A Liouville type theorem for poly-harmonic Dirichlet problems in a half space

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Received 26 November 2011; accepted 30 January 2012
Available online 9 February 2012
Communicated by Erwin Lutwak

Abstract

In this paper, we consider the following Dirichlet problem for poly-harmonic operators on a half space $\mathbb{R}^n_+$:

\[
\begin{aligned}
(-\Delta)^m u &= u^p, & \text{in } \mathbb{R}^n_+, \\
u &= \frac{\partial u}{\partial x_n} = \frac{\partial^2 u}{\partial x_n^2} = \cdots = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0, & \text{on } \partial \mathbb{R}^n_+.
\end{aligned}
\]  

(1)

First, under some very mild growth conditions, we show that problem (1) is equivalent to the integral equation

\[
u(x) = \int_{\mathbb{R}^n_+} G(x, y) u^p \, dy,
\]  

where $G(x, y)$ is the Green’s function on the half space.

Then, by combining the method of moving planes in integral forms with some new ideas, we prove that there is no positive solution for integral equation (2) in both subcritical and critical cases. This partially
solves an open problem posed by Reichel and Weth (2009) [40]. We also prove non-existence of weak solutions for problem (1).

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Keywords: Liouville theorems; Half space; Higher order; Poly-harmonic operators; Dirichlet problems; Equivalence; Integral equations; Green’s functions; Method of moving planes in integral forms; Non-existence of solutions

1. Introduction

Let

\[ R^n_+ = \{ x = (x_1, \ldots, x_n) \in R^n \mid x_n > 0 \} \]

be the upper half Euclidean space.

Consider the Dirichlet problem for poly-harmonic operators

\[
\begin{cases}
(-\Delta)^m u = u^p, & \text{in } R^n_+ , \\
u = \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0, & \text{on } \partial R^n_+ ,
\end{cases}
\]  

(3)

where \( m \) is any positive integer, \( 2m < n \), and \( 1 < p \leq \frac{n+2m}{n-2m} \).

First, we show that, under a mild growth condition on \( u \), (3) is equivalent to the integral equation

\[
u(x) = \int_{R^n_+} G^+(x, y) u^p(y) \, dy ,
\]

(4)

where

\[
G^+_{\infty}(x, y) = \frac{c_n}{|x - y|^{n-2m}} \int_0^{\frac{|x - y|^{n-2m}}{z^{n-2m}}} \frac{z^{m-1}}{(z + 1)^{n/2}} \, dz
\]

(5)

is the Green’s function in \( R^n_+ \) with the same Dirichlet boundary conditions.

Then, by using the method of moving planes in integral forms, we prove that integral equation (4) possesses no positive solutions. It is well known that this kind of Liouville theorem plays an important role in a priori estimates of solutions for the corresponding family of equations either on domains or on Riemannian manifolds with boundary.

The same problem (3) has been considered by Reichel and Weth [40]. They proved that there are no bounded classical solutions. Then they posed an open problem:

**Can the boundedness assumption on \( u \) be removed?**

In this paper, we partially answer their open question. For the equivalence, Reichel and Weth proved
**Proposition 1.** Suppose that $u \in C^{2m-1}(\overline{R^n_+}) \cap W_{loc}^{2m,p}(R^n_+)$, $p > \frac{n}{2m}$ is a function with the following properties:

(i) $u$ and all partial derivatives of $u$ of order less than or equal to $2m-1$ are bounded.
(ii) $u$ satisfies Dirichlet boundary conditions in (3).
(iii) $(-\Delta)^m u \in L^p_{loc}(R^n_+)$ is non-negative in $R^n_+$.

Then

$$u(x) = \int_{R^n_+} G^+_\infty(x,y)(-\Delta)^m u(y) \, dy, \quad x \in R^n_+.$$ 

In this paper, we removed their boundedness assumptions on $u$ and all its derivatives and replace it by a much weaker one. Some new ideas are involved.

