Journal of Algebra 335 (2011) 1–17

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Restrictive, split and unital quasi-Jordan algebras

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ARTICLE INFO

Article history:
Received 8 November 2009
Available online 16 April 2011
Communicated by Efim Zelmanov

MSC:
17A30
17B40
17C99

Keywords:
Dialgebras
Leibniz algebras
Quasi-Jordan algebras

ABSTRACT

It is well known that by means of the right and left products of an associative dialgebra we can build a new product over the same vector space with respect to which it becomes a right version of a Jordan algebra (in fact, this new product is right commutative) called quasi-Jordan algebra. Recently, Kolesnikov and Bremner independently have discovered an interesting property of this new product. As the results of this paper indicate, when the said property is incorporated as an axiom in the definition of quasi-Jordan algebra then in a natural way one can introduce and study concepts in this new structure such as derivations (in particular inner derivations), quadratic representations, and the structure groups of a quasi-Jordan algebra.

1. Introduction

It is well known that any associative algebra A becomes a Lie algebra under the skew-symmetric product (Lie bracket) $[x, y] := xy - yx$ and at the same time it becomes a Jordan algebra with respect to the product $x \bullet y := \frac{1}{2}(xy + yx)$. On the other hand, we recall that from the works of J. Tits [12], I. Kantor [4] and M. Koecher [7] it follows that any Jordan algebra can be imbedded into a Lie algebra. It is well known that the derivations of the Jordan algebra play a fundamental role in this construction.

In 1993, J.L. Loday introduced the notion of Leibniz algebras (see [8]), which is a generalization of the Lie algebras where the skew-symmetricity of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. Loday also showed that the relationship between Lie algebras and associative algebras translates into an analogous relationship between Leibniz algebras and the so-called associative dialgebras (see [8]) which are a generalization of associative algebras possessing

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.03.029

two products. In particular Loday showed that any dialgebra $(D, \dashv, \vdash)$ becomes a Leibniz algebra D_{Leib} under the Leibniz bracket $[x, y] := x \dashv y - y \vdash x$ (see [8] and [9]). This fact is a trivial calculation using the defining identities for associative dialgebras.

However, there exists a more deep relation between a Lie algebra and certain associative algebra for which it also has been pointed an analogue by Loday in the context of his work about Leibniz algebras and associative dialgebras that we pass to discuss. A main tool in the Lie algebra theory is another algebra obtained by means of a quotient construction. This algebra constitutes an associative algebra and is called universal enveloping algebra of a Lie algebra. It is important because allows us to translate questions about Lie algebras into corresponding questions about associative algebras. Let L be a Lie algebra and let $U(L)$ denote its universal enveloping algebra, then one can introduce $U(L)$ as follows. Since L is a vector space, then it is possible to construct the associative algebra $T(L) = \mathbb{F} \dotplus L \dotplus L \otimes L \dotplus \cdots$, which is its contravariant tensor algebra. In $T(L)$ one considers the two-sided ideal K generated by the set of all elements of the form

$$[x, y] = (x \otimes y - y \otimes x),$$

where x and y are elements of L. In this form, the ideal K contains the differences between Lie algebra products and the corresponding commutators in the associative algebra $T(L)$. If next we consider the associative quotient algebra $U(L) = T(L)/K$ then its Lie products will not be distinguished from commutators since they belong to the same coset. Now as any associative algebra, we can make $U(L)$ a Lie algebra using the commutator operation as the Lie product. When one does this, we can consider L to be injective homomorphically into $U(L)$, considered as a Lie algebra. If $\{x_i\}$ is a basis for L, then the monomials of the form

$$x_{i_1} \otimes x_{i_2} \otimes \cdots x_{i_n}, \quad n = 0, 1, 2, \ldots,$$

where we take the trivial monomial 1 in the case $n = 0$, span $T(L)$, and hence their cosets span $U(L)$. A result of Poincaré, Birkhoff and Witt is that if we only take monomials having their indices i_j in ascending order, allowing repetition, then the cosets of these monomials, again including 1, form a basis for $U(L)$. Remarkably, Loday proved that the universal enveloping algebra of a Leibniz algebra has the structure of a dialgebra, see [8] and [9].

We would like to observe that the referee has informed the author that the first discussion of Leibniz algebras was in fact given by C. Cuvier in 1994. Thus, the paper [3] should be mentioned in addition to that of Loday.

For convenience of the reader, we include here the definition of dialgebra which is a generalization of associative algebras, with two operations.

Definition 1. A **dialgebra** over a field K is a K-vector space D equipped with two associative products

$$\dashv : D \times D \to D,$$

$$\vdash : D \times D \to D$$

satisfying the identities:

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{2}$$

$$(x \vdash y) \vdash z = (x \dashv y) \vdash z. \tag{3}$$

Very recently, Velasquez and Felipe introduced the notion of quasi-Jordan algebras which may have, with respect to the Leibniz algebras, a relationship similar to those existing among the Jordan algebras and the Lie algebras. In fact, in [13] they attach a quasi-Jordan algebra L_x to any ad-nilpotent element x with an index of nilpotence 3 (Q-Jordan element) in a Leibniz algebra L. Thus, the quasi-Jordan algebras are a generalization of Jordan algebras but where the commutative law is changed by a quasi-commutative identity and a special form of the Jordan identity is retained.

It should be indicated that in the above mentioned paper [13], the authors establish a few results about the relationship between Jordan algebras and quasi-Jordan algebras; moreover, they compare the quasi-Jordan algebras with some known structures. For instance, the Jordan and Perm algebras are obvious examples of quasi-Jordan algebras. On the contrary, the noncommutative Jordan algebras are not in general quasi-Jordan algebras. At the same time, we remember that the authors pointed there a way in which one can use the notion of Jordan bimodule to construct interesting quasi-Jordan algebras.

In [14], the notions of the annihilator ideal and split structure are studied in detail for both dialgebras and quasi-Jordan algebras. The authors also provide methods for additional units in the two structures. As a consequence, the notion of regular element receives special attention of the authors of this article.

The objects of study of this article are the derivations (in particular, inner derivations), quadratic representations and the structure group of a quasi-Jordan algebra. For this purpose, we use a property recently discovered by Kolesnikov and independently by Bremner of the Jordan di-product constructed from the right and left product over a dialgebra.

2. Preliminaries on quasi-Jordan algebras

In [13] the aim of the work was to discover a new generalization of Jordan algebras. This new structure, named quasi-Jordan algebra, can be noncommutative although it is not in general equivalent to a noncommutative Jordan algebra and satisfies a particular noncommutative version of the Jordan identity. As was seen in the mentioned paper, the quasi-Jordan algebras appear in the study of the product

$$x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x), \tag{4}$$

where x and y are elements in a dialgebra D over a field K of characteristic different from 2.

We give the following definition (see [13]).

