can show that there exists $i \in [r]$ and a partition $V(G_i) = \bigcup_{j=1}^{s} D^j$ such that $G_i[D^j] \subseteq K^j_1[C'_i \cap V(K^j_1)]$ for every j. To get an induced copy of G_i, one has to insert some bipartite graphs between the subgraphs $G_i[D^j]$. Choose any family of $\binom{s}{2}$ bipartite graphs which, when inserted between the copies K^j_1, would connect the subgraphs $G_i[D^j]$ contained in C'_i to form a copy of G_i. This family determines a type τ of weights in the auxiliary graph A.

Finally, return to the graph K and choose one copy of each $K^j_{l_j}$, $l_j \leqslant N_j$, in such a way that they do correspond to the type τ in A. This yields an induced copy of G_i in color C_i. ∎

Proof of Theorem 2. We show first that (ii) implies (i). To this end, let $F \rightarrow (G_1, ..., G_r)$. Consider an arbitrary coloring of F with s colors and let F_j, $1 \leqslant j \leqslant s$, denote the graph induced by the jth color. If for every $j = 1, ..., s$ we have $F_j \nrightarrow (G_i \colon i \in B_j)$, then we can color F with r colors not creating a copy of G_i in the ith color, contradicting the fact that $F \rightarrow (G_1, ..., G_r)$. Thus, for some $j_0 = 1, 2, ..., s$, we must have $F_{j_0} \rightarrow (G_i \colon i \in B_{j_0})$. But (ii) implies that F_{j_0} contains a copy of H_{j_0}, and consequently $F \rightarrow (H_1, ..., H_s)$.

Assume now that (ii) does not hold. We shall construct a graph K such that $K \rightarrow (G_1, ..., G_r)$ but $K \nrightarrow (H_1, ..., H_s)$, thus negating (i).

For every partition $\mathscr{B} = (B_1, ..., B_s)$ of $\{1, ..., r\}$, let $F = F(\mathscr{B})$ be any graph such that $F \rightarrow (G_i \colon i \in B_j)$ and $F \not\supseteq H_j$ for some j. This index j will be denoted by $j(\mathscr{B})$. Set $g = \max\{|G_i| \colon 1 \leqslant i \leqslant r\}$. For every $j = 1, ..., s$, let K^j be the graph which contains as components precisely rg copies of each graph $F(\mathscr{B})$ such that $j(\mathscr{B}) = j$. If for some j, we have $j(\mathscr{B}) \neq j$ for all $\mathscr{B}$, then we define K^j as a graph of rg isolated vertices. Finally, set $K = K^1 \oplus K^2 \oplus \cdots \oplus K^s$.

Note that since $K^j \not\supseteq H_j$, one can color the vertices of K^j by the jth color yielding $K \not\rightarrow (H_1, \ldots, H_s)$. In order to complete the proof, we need to show that $K \rightarrow (G_1, \ldots, G_r)$. Consider any coloring of K with r colors. Let B_j, $j = 1, 2, \ldots, r$, be the set of colors which appear in $V(K^j)$ at least g times. Let us call these colors frequent in K^j. Note that $B_j \neq \emptyset$ and that no color i can be frequent in more than one subgraph K^j. Indeed, otherwise the complete bipartite graph spanned between two such K^j's would contain a copy of G_i in color i. Furthermore, if a color is not frequent in any K^j, we put it into B_1 so that $B_1, \ldots, B_s$ form a partition $\mathscr{B}_0$ of $\{1, 2, \ldots, r\}$. Set $F_0 = F(\mathscr{B}_0)$ and $j_0 = j(\mathscr{B}_0)$. As K^{j_0} contains at least rg copies of F_0, and only fewer than rg vertices of K^{j_0} can be colored by nonfrequent colors, at least one of these copies is colored exclusively with the colors from B_{j_0}. However, $F_0 \rightarrow (G_i\colon i \in B_{j_0})$, and so one of the graphs G_i, $i \in B_{j_0}$, is colored with the ith color. The assertion follows. ∎

Proof of corollary. For two families of graphs $\mathbf{G} = \{G_1, \ldots, G_r\}$ and $\mathbf{G}' = \{G'_1, \ldots, G'_{r'}\}$, we write $\mathbf{G} > \mathbf{G}'$ if there exists a one-to-one function $\rho\colon \{1, \ldots, r'\} \rightarrow \{1, \ldots, r\}$ such that $G'_i \subseteq G_{\rho(i)}$ for $i = 1, \ldots, r'$ and $\sum_{i=1}^{r} |G_i| > \sum_{i=1}^{r'} |G'_i|$. Let us make the following observation.