**Theorem 1.** Suppose that $u \in C^{2m-1}(\overline{R^n_+}) \cap W_{loc}^{2m,p}(R^n_+)$, $p > \frac{n}{2m}$ is a function with the following properties:

(i) For $|\alpha| = m - 1$, $m \geq 2$,

$$|D^\alpha u| = O(|y|^a), \quad \text{for large } |y|, \text{ and for some } 0 < a < 1.$$  \hspace{1cm} (6)

(ii) $u$ satisfies equation and Dirichlet boundary conditions (3).
(iii) $(-\Delta)^m u \in L^p_{loc}(R^n_+)$ is non-negative in $R^n_+$.

Then

$$u(x) = \int_{R^n_+} G^+_\infty(x,y)(-\Delta)^m u(y) \, dy, \quad x \in R^n_+.$$ 

As an immediate consequence, we have

**Corollary 1.** If $u$ is a positive classical solution of PDE (3) satisfying (6), then a constant multiple of $u$ is a solution of integral equation (4). Condition (6) is not needed when $m = 1$.

It is also easy to show

**Theorem 2.** If $u$ is a positive smooth solution of integral equation (4), then a constant multiple of $u$ satisfies PDE (3).

Due to the equivalence between PDE (3) and integral equation (4), in order to establish Liouville theorems for (3), we only need to work on integral equation (4). We prove

**Theorem 3.** Assume $\frac{n}{n-2m} < p \leq \frac{n+2m}{n-2m}$. If $u \in L^n_{\infty} (R^n_+)$ is a non-negative solution of (4), then $u(x) \equiv 0$. 
This together with Corollary 1 implies

**Corollary 2.** Assume \( \frac{n}{n-2m} < p \leq \frac{n+2m}{n-2m} \). If \( u \) is a non-negative classical solution of (3) satisfying (6), then \( u \equiv 0 \). For \( m = 1 \), condition (6) is not needed for the conclusion to be valid.

**Remark 1.**

(i) Under the assumption that \( u \) be a classical bounded solution of (3) as given in [40], from the equation, one derives immediately that all the partial derivatives of \( u \) up to the order \( 2m - 1 \) are bounded. Hence we partially answer the open question posed by Reichel and Weth. Later, in [41], Reichel and Weth removed the boundedness assumption on \( u \) in the subcritical case by using a doubling lemma. Although in the statement of their theorem, the critical case is included. However, we believe that is a typo and they probably have noticed it by now. Because in their proof, they needed to use the non-existence of positive bounded solutions for the same equation in both \( \mathbb{R}^n \) and \( \mathbb{R}^n_+ \). However, it is well known that, in the critical case, there are a family of solutions for the equation in \( \mathbb{R}^n \).

(ii) When \( m = 1 \), Gidas and Spruck [20] proved the non-existence of positive solutions for (3), which is a special case of Corollary 2. Although they only stated the result for the subcritical case, their proof works for critical case as well.

(iii) Fang and Zhang [18] and Lu and Zhu [35] considered integral equation (4) with more general function \( f(u) \) instead of \( u^p \). However, to show symmetry and non-existence of positive solutions, they needed to assume some global integrability conditions.

To prove Theorem 3, we use the method of moving planes in integral forms. It is completely different from the traditional methods of moving planes used for PDEs. Instead of using local properties of the differential operators, such as maximum principles, we exploited global properties and estimated certain integral norms. One remarkable advantage of this method is that it works for all real values of \( \alpha \) indiscriminately. For people who once applied the method of moving planes for equations involving Laplacians would notice that it becomes much more difficult to consider higher powers of Laplacian because there is no corresponding maximum principles, and let alone fractional powers of Laplacian.

To apply the method of moving planes in integral forms, one usually needs to assume some global integrability on the solution \( u \). Here we introduce a new idea to weaken this condition remarkably. By properly using Kelvin transforms, we only need to assume that \( u \) is locally integrable. To ensure that the half space \( \mathbb{R}^n_+ \) is invariant under the inversion, we need to place the centers at boundary \( \partial \mathbb{R}^n_+ \). For a point \( z_0 \in \partial \mathbb{R}^n_+ \), we consider

\[
\bar{u}(x) = \frac{1}{|x-z_0|^{n-2m}} u \left( \frac{x-z_0}{|x-z_0|^2} + z_0 \right),
\]

the Kelvin type transform of \( u(x) \). We consider two possibilities.