Definition 2. By a **quasi-Jordan algebra** we mean a vector space $\Im$ over a field K of a characteristic different of 2 equipped with a bilinear product $\triangleleft : \Im \times \Im \to \Im$ that satisfies

$$x \triangleleft (y \triangleleft z) = x \triangleleft (z \triangleleft y) \quad \text{(right commutativity)}, \tag{5}$$

$$(y \triangleleft x) \triangleleft x^2 = \left(y \triangleleft x^2\right) \triangleleft x \quad \text{(right Jordan identity)}, \tag{6}$$

for all $x, y, z \in \Im$, where $x^2 = x \triangleleft x$.

In [13] it was shown that all Q-Jordan elements in a Leibniz algebra are associated to quasi-Jordan algebras.

Next, we provide a few examples of quasi-Jordan algebras.

Example 1. First, if we translate the quasi-multiplication (Jordan product) to the dialgebra framework, we obtain a new algebraic structure of Jordan type. Let D be a dialgebra over a field K of the characteristic different from 2. We define the product $\triangleleft : D \times D \to D$ by

$$x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x),$$

for all $x, y \in D$. Simple calculations show that the product $\triangleleft$ satisfies the identities (5) and (6), but the product $\triangleleft$ is noncommutative in general. It follows that $(D, \triangleleft)$ is a quasi-Jordan algebra. The product $\triangleleft$ of this example is called the Jordan di-product for dialgebras.

On the other hand, if D is a unital dialgebra, with a specific bar-unit e, we have that $x \triangleleft e = x$, for all x in D. This implies that e is a right unit for the quasi-Jordan algebra $(D, \triangleleft)$.

A **right unit** in a quasi-Jordan algebra $\Im$ is an element e in $\Im$ such that $x \triangleleft e = x$, for all $x \in \Im$. Let $\Im$ be a quasi-Jordan algebra, if there is an element ϵ in $\Im$ such that $\epsilon \triangleleft x = x$ then $\Im$ is a classical Jordan algebra and ϵ is a unit. For this reason, we only consider right units over quasi-Jordan algebras. It is possible to attach a unit to any Jordan algebra, but in quasi-Jordan algebras the problem of attaching a right unit is an open problem. Additionally, the right units in quasi-Jordan algebras are not unique.

Definition 3. Let J be a Jordan algebra and let M be a vector space over the same field as J. Then M is a Jordan bimodule for J in the case when there are two bilinear compositions $(m, a) \mapsto ma$ and $(m, a) \mapsto am$, for all $m \in M$ and $a \in J$, satisfying

$$ma = am$$

and

$$\left(a^2, m, a\right) = \left(a^2, b, m\right) + 2(ma, b, a) = 0,$$

for all $m \in M$ and $a, b \in J$, where (a, b, c) denotes the associator.

The following examples can be find in [13] and [14].

Example 2. Let J be a Jordan algebra and let M be a Jordan bimodule. A linear map $f : M \to J$ is called J**-equivariant** over M if $f(am) = af(m)$, for all $m \in M$ and $a \in J$. If f is a J-equivariant map over M, then we define the product $\triangleleft : M \times M \to M$ by

$$m \triangleleft n = f(n)m, \quad \text{for all } m, n \in M,$$

then $(M, \triangleleft)$ is a quasi-Jordan algebra.

Example 3. Let J be a Jordan algebra and let M be a Jordan bimodule over J. We consider the direct sum $\Im := M \oplus J$ and we define the product $\triangleleft$ over $\Im$ by

$$(u + x) \triangleleft (v + y) = uy + x \bullet y,$$

for all $u, v \in M$ and $x, y \in J$. Then $(\Im, \triangleleft)$ is a quasi-Jordan algebra. We call this algebra the **demisemidirect** product of M with J.

One can construct a quasi-Jordan algebra with the assistance of a vector space and its Jordan algebra of linear transformations. In this sense we have

Example 4. Let V be a vector space over a field K with a characteristic different from 2 and let $gl^+(V)$ be a Jordan algebra of linear transformations over V with a product defined by

$$A \bullet B = \frac{1}{2}(AB + BA),$$

where AB denotes the composition of the maps A and B. We consider the vector space $gl^+(V) \times V$ and we define the product $\lhd : (gl^+(V) \times V) \times (gl^+(V) \times V) \to gl^+(V) \times V$ by

$$(A, u) \lhd (B, v) = (A \bullet B, Bu),$$

for all $A, B \in gl(V)$ and $u, v \in V$. This product satisfies the identities (5) and (6). Moreover, this quasi-Jordan algebra is power-associative and (Id, v), where Id denotes the identity map over V, is a right unit which is not a left unit for all $v \in V$.

Example 5. Let V be a 2-dimensional vector space with a base $\{e_1, e_2\}$. If we define the product $\lhd : V \times V \to V$ with respect to e_1 and e_2 by $e_i \lhd e_j = e_i$, for $i = 1, 2$, and extend the product to V for linearity, we have that $(V, \lhd)$ is a noncommutative quasi-Jordan algebra.

For more details on the concepts and results that we discuss below we refer the reader to [14].

For a quasi-Jordan algebra $\Im$ we introduce

$$Z^r(\Im) = \{z \in \Im \mid x \lhd z = 0, \ \forall x \in \Im\}.$$

We denote by $\Im^{\mathrm{ann}}$ the subspace of $\Im$ spanned by elements of the form $x \lhd y - y \lhd x$, with $x, y \in \Im$. We have that $\Im$ is a Jordan algebra if and only if $\Im^{\mathrm{ann}} = \{0\}$. It follows from the right commutativity (5) that in any quasi-Jordan algebra we have

$$x \lhd (y \lhd z - z \lhd y) = 0.$$

The last identity implies

$$\Im^{\mathrm{ann}} \subset Z^r(\Im).$$

One can prove that both $\Im^{\mathrm{ann}}$ and $Z^r(\Im)$ are two-sided ideals of $\Im$. The ideal $\Im^{\mathrm{ann}}$ is called the annihilator ideal of the quasi-Jordan algebra $\Im$. On the other hand, we recall that if $\Im$ is a unital quasi-Jordan algebra, with a specific right unit e, then (see [13])

$$\Im^{\mathrm{ann}} = Z^r(\Im),$$

$$\Im^{\mathrm{ann}} = \{x \in \Im \mid e \lhd x = 0\}$$

and

$$U_r(\Im) = \left\{x + e \;\middle|\; x \in \Im^{\mathrm{ann}}\right\}$$

Quotienting the quasi-Jordan algebra $\Im$ by the ideal $\Im^{\mathrm{ann}}$ gives a Jordan algebra denoted by $\Im_{\mathrm{Jor}}$. Moreover, the ideal $\Im^{\mathrm{ann}}$ is the smallest two-sided ideal of $\Im$ such that $\Im/\Im^{\mathrm{ann}}$ is a Jordan algebra.

Definition 4. Let $\Im$ be a quasi-Jordan algebra and let I be an ideal in $\Im$ such that $\Im^{\mathrm{ann}} \subset I \subset Z^r(\Im)$. We say that $\Im$ is **split** over I if there is a subalgebra J of $\Im$ such that $\Im = I \oplus J$, as a direct sum of subspaces.