CLAIM *If* $(\mathbf{G}) \leftrightarrow (\mathbf{H})$ *and* $\mathbf{G} > \mathbf{G}'$, *then* $\mathbf{G}' \not\rightarrow \mathbf{H}$.

Proof. If $\mathbf{G}' \rightarrow \mathbf{H}$, then $\mathbf{G}' \rightarrow \mathbf{H} \rightarrow \mathbf{G}$ which contradicts (1) as well as Theorem 1. ∎

Now suppose that $(G_1, \ldots, G_r) \rightarrow (H_1, \ldots, H_s)$ and all graphs $H_1, \ldots, H_s$ are 2-connected. Theorem 1 and the claim imply that all subsets A_i defined in Theorem 1 must be nonempty, and so $s \geqslant r$. Furthermore, note that if $G \rightarrow (F_1, \ldots, F_t)$ and all graphs $F_1, \ldots, F_t$ are 2-connected, then either G is 2-connected or it contains a 2-connected subgraph G' such that $G' \rightarrow (F_1, \ldots, F_t)$ and $|G'| < |G|$. Hence, again by Theorem 1 and the claim (and because no A_i is empty), it follows that all graphs $G_1, \ldots, G_r$ are 2-connected. Thus, by symmetry, $r \geqslant s$ and, consequently, $r = s$. Therefore, all A_i's must be singletons and, using the symmetry again, we arrive at $\mathbf{G} = \mathbf{H}$.

Assume now that the graphs $G_1, \ldots, G_r$ are bipartite. We first argue that if $(G_1, \ldots, G_r) \rightarrow (H_1, \ldots, H_s)$, then each of the graphs $H_1, \ldots, H_s$ must be bipartite as well. Indeed, let $g = \max\{|G_i|\colon 1 \leqslant i \leqslant r\}$ and $h = \max\{|H_j|\colon 1 \leqslant j \leqslant s\}$, and let F be a graph such that the chromatic number of F is larger than r and F contains no cycles shorter than $h+1$. Construct a new graph $\tilde{F}$ by replacing each vertex of F by an independent set of size rg and each of the edges of F by a complete bipartite graph. Then, each coloring of $\tilde{F}$ with r colors leads to a monochromatic bipartite graph with at least g

vertices in each of the bipartition classes and, consequently, $\tilde{F} \to (G_1, \ldots, G_r)$. Hence, also $\tilde{F} \to (H_1, \ldots, H_s)$, and, in particular, $\tilde{F} \supset H_j$, $j = 1, \ldots, s$. On the other hand, each subgraph of $\tilde{F}$ on at most h vertices is bipartite, so all graphs $H_1, \ldots, H_s$ are bipartite.

Now the rest of the proof is almost immediate. If $(\mathbf{G}) \leftrightarrow (\mathbf{H})$, then, by (2), we have $r = s$. From Theorem 2 and the claim, we deduce that all B_i's defined in Theorem 2(ii) must be singletons. Using symmetry, we conclude that $\mathbf{G} = \mathbf{H}$. ∎

3. FINAL REMARKS

The most interesting open question concerning vertex Ramsey properties of families of graphs is whether for every two families $\mathbf{G}$ and $\mathbf{H}$ of connected nontrivial graphs the condition $(\mathbf{G}) \leftrightarrow (\mathbf{H})$ implies $\mathbf{G} = \mathbf{H}$, i.e., if we can omit the additional assumptions (i) and (ii) in the Corollary.

Another challenging task is to investigate the much more involved case of edge coloring. In particular, we ask if for 2-connected graphs results similar to Theorems 1 and 1* and the corollary hold. Let us note, however, that the behavior of bipartite graphs is quite different when we color the edges instead of the vertices. In order to see this let S_k denote a star of k rays and let "$\to_e$" and "$\leftrightarrow_e$" denote the edge-coloring counterparts of "$\to$" and "$\leftrightarrow$." Note that the pigeonhole principle and Petersen's theorem, which states that any graph F with maximum degree $2k$ can be decomposed into k subgraphs of maximal degrees at most 2, imply that

$$F \to_e (S_{2k_1+1}, \ldots, S_{2k_s+1})$$

if and only if the maximum degree of F is larger that $2 \sum_{i=1}^{s} k_i$. Hence, for instance,

$$(S_7, S_7, S_7, S_7) \leftrightarrow_e (S_9, S_9, S_9),$$

which seems to suggest that for edge colorings no result similar to Theorem 2 holds.

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