(i) **There is a** \( z_0 \in \partial \mathbb{R}^n_+ \) **such that** \( \bar{u}(x) \) **is not singular at** \( z_0 \). **In this case,** \( u \) **is globally integrable, and we move the planes in the direction of** \( x_n \)-axis **to show that the solution** \( u \) **is monotone increasing in** \( x_n \).

(ii) **For all** \( z_0 \in \partial \mathbb{R}^n_+ \), \( \bar{u}(x) \) **is singular at** \( z_0 \). **In this case, we move the planes in** \( x_1, \ldots, x_{n-1} \) **directions to show that** \( \bar{u} \) **is axially symmetric about the line that is parallel to** \( x_n \)-axis **and passing through** \( z_0 \). **This implies further that** \( u \) **depends on** \( x_n \) **only.
In both cases, we will be able to derive a contradiction. We believe that this idea can be applied to many other problems.

We also study non-existence of weak solutions. As usual, we say that $u$ is a weak solution of (3) in Sobolev space $H^m_0(R^n_+)$, if it satisfies

$$\langle u, v \rangle_m = \int_{R^n_+} u^p(x)v(x)dx, \quad \forall v \in H^m_0(R^n_+),$$

where

$$\langle u, v \rangle_m = \begin{cases} \int_{R^n_+} \Delta^m u(x) \cdot \Delta^m v(x)dx, & m \text{ even}, \\ \int_{R^n_+} (\nabla \Delta^{m-1} u(x)) \cdot (\nabla \Delta^{m-1} v(x))dx, & m \text{ odd} \end{cases}$$

is an inner product in $H^m_0(R^n_+)$. When consider weak solutions of (3), the growth condition (6) can be replaced by $u \in H^m_0(R^n_+)$, then the equivalence result still holds.

**Theorem 4.** Assume $2m < n$ and $1 < p \leq \frac{n+2m}{n-2m}$. If $u \in H^m_0(R^n_+)$ is a weak solution of partial differential equations (3), then a constant multiple of $u$ satisfies integral equation (4).

This together with Theorem 3 implies

**Corollary 3.** Assume $2m < n$ and $\frac{n}{n-2m} < p \leq \frac{n+2m}{n-2m}$. If $u \in H^m_0(R^n_+)$ is a non-negative weak solution of (3), i.e. if $u$ satisfies (7), then $u \equiv 0$.

For more related results, please see [1,2,4–10,12–17,21–34,36–39,42] and the references therein.

This paper is arranged as follows. In Section 2, we will establish the equivalence between the integral equations and PDEs and thus prove Theorems 1, 2, and 4. In Section 3, we will use the method of moving planes in integral forms and Kelvin transforms to prove Theorem 3 – the non-existence of positive solutions for integral equation (4).

We use $C$ to denote various positive constants.

2. The equivalence between integral equations and PDEs

2.1. The proof of Theorem 1

In this section we prove

**Theorem 2.1.** Suppose that $u \in C^{2m-1}(\overline{R^n_+}) \cap W^{2m,p}_{loc}(R^n_+), \quad p > \frac{n}{2m}$ is a function with the following properties:

(i) For $|\alpha| = m - 1, m \geq 2$,

$$|D^\alpha u| = O(|y|^a), \quad \text{for large } |y|, \text{ and for some } 0 < a < 1.$$
(ii) \( u \) satisfies equation and Dirichlet boundary conditions (3).  
(iii) \((-\Delta)^m u \in L^p_{\text{loc}}(R^n_+)\) is non-negative in \( R^n_+ \).

Then

\[
u(x) = \int_{R^n_+} G^+_\infty(x, y)(-\Delta)^m u(y) \, dy, \quad x \in R^n_+.
\]

In [3], Boggio obtained the Green’s function of the operator \((-\Delta)^m\) with Dirichlet boundary conditions on the unit ball \( B_1 = \{x \in R^n: |x| < 1\} \):

\[
G_1(x, y) = \frac{c_n}{|x - y|^{n-2m}} \int_0^{\varphi_1(x, y)} \frac{z^{m-1}}{(1 + z)^{n/2}} \, dz, \quad x, y \in B_1,
\]

where

\[
\varphi_1(x, y) = \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2},
\]

and \( c_n \) is a normalization constant.