This class of quasi-Jordan algebras is important because there is a special relationship between quasi-Jordan algebras and split quasi-Jordan algebras. This relationship shows that every quasi-Jordan algebra is isomorphic to a subalgebra of a split quasi-Jordan algebra (see [14] for more details).

It is clear from the previous definition that if $\Im$ is split over an ideal I with complement J, then J is a Jordan algebra with respect to the product $\lhd$ restricted to J. This is equivalent to saying that $(J, \lhd|_J)$ is a Jordan algebra. In fact, let $x, y \in J$, then $x \lhd y, \ y \lhd x \in J$ and this implies $x \lhd y - y \lhd x \in$

$I \cap J = \{0\}$, i.e. $\lhd|_J$ is commutative and therefore the right Jordan identity over $\Im$ implies that $(J, \lhd|_J)$ is a Jordan algebra.

Additionally, for $u, v \in I$ and $x, y \in J$ we have

$$(u + x) \lhd (v + y) = u \lhd y + x \lhd y,$$

because $I \subset Z^r(\Im)$.

Let $\Im$ be a quasi-Jordan algebra and let I be an ideal of $\Im$ such that $\Im^{\mathrm{ann}} \subset I \subset Z^r(\Im)$. Then $\Im$ is split over I if and only if $\Im$ is the demisemidirect product (in the sense of Example 3) of I with a Jordan algebra J.

We suppose that $\Im$ is a split quasi-Jordan algebra with a specific right unit e. Because, $\Im^{\mathrm{ann}} = Z^r(\Im)$ we have that there is a Jordan algebra J such that $\Im = \Im^{\mathrm{ann}} \oplus J$. Moreover, the Jordan algebra J is isomorphic to the Jordan algebra $\Im_{\mathrm{Jor}}$.

Because $e \in \Im$ is a right unit in $\Im$, there are elements $x \in \Im^{\mathrm{ann}}$ and $\epsilon \in J$ such that $e = x + \epsilon$. If $y + a \in \Im$, with $y \in \Im^{\mathrm{ann}}$ and $a \in J$, we have

$$y + a = (y + a) \lhd e = (y + a) \lhd (x + \epsilon) = y \lhd \epsilon + a \lhd \epsilon = (y + a) \lhd \epsilon.$$

The last identity implies that ϵ is a right unit in $\Im$ and it is a unit in the Jordan algebra J. Also, ϵ is the only element in J such that $x + \epsilon$ is a right unit in $\Im$.

It implies that the right units in a split quasi-Jordan algebra are of the form $i + \epsilon$, where $i \in \Im^{\mathrm{ann}}$ and ϵ is the unique unit of a unital Jordan algebra, hence $U_r(\Im) = \Im^{\mathrm{ann}} \oplus \{\epsilon\}$.

The reciprocal of this characterization is not true, that is, a split quasi-Jordan algebra with unital Jordan part, need not necessarily have a right unit (see [14] for more details).

In [1] Bremner proved that the product (4) over a dialgebra satisfies the relation

$$\bigl(x, y^2, z\bigr) = 2(x, y, z) \lhd y, \tag{7}$$

where $(u, v, w) = (u \lhd v) \lhd w - u \lhd (v \lhd w)$.

The equality (7) was first obtained by Kolesnikov in other context (see [5]), therefore we will call it **the Kolesnikov–Bremner identity** (or KB identity) for the Jordan di-product (4).

We are in a position to introduce the following concept.

Definition 5. A quasi-Jordan algebra for which (7) holds is called **restrictive quasi-Jordan algebra**.

Since any associative algebra is a dialgebra for which the left and right product coincides, then it follows that any special Jordan algebra is a restrictive quasi-Jordan algebra. On the other hand, the referee has pointed to the author that the product in any Jordan algebra also satisfies the KB identity. We thank the referee for suggesting us the following proof of this fact which we present here for completeness of the article.

To check that the KB identity holds in every Jordan algebra, we start with the Jordan identity:

$$\bigl(x, y, x^2\bigr),$$

then we expand the associator, linearize the identity, and use commutativity to obtain (for brevity we omit the symbol for the product)

$$2(wy)(xz) + 2(xy)(wz) + 2(zy)(wx) - 2w\bigl(y(xz)\bigr) - 2x\bigl(y(wz)\bigr) - 2z\bigl(y(wx)\bigr) = 0,$$

just now, we divide by -2 and one more time we use commutativity to obtain

$$\bigl((wx)y\bigr)z + \bigl((wz)y\bigr)x + \bigl((xz)y\bigr)w - (wx)(yz) - (wy)(xz) - (wz)(xy) = 0,$$

let us denote the left side of this equation by $J(w,x,y,z)$. One can see that

$$J(w,x,z,y)-J(w,y,x,z)=\big((wx)z\big)y-\big((wy)x\big)z+\big((wy)z\big)x-\big((wz)x\big)y$$
$$+\big((xy)z\big)w-\big((yz)x\big)w. \tag{8}$$

The following step consists in starting again with the KB identity:

$$\big(x,y^2,z\big)=2(x,y,z)y,$$

then, in this equality we expand the associators, use commutativity, linearize the identity, and collect terms on the left side to get

$$2\big((wy)x\big)z-2\big((wy)z\big)x-2\big((xw)z\big)y-2\big((xy)z\big)w+2\big((wz)x\big)y+2\big((yz)x\big)w=0,$$

let us divide by -2 and finally let us use commutativity again to obtain

$$\big((wx)z\big)y-\big((wy)x\big)z+\big((wy)z\big)x-\big((wz)x\big)y+\big((xy)z\big)w-\big((yz)x\big)w=0. \tag{9}$$

Then the left side of (9) coincides with (8).

Remark 6. Bremner discovered (7) by means of a computer-assisted method to study identities of certain degree in non-associative algebraic structures, while Kolesnikov arrived at this in his before mentioned work, where he introduced and studied the concept of a variety of dialgebras which this author found to be closely related to the notion of a variety of conformal algebras.

Another reference, extending the work of Kolesnikov, on the algorithm for converting algebras identities to dialgebras identities, is a paper of A.P. Pozhidaev [11].

Let $(A,\lhd)$ be an algebra. We recall that the powers of $u\in A$ are defined by $u^1=u$, $u^{m+1}=u^m\lhd u$ for $m\geqslant 1$. We say that $(A,\lhd)$ is a power-associative algebra if the equality $u^r\lhd u^s=u^{r+s}$ holds for all $u\in A$, $r\geqslant 1$ and $s\geqslant 1$. The referee raised the important question of whether every quasi-Jordan algebra is power-associative. The answer to this question turns out to be negative, as follows from the following counterexample: Let $\Im_4$ be the 4-dimensional split quasi-Jordan algebra (it is not a Jordan algebra) whose multiplication table is

$\lhd$	i	j	a	b
i	0	0	i	j
j	0	0	j	i
a	0	0	b	a
b	0	0	a	b

Then, $\Im_4=\Im_4^{\mathrm{ann}}\oplus J_4$ where $\Im_4^{\mathrm{ann}}=\langle\{i,j\}\rangle$ and J_4 is the 2-dimensional Jordan algebra generated by $\{a,b\}$. It is easy to show that $(i+b)^5=(i+b)$, $(i+b)^3=(i+b)$ and $(i+b)^2=(j+b)$. Hence

$$(i+b)^3\lhd(i+b)^2=(i+b)\lhd(j+b)=(j+b),$$

it follows that $(i+b)^3\lhd(i+b)^2\neq(i+b)^5$. Thus, the quasi-Jordan algebra $\Im_4$ is not power-associative.

From now on, we abandon the review about quasi-Jordan algebras.

3. Derivations in restrictive, split and unital quasi-Jordan algebras

In the remainder of the article we suppose that $\Im$ is a split and unital quasi-Jordan algebra, thus there is a unital Jordan algebra J such that $\Im = \Im^{\text{ann}} \oplus J$. We denote the unit of J by ϵ. As we already know ϵ will be a unit of all $\Im$.

The elements of $\Im$ should be presented as pairs, that is, $(i,a), (j,b), (k,c), \ldots$ etc. We remember that the product of two elements is $(i,a) \lhd (j,b) = (i \lhd b, ab)$, where $i \lhd b \in \Im^{\text{ann}}$ and $ab \in J$. Thus, we have defined a linear transformation $R_{(j,b)}$ over all $\Im$ in the way $R_{(j,b)}(i,a) = (i,a) \lhd (j,b) = (i \lhd b, ab)$ which can be presented using a pair of linear transformations, $R_b : \Im^{\text{ann}} \to \Im^{\text{ann}}$ and $L_b : J \to J$ where $R_b i = i \lhd b$ and $L_b a = ba$. It is clear that through the obvious action, we have $R_{(j,b)}(i,a) = (i,a) \lhd (j,b) = (R_b, L_b)(i,a)$.

Proposition 7. *Let $\Im = \Im^{\text{ann}} \oplus J$ be a split and unital quasi-Jordan algebra. Then*

$$[R_d, R_{bc}] + [R_b, R_{dc}] + [R_c, R_{bd}] = 0, \tag{10}$$

and

$$[L_d, L_{bc}] + [L_b, L_{dc}] + [L_c, L_{bd}] = 0, \tag{11}$$

for all $b, c, d \in J$. Here $[\cdot,\cdot]$ represents the brackets in End$(\Im^{\text{ann}})$ and End(J) respectively.

Proof. The Jordan identity (6), in our case acquires the form of two equations, the first

$$(i \lhd b) \lhd b^2 = \left(i \lhd b^2\right) \lhd b, \tag{12}$$

for every $i \in \Im^{\text{ann}}$ and all $b \in J$. Moreover, the second

$$(ab)b^2 = \left(ab^2\right)b, \tag{13}$$

for any $a, b \in J$.

Now, since J is a Jordan algebra, then it is a well-known fact that the equality (11) is obtained from (13) by means of a double process of linearization. On the other hand, it is easy to see that the proof of (10) follows a similar path. In fact, if we just execute a similar double process of linearization to Eq. (12) we obtain (10). $\square$

Note that in Proposition 13 we did not assume that $\Im$ is restrictive.

From here on we assume $\Im = \Im^{\text{ann}} \oplus J$ is a restrictive, split and unital quasi-Jordan algebra. The unit of J will be denoted by ϵ.

Theorem 8. *In $\Im$ we have*

$$R_{([L_b, L_a]c)} = \left[[R_b, R_a], R_c\right], \tag{14}$$

for all $a, b, c \in J$.

Proof. The KB identity (7) implies that

$$\left(i \lhd a^2\right) \lhd b - i \lhd \left(a^2 b\right) = 2\left((i \lhd a) \lhd b\right) \lhd a - 2(i \lhd ab) \lhd a, \tag{15}$$

for all $i \in \Im^{\text{ann}}$ and any $a, b \in J$. Now, linearizing Eq. (15) only once, we obtain (14). Next, we develop this calculation in detail. But first, we shall show that (15) holds. If $(j,y) \in \Im$ then $(j,y) \lhd (j,y) = (j \lhd y, y^2)$. Thus, we have

$$
\begin{aligned}
\big((i,x),(j,y)^2,(k,z)\big) &= \big((i,x),\big(j \triangleleft y, y^2\big),(k,z)\big) \\
&= \big((i,x) \triangleleft \big(j \triangleleft y, y^2\big)\big) \triangleleft (k,z) - (i,x) \triangleleft \big(\big(j \triangleleft y, y^2\big) \triangleleft (k,z)\big) \\
&= \big(i \triangleleft y^2, xy^2\big) \triangleleft (k,z) - (i,x) \triangleleft \big((j \triangleleft y) \triangleleft z, y^2 z\big) \\
&= \big(\big(i \triangleleft y^2\big) \triangleleft z, \big(xy^2\big)z\big) - \big(i \triangleleft \big(y^2 z\big), x\big(y^2 z\big)\big) \\
&= \big(\big(i \triangleleft y^2\big) \triangleleft z - i \triangleleft \big(y^2 z\big), \big(xy^2\big)z - x\big(y^2 z\big)\big),
\end{aligned} \tag{16}
$$

on the other hand,

$$
\begin{aligned}
2\big((i,x),(j,y),(k,z)\big) \triangleleft (j,y) &= 2\big((i \triangleleft y) \triangleleft z - i \triangleleft yz, (xy)z - x(yz)\big) \triangleleft (j,y) \\
&= 2\big(\big((i \triangleleft y) \triangleleft z - i \triangleleft yz\big) \triangleleft y, \big((xy)z\big)y - \big(x(yz)\big)y\big),
\end{aligned} \tag{17}
$$

equating the first components of (16) and (17) we arrive at (15). We are now ready to linearize the equality (15).

Let L_1, L_2, L_3 and L_4 denote the linearization of the terms of Eq. (15) starting from the left. One can see that if $a \to a + \alpha c$ and we take the coefficients of α then

$$
L_1 = 2\big(i \triangleleft (ac)\big) \triangleleft b, \qquad L_2 = 2\big(i \triangleleft (ac)b\big), \tag{18}
$$

further,

$$
L_3 = 2\big((i \triangleleft c) \triangleleft b\big) \triangleleft a + 2\big((i \triangleleft a) \triangleleft b\big) \triangleleft c, \tag{19}
$$

and finally,

$$
L_4 = 2\big(i \triangleleft (cb)\big) \triangleleft a + 2\big(i \triangleleft (ab)\big) \triangleleft c. \tag{20}
$$

Now since $L_1 - L_2 = L_3 - L_4$, then together with (18)–(20) the identity gives

$$
R_{(ac)b} = R_b R_{ac} - R_a R_b R_c - R_c R_b R_a + R_a R_{cb} + R_c R_{ab}, \tag{21}
$$

exchange in (21) the roles of a and b

$$
R_{(bc)a} = R_a R_{bc} - R_b R_a R_c - R_c R_a R_b + R_b R_{ca} + R_c R_{ab}, \tag{22}
$$

hence, taking into account that a, b, c live in a Jordan algebra, from (21) and (22) it follows that

$$
R_{([L_b, L_a]c)} = \big[[R_b, R_a], R_c\big]. \tag{23}
$$

The theorem is proved. $\square$

Note that because J is a Jordan algebra, then for any $a, b, c \in J$ we also have

$$
L_{([L_b, L_a]c)} = \big[[L_b, L_a], L_c\big]. \tag{24}
$$

The KB identity is really useful. In fact, we have

Remark 9. In the general case, that is, when $\Im$ is a restrictive quasi-Jordan algebra, it does not necessarily split, neither is it necessarily unital, the process of linearization of Eq. (7) leads to the following

$$R_{(y \lhd w) \lhd z} = R_z R_{y \lhd w} - R_y R_z R_w + R_y R_{w \lhd z} - R_w R_z R_y + R_w R_{y \lhd z}, \tag{25}$$

for all $y, z, w \in \Im$. Here $R_x y = y \lhd x$. Hence, also

$$R_{(z \lhd w) \lhd y} = R_y R_{z \lhd w} - R_z R_y R_w + R_z R_{w \lhd y} - R_w R_y R_z + R_w R_{z \lhd y}, \tag{26}$$

now, it should be noted that $R_{(y \lhd w) \lhd z} = R_{(w \lhd y) \lhd z}$, $R_{(z \lhd w) \lhd y} = R_{(w \lhd z) \lhd y}$, $R_{y \lhd w} = R_{w \lhd y}$, $R_{z \lhd w} = R_{w \lhd z}$ and $R_{z \lhd y} = R_{y \lhd z}$. Using these equalities by subtracting (26) from (25) we obtain

$$R_{[R_z, R_y]w} = \big[[R_z, R_y], R_w\big]. \tag{27}$$

Next, we pay attention to the notion of derivation over a restrictive, split and unital quasi-Jordan algebra $\Im = \Im^{\mathrm{ann}} \oplus J$.

It is obvious that $(0, \epsilon)$ is a unit of $\Im$.

Definition 10. A linear transformation D on $\Im$ (that is $D \in End\,\Im$) is called a derivation if for all $(i, a), (j, b) \in \Im$

$$D\big((i, a) \lhd (j, b)\big) = \big(D(i, a)\big) \lhd (j, b) + (i, a) \lhd \big(D(j, b)\big). \tag{28}$$

Since we can write $D = (D_1, D_2)$, where $D_1 : \Im^{\mathrm{ann}} \to \Im^{\mathrm{ann}}$ and $D_2 : J \to J$ then (28) is equivalent to the two following equations

$$D_1(i \lhd b) = (D_1 i) \lhd b + i \lhd (D_2 b), \tag{29}$$

for all $i \in \Im^{\mathrm{ann}}$ and every $b \in J$, moreover

$$D_2(ab) = (D_2 a)b + a(D_2 b), \tag{30}$$

for all $a, b \in J$.

It is clear that (29) and (30) can be written in this way

$$R_{D_2 b} = [D_1, R_b], \qquad L_{D_2 b} = [D_2, L_b]. \tag{31}$$

Reciprocally, (31) implies that $D = (D_1, D_2)$ is a derivation of $\Im$. Note that (28) is the same as

$$R_{D(j,b)} = [D, R_{(j,b)}]. \tag{32}$$

In other words, D is a derivation if and only if (32) holds.

Corollary 11. *$D = ([R_a, R_b], [L_a, L_b])$ is a derivation of $\Im$.*

Proof. It is well known that for all $a, b \in J$, $[L_a, L_b]$ is a derivation. Hence, the corollary follows from (31) and Theorem 8. $\square$

Since in any derivation $D = (D_1, D_2)$, D_2 is a derivation over the Jordan algebra J, then $D_2 \epsilon = 0$. It follows that $D(0, \epsilon) = (0, 0)$.

We define $Der(\Im^{\mathrm{ann}})$ as the subset of $End(\Im^{\mathrm{ann}})$ of all D_1 for which (29) is satisfied. Therefore, if $Der(\Im)$ is the set of all derivations of $\Im$, then $Der(\Im) = (Der(\Im^{\mathrm{ann}}), Der(J))$. It is easy to show that over $End(\Im) = (End(\Im^{\mathrm{ann}}), End(J))$, the product $[(W_1, A_1), (W_2, A_2)] = ([W_1, W_2], [A_1, A_2])$ is a Lie bracket. Here $[W_1, W_2] \in End(\Im^{\mathrm{ann}})$ and $[A_1, A_2] \in End(J)$ are the Lie products in $End(\Im^{\mathrm{ann}})$ and $End(J)$ respectively. Observe also that

$$\big[(W_1, A_1), (W_2, A_2)\big] = (W_1, A_1)(W_2, A_2) - (W_2, A_2)(W_1, A_1),$$

under the usual product in $(End(\Im^{\mathrm{ann}}), End(J))$.

After introducing the necessary definitions, we are able to ensure that

$$R_{([R_{(i,a)}, R_{(j,b)}](k,c))} = \big[[R_{(i,a)}, R_{(j,b)}], R_{(k,c)}\big], \tag{33}$$

which follows directly from (14) and (24).

Obviously, $Der(\Im)$ is a vectorial subspace of $End(\Im)$. Moreover,

Theorem 12. *Let us assume that $D = (D_1, D_2)$ and $\widehat{D} = (\widehat{D}_1, \widehat{D}_2)$ are derivatives of $\Im$. Then $[D, \widehat{D}]$ is also a derivation. Therefore, $Der(\Im)$ is a Lie algebra.*

Proof. For any $b \in J$, we have

$$\begin{aligned} R_{[D_2, \widehat{D}_2]b} = R_{D_2\widehat{D}_2b - \widehat{D}_2D_2b} &= R_{D_2\widehat{D}_2b} - R_{\widehat{D}_2D_2b} \\ &= [D_1, R_{\widehat{D}_2b}] - [\widehat{D}_1, R_{D_2b}] \\ &= \big[D_1, [\widehat{D}_1, R_b]\big] - \big[\widehat{D}_1, [D_1, R_b]\big] \\ &= \big[D_1, [\widehat{D}_1, R_b]\big] + \big[\widehat{D}_1, [R_b, D_1]\big] \\ &= \big[[D_1, \widehat{D}_1], R_b\big]. \end{aligned} \tag{34}$$

Noting now that both D_2 and $\widehat{D}_2$ are derivations over the Jordan algebra J, we then can state that

$$L_{[D_2, \widehat{D}_2]b} = \big[[D_2, \widehat{D}_2], L_b\big], \tag{35}$$

Thus, from (31), (34) and (35) it follows that $[D, \widehat{D}_1] = ([D_1, \widehat{D}_1], [D_2, \widehat{D}_2])$ belongs to $Der(\Im)$. $\square$

Note that from (34) and (35) one concludes that

$$R_{[D, \widehat{D}](j,b)} = \big[[D, \widehat{D}], R_{(j,b)}\big], \tag{36}$$

for every $D, \widehat{D} \in Der(\Im)$ and all $(j, b) \in \Im$. It follows (32). However, it can be proved directly. In fact, $[D, \widehat{D}] = ([D_1, \widehat{D}_1], [D_2, \widehat{D}_2])$ and remembering now that $R_{(j,b)} = (R_b, L_b)$ then we have

$$\begin{aligned} R_{[D, \widehat{D}](j,b)} = R_{([D_1, \widehat{D}_1], [D_2, \widehat{D}_2])(j,b)} &= R_{([D_1, \widehat{D}_1]j, [D_2, \widehat{D}_2]b)} \\ &= (R_{[D_2, \widehat{D}_2]b}, L_{[D_2, \widehat{D}_2]b}) = \big(\big[[D_1, \widehat{D}_1], R_b\big], \big[[D_2, \widehat{D}_2], L_b\big]\big) \\ &= \big[[D, \widehat{D}], (R_b, L_b)\big] = \big[[D, \widehat{D}], R_{(j,b)}\big]. \end{aligned}$$

Definition 13. The derivations of the form $D = [R_{(i,a)}, R_{(j,b)}] = [(R_a, L_a), (R_b, L_b)] = ([R_a, R_b], [L_a, L_b])$ will be called **inner derivations**.

We consider the **formal quadratic representation**

$$\begin{aligned} F(i,a) &= 2R^2_{(i,a)} - R_{(i,a)^2} = 2(R_a, L_a)^2 - R_{(i \triangleleft a, a^2)} \\ &= \big(2R_a^2 - R_{a^2}, 2L_a^2 - L_{a^2}\big) \\ &= \big(Q(a), P(a)\big). \end{aligned} \tag{37}$$

The quadratic representation $Q(.) : \mathfrak{I}^{\text{ann}} \to \mathfrak{I}^{\text{ann}}$ over the annihilator ideal $\mathfrak{I}^{\text{ann}}$ is an entirely new object and so it should be investigated in detail. On the contrary, the quadratic representation $P(a)$ on J is well studied.

We define

$$Q(a,b) = R_a R_b + R_b R_a - R_{ab}, \tag{38}$$

where $a, b \in J$. Clearly $Q(a) = Q(a,a)$.

We remember that $\mathfrak{I} = \mathfrak{I}^{\text{ann}} \oplus J$ is a restrictive, split and unital quasi-Jordan algebra.

Proposition 14. *Let $D = (D_1, D_2)$ be a derivation. Then*

$$2Q(D_2 a, a) = \big[D_1, Q(a)\big], \tag{39}$$

for all $a \in J$.

Proof. From (31) we know that if $a \in J$

$$R_{D_2 a} = [D_1, R_a], \tag{40}$$

hence

$$2Q(D_2 a, a) = 2(R_{D_2 a} R_a + R_a R_{D_2 a} - R_{(D_2 a)a}). \tag{41}$$

It shall be noted that $2(D_2 a)a = D_2 a^2$. Therefore, it follows from (41) that

$$\begin{aligned} 2Q(D_2 a, a) &= 2[D_1, R_a]R_a + 2R_a[D_1, R_a] - [D_1, R_{a^2}] \\ &= D_1\big(2R_a^2\big) - \big(2R_a^2\big)D_1 - [D_1, R_{a^2}] \\ &= \big[D_1, 2R_a^2 - R_{a^2}\big] = \big[D_1, Q(a)\big]. \qquad \square \end{aligned} \tag{42}$$

In addition to Proposition 14, from the classical theory of Jordan algebras it follows that

$$2P(D_2 a, a) = \big[D_2, P(a)\big], \tag{43}$$

for all $a \in J$.

Reciprocally, we have

Theorem 15. *Let $D_1 : \mathfrak{I}^{\text{ann}} \to \mathfrak{I}^{\text{ann}}$, $D_2 : J \to J$ be two transformations given, such that, D_1 and D_2 are linear and for all $a \in J$*

$$2Q(D_2 a, a) = \big[D_1, Q(a)\big], \qquad 2P(D_2 a, a) = \big[D_2, P(a)\big], \tag{44}$$

then $D = (D_1, D_2)$ is a derivation over $\mathfrak{I}$.

Proof. In fact, the second equality of (44) implies that D_2 is a derivation over J, so $D_2\epsilon = 0$. Now, in the first equality we replace a by $a + \alpha\epsilon$ and collect linear terms. We obtain $R_{D_2 a} = [D_1, R_a]$. It shows that $D = (D_1, D_2)$ belongs to $Der(\Im)$. □

Let us define

$$F\big((i,a),(j,b)\big) = R_{(i,a)}R_{(j,b)} + R_{(j,b)}R_{(i,a)} - R_{(i,a)\triangleleft(j,b)}, \tag{45}$$

then it is easy to show that $F((i,a),(j,b)) = (Q(a,b), P(a,b))$ where $P(a,b) = L_aL_b + L_bL_a - L_{ab}$ is a very well-known object in the theory of Jordan algebras.

We have the following result.

Theorem 16. *Let $D = (D_1, D_2) \in (End(\Im^{\mathrm{ann}}), End(J))$ be given, then D is a derivation over $\Im$ if and only if*

$$2F\big(D(i,a),(i,a)\big) = \big[D, F(i,a)\big], \tag{46}$$

for any $(i,a) \in \Im$.

Proof. Let us suppose that $D = (D_1, D_2)$ is a derivation then

$$\begin{aligned} 2F\big(D(i,a),(i,a)\big) &= 2(R_{D(i,a)}R_{(i,a)} + R_{(i,a)}R_{D(i,a)} - R_{D(i,a)\triangleleft(i,a)}) \\ &= 2(R_{(D_1 i, D_2 a)}R_{(i,a)} + R_{(i,a)}R_{(D_1 i, D_2 a)} - R_{(D_1 i \triangleleft a, D_2 a.a)}), \end{aligned}$$

but

$$\begin{aligned} R_{(D_1 i, D_2 a)}R_{(i,a)} &= (R_{D_2 a}, L_{D_2 a})(R_a, L_a) = (R_{D_2 a}R_a, L_{D_2 a}L_a), \\ R_{(i,a)}R_{(D_1 i, D_2 a)} &= (R_a, L_a)(R_{D_2 a}, L_{D_2 a}) = (R_a R_{D_2 a}, L_a L_{D_2 a}), \end{aligned}$$

and

$$R_{(D_1 i \triangleleft a, D_2 a.a)} = (R_{D_2 a.a}, L_{D_2 a.a}).$$

Hence

$$2F\big(D(i,a),(i,a)\big) = 2(A,B),$$

where

$$\begin{aligned} 2A &= 2R_{D_2 a}R_a + 2R_a R_{D_2 a} - 2R_{D_2 a.a} \\ &= 2[D_1, R_a]R_a + 2R_a[D_1, R_a] - R_{D_2 a^2} \\ &= \big[D_1, 2R_a^2\big] - R_{D_2 a^2} \\ &= \big[D_1, Q(a)\big]. \end{aligned}$$

In the same way we can see that $2B = [D_2, P(a)]$. Hence,

$$2F\big(D(i,a),(i,a)\big) = \big(\big[D_1, Q(a)\big], \big[D_2, P(a)\big]\big). \tag{47}$$

On the other hand,

$$\begin{aligned}\big[D, F(i,a)\big] &= \big[(D_1, D_2), 2R^2_{(i,a)} - R_{(i,a)^2}\big] \\ &= \big[(D_1, D_2), \big(Q(a), P(a)\big)\big] \\ &= \big(\big[D_1, Q(a)\big], \big[D_2, P(a)\big]\big). \end{aligned} \tag{48}$$

It follows from (47) and (48) that (46) holds.

Reciprocally, if (46) holds then

$$2\big(Q(D_2a, a), P(D_2a, a)\big) = \big(\big[D_1, Q(a)\big], \big[D_2, P(a)\big]\big).$$

It implies that $2Q(D_2a, a) = [D_1, Q(a)]$ and $2P(D_2a, a)[D_2, P(a)]$. In other words $D = (D_1, D_2)$ is a derivation over $\Im$. □

4. The structure group of a quasi-Jordan algebra

As before, $\Im = \Im^{\mathrm{ann}} \oplus J$ is a restrictive, split and unital quasi-Jordan algebra. In this section we introduce and study the structure group of this type of quasi-Jordan algebras. The unit in J is denoted by ϵ.

Let $\Phi : \Im \to \Im$ be such that if $\Phi = (\Phi_1, \Phi_2)$ then both $\Phi_1 : \Im^{\mathrm{ann}} \to \Im^{\mathrm{ann}}$ and $\Phi_2 : J \to J$ are bijective. By $\Gamma(\Im)$ we denote the set of these Φ for which there is $\Phi^\sharp = (\Phi_1^\sharp, \Phi_2^\sharp)$ such that for all $a \in J$

$$Q(\Phi_2 a) = \Phi_1 Q(a)\Phi_1^\sharp, \qquad P(\Phi_2 a) = \Phi_2 P(a)\Phi_2^\sharp, \tag{49}$$

and

$$Q\big(\Phi_2^\sharp a\big) = \Phi_1^\sharp Q(a)\Phi_1, \qquad P\big(\Phi_2^\sharp a\big) = \Phi_2^\sharp P(a)\Phi_2. \tag{50}$$

Lemma 17. *Let us suppose that $\Phi \in \Gamma(\Im)$, then for any $(i,a) \in \Im$*

$$F\big(\Phi(i,a)\big) = \Phi F(i,a)\Phi^\sharp, \qquad F\big(\Phi^\sharp(i,a)\big) = \Phi^\sharp F(i,a)\Phi. \tag{51}$$

Proof. It is very easy, so it will be omitted. □

Taking in (49) and (50), $a = \epsilon$ we obtain

$$\Phi_1^\sharp = \Phi_1^{-1} Q(\Phi_2\epsilon), \qquad \Phi_2^\sharp = \Phi_2^{-1} P(\Phi_2\epsilon), \tag{52}$$

that is $\Phi^\sharp = \Phi^{-1}F(\Phi(0,\epsilon))$, because $F(0,\epsilon) = Id$. Hence, it shows that $\Phi^\sharp$ is uniquely determined by Φ. Since (49) and (50) are recovered by exchanging the roles of Φ and $\Phi^\sharp$, it follows that $\Phi^\sharp \in \Gamma(\Im)$. Moreover, $(\Phi^\sharp)^\sharp = \Phi$.

Consider $\Phi, \Pi \in \Gamma(\Im)$ then replacing a by $\Pi_2 a$ in the first equation of (49) we see that

$$Q(\Phi_2\Pi_2 a) = \Phi_1 Q(\Pi_2 a)\Phi_1^\sharp = \Phi_1\Pi_1 Q(a)\Pi_1^\sharp\Phi_1^\sharp, \tag{53}$$

just as we did before in the first equation of (49) we obtain

$$P(\Phi_2\Pi_2 a) = \Phi_2 P(\Pi_2 a)\Phi_2^\sharp = \Phi_2\Pi_2 P(a)\Pi_2^\sharp\Phi_2^\sharp. \tag{54}$$

On the other hand, from (50)

$$Q\left(\Phi_2^\sharp \Pi_2^\sharp a\right) = \Phi_1^\sharp Q\left(\Pi_2^\sharp a\right)\Phi_1 = \Phi_1^\sharp \Pi_1^\sharp Q(a)\Pi_1\Phi_1, \tag{55}$$

and

$$P\left(\Phi_2^\sharp \Pi_2^\sharp a\right) = \Phi_2^\sharp P\left(\Pi_2^\sharp a\right)\Phi_2 = \Phi_2^\sharp \Pi_2^\sharp P(a)\Pi_2\Phi_2. \tag{56}$$

Thus, we are able to infer that if $\Phi, \Pi \in \Gamma(\Im)$ then $\Phi\Pi \in \Gamma(\Im)$ and $(\Phi\Pi)^\sharp = \Pi^\sharp\Phi^\sharp$.

Since $\Phi^\sharp \in \Gamma(\Im)$ provided that $\Phi \in \Gamma(\Im)$ then from $Q(\Phi_2(\Phi_2^{-1}a)) = \Phi_1 Q(\Phi_2^{-1}a)\Phi_1^\sharp$ it follows that $\Phi_1^{-1}Q(a)(\Phi_1^\sharp)^{-1} = Q(\Phi_2^{-1}a)$. Also, one can see that $\Phi_2^{-1}P(a)(\Phi_2^\sharp)^{-1} = P(\Phi_2^{-1}a)$.

In a quite similar form we can obtain that $Q((\Phi_2^\sharp)^{-1}a) = (\Phi_1^\sharp)^{-1}Q(a)\Phi_1^{-1}$ and $P((\Phi_2^\sharp)^{-1}a) = (\Phi_2^\sharp)^{-1}P(a)\Phi_2^{-1}$.

Thus, we conclude that if $\Phi \in \Gamma(\Im)$ then $\Phi^{-1} = (\Phi_1^{-1}, \Phi_2^{-1}) \in \Gamma(\Im)$ and $(\Phi^{-1})^\sharp = (\Phi^\sharp)^{-1}$. Hence, $\Gamma(\Im)$ is a group which will be called the **structure group** of $\Im$. The map $\Phi \to \Phi^\sharp$ is an involution over $\Gamma(\Im)$.

Next, we will study an important subgroup of $\Gamma(\Im)$. First of all, we observe that for every $a, b \in J$

$$Q(a+b) = Q(a) + Q(b) + 2Q(a,b), \qquad P(a+b) = P(a) + P(b) + 2P(a,b). \tag{57}$$

Theorem 18. *Suppose that $\Phi \in \Gamma(\Im)$ and $(i,a), (j,b) \in \Im$, then*

$$F\left(\Phi(i,a), \Phi(j,b)\right) = \Phi F\left((i,a),(j,b)\right)\Phi^\sharp, \tag{58}$$

and

$$F\left(\Phi^\sharp(i,a), \Phi^\sharp(j,b)\right) = \Phi^\sharp F\left((i,a),(j,b)\right)\Phi. \tag{59}$$

Proof. First, we prove (58). For this, we shall use (57). In fact,

$$\begin{aligned} F\left(\Phi(i,a), \Phi(j,b)\right) &= F\left((\Phi_1 i, \Phi_2 a), (\Phi_1 j, \Phi_2 b)\right) \\ &= \left(Q(\Phi_2 a, \Phi_2 b), P(\Phi_2 a, \Phi_2 b)\right) \\ &= (H, I), \end{aligned}$$

now

$$H = \Phi_1\left(\frac{Q(a+b) - Q(a) - Q(b)}{2}\right)\Phi_1^\sharp = \Phi_1 Q(a,b)\Phi_1^\sharp,$$

and

$$L = \Phi_2\left(\frac{P(a+b) - P(a) - P(b)}{2}\right)\Phi_2^\sharp = \Phi_2 P(a,b)\Phi_2^\sharp.$$

We deduce that $(H, I) = \Phi F((i,a),(j,b))\Phi^\sharp$. Thus, we have proved (58). The proof of (59) is similar and this is left to the reader. □

We say that $\Phi \in \Gamma(\Im)$ is an automorphism if $\Phi((i,a) \lhd (j,b)) = \Phi(i,a) \lhd \Phi(j,b)$. We will denote the set of all automorphisms by $Aut(\Im)$. It is easy to show that $\Phi = (\Phi_1, \Phi_2) \in Aut(\Im)$ if and only if for $i \in \Im^{\mathrm{ann}}$ and $a, b, c \in J$

$$\Phi_1(i \lhd a) = \Phi_1 i \lhd \Phi_2 a, \qquad \Phi_2 ab = (\Phi_2 a)(\Phi_2 b).$$

One can also check that if $\Phi, \Pi \in Aut(\Im)$ then $\Phi\Pi \in Aut(\Im)$ and $\Phi^{-1} \in Aut(\Im)$. Hence, $Aut(\Im)$ is a subgroup of $\Gamma(\Im)$.

We have a simple characterization of the elements of $Aut(\Im)$.

Theorem 19. *Let Φ be an element of $\Gamma(\Im)$. Then $\Phi \in Aut(\Im)$ if and only if $\Phi(0,\epsilon) = (0,\epsilon)$ and $\Phi_1^\sharp(i \lhd a) = \Phi_1^\sharp i \lhd \Phi_2^\sharp a$ for all $i \in \Im^{\mathrm{ann}}$ and every $a \in J$. Moreover, in this case $\Phi^\sharp = \Phi^{-1}$.*

Proof. Let Φ be an element of $Aut(\Im)$, then $\Phi R_{(j,b)}(i,a) = R_{\Phi(j,b)}\Phi(i,a)$. It implies that $\Phi R_{(j,b)}\Phi^{-1} = R_{\Phi(j,b)}$ and hence

$$\begin{aligned}\Phi F(i,a)\Phi^{-1} &= \Phi\big(2R^2_{(i,a)}\big)\Phi^{-1} - \Phi R_{(i,a)^2}\Phi^{-1} \\ &= 2\big(\Phi R_{(i,a)}\Phi^{-1}\big)\big(\Phi R_{(i,a)}\Phi^{-1}\big) - R_{\Phi(i,a)^2} \\ &= 2R^2_{\Phi(i,a)} - R_{(\Phi(i,a))^2} = F\big(\Phi(i,a)\big).\end{aligned} \tag{60}$$

On the other hand, observe that Lemma 17 shows us that $\Phi : \Im \to \Im$ bijective, belongs to $\Gamma(\Im)$ if and only if there is $\Phi^\sharp$ such that

$$F\big(\Phi(i,a)\big) = \Phi F(i,a)\Phi^\sharp, \qquad F\big(\Phi^\sharp(i,a)\big) = \Phi^\sharp F(i,a)\Phi.$$

But we just have proved that $F(\Phi(i,a)) = \Phi F(i,a)\Phi^{-1}$. Also, from here, it follows that $F(\Phi^{-1}(i,a)) = \Phi^{-1}F(i,a)\Phi$. Thus $\Phi^\sharp = \Phi^{-1}$. Since $\Phi_1 \in End(\Im^{\mathrm{ann}})$ and $F(0,\epsilon) = Id_{|\Im}$ the equality $\Phi R_{(j,b)}\Phi^{-1} = R_{\Phi(j,b)}$ implies that $\Phi_2\epsilon = \epsilon$. Thus $\Phi(0,\epsilon) = (0,\epsilon)$. Finally, as $\Phi^\sharp = \Phi^{-1} \in Aut(\Im)$ then $\Phi_1^\sharp(i \lhd a) = \Phi_1^\sharp i \lhd \Phi_2^\sharp a$.

Reciprocally, if $\Phi \in \Gamma(\Im)$ is such that $\Phi(0,\epsilon) = (0,\epsilon)$. Then by definition there exists $\Phi^\sharp$ for which (51) holds for all $(i,a) \in \Im$. Thus,

$$\begin{aligned}\big(0, \big(\Phi_2^\sharp a\big)^2\big) &= F\big(\Phi^\sharp(i,a)\big)(0,\epsilon) = \Phi^\sharp F(i,a)\Phi(0,\epsilon) \\ &= \Phi^\sharp F(i,a)(0,\epsilon) = \Phi^\sharp\big(0,a^2\big) = \big(0, \Phi_2^\sharp a^2\big).\end{aligned}$$

Hence, $(\Phi_2^\sharp a)^2 = \Phi_2^\sharp a^2$ for all $a \in J$. Now linearizing this last equality we arrive at the following $\Phi_2^\sharp(ab) = (\Phi_2^\sharp a)(\Phi_2^\sharp b)$. Combining the last equation and the second hypothesis we deduce that $\Phi^\sharp$ is an automorphism. Now from the first part of the proof we have that $\Phi = (\Phi^\sharp)^\sharp = (\Phi^\sharp)^{-1}$. Thus, $\Phi \in Aut(\Im)$. □

5. Concluding remarks

Out below we propose a few possible directions of work:

1) Suppose $\mathbb{F}$ is a field of characteristic different of 2 or 3. Tits, Kantor and Koecher have given a construction of Lie algebras over $\mathbb{F}$ from Jordan algebras. Thus, an interesting problem should be the construction of Leibniz algebras from quasi-Jordan algebras (see [6] and [10] for more details).

2) We believe it is necessary to initiate a systematic study of the role of quasi-Jordan algebras in analysis and differential geometry. For instance, what could be the analogue of the concept of

cone over a Euclidean quasi-Jordan algebra and what could be his relationship with optimization problems.

3) Establish a relationship between the quasi-Jordan algebras and some type of Capelli identities.

Acknowledgments

I thank Murray Bremner for sending me his preprints [1] and [2]. This research was supported in part under CONACYT grant 106923 and the Cuban National Research Project "Theory and efficient algorithms for the solution of problems in operator algebra and geometric modeling".

The author also wishes to thank the referee for his (or her) accurate and valuable comments.

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