Entirely similar to [40], on the ball \( B_R = \{x \in R^n: |x| < R\} \), the Green’s function can be expressed as

\[
G_R(x, y) = \frac{1}{R^{n-2m}} G_1 \left( \frac{x}{R}, \frac{y}{R} \right) = \frac{c_n}{|x - y|^{n-2m}} \int_0^{\varphi_R(x, y)} \frac{z^{m-1}}{(1 + z)^{n/2}} \, dz.
\]

with

\[
\varphi_R(x, y) = \frac{(R^2 - |x|^2)(R^2 - |y|^2)}{R^2 |x - y|^2}.
\]

Denote \( p_R := (0, 0, \ldots, R) \in R^n_+ \), and \( B_R(p_R) := \{x \in R^n_+: |x - p_R| < R\} \), the ball of radius \( R \) centered at \( p_R \). Let

\[
\varphi_R^+(x, y) = \frac{(R^2 - |x - p_R|^2)(R^2 - |y - p_R|^2)}{R^2 |x - y|^2}.
\]

Then we can write the Green’s function on \( B_R(p_R) \) as

\[
G_R^+(x, y) = \frac{1}{R^{n-2m}} G_1 \left( \frac{x - p_R}{R}, \frac{y - p_R}{R} \right) = \frac{c_n}{|x - y|^{n-2m}} \int_0^{\varphi_R^+(x, y)} \frac{z^{m-1}}{(1 + z)^{n/2}} \, dz.
\]
From [40], we obtain the Green’s function on $\mathbb{R}^n_+$ with Dirichlet boundary conditions:

$$
G^+_\infty(x, y) = \frac{c_n}{|x - y|^{n - 2m}} \int_0^{\frac{4\pi y_n}{|x - y|^2}} \frac{z^{m - 1}}{(z + 1)^{n/2}} dz.
$$

The following two lemmas play important roles in our proof.

**Lemma 1** (Reichel–Weth). The Green’s function $G^+_R$ on $B_R(P_R)$ converges pointwise and monotonically to the Green’s function $G^+_\infty$ on $\mathbb{R}^n_+$.

**Lemma 2** (Reichel–Weth). Let $G$ be the Green’s function of $(-\Delta)^m$ with Dirichlet boundary condition on arbitrary ball $B \subset \mathbb{R}^n$ with exterior unit normal $\nu$ on $\partial B$. For any function $u \in C^{2m-1}(\overline{B}) \cap W^{2m,p}(B)$ with $p > \frac{n}{2m}$, one has the following Poisson–Green representation for $x \in B$: for $m$ even,

$$
u(x) = \sum_{i=0}^{m-1} \int_{\partial B} \left( \Delta^i u \frac{\partial}{\partial \nu} \Delta^{m-i-1} G(x, y) - \Delta^{m-i-1} G(x, y) \frac{\partial}{\partial \nu} \Delta^i u(y) \right) dS_y + \int_{\partial B} G(x, y)(-\Delta)^m u(y) dy, \quad (9)
$$

and for $m$ odd,

$$
u(x) = -\sum_{i=0}^{m-1} \int_{\partial B} \left( \Delta^i u \frac{\partial}{\partial \nu} \Delta^{m-i-1} G(x, y) - \Delta^{m-i-1} G(x, y) \frac{\partial}{\partial \nu} \Delta^i u(y) \right) dS_y - \int_{\partial B} \Delta^{m-i-1} u \frac{\partial}{\partial \nu} \Delta^{m-i-1} G(x, y) dS_y + \int_{\partial B} G(x, y)(-\Delta)^m u(y) dy. \quad (10)
$$

In [19, Lemma 3.4], Grunau and Sweers established the following estimates for the poly-harmonic Green’s function $G^1(x, y)$ on the unit ball if $|\alpha| = k \geq m$ and $x \in B_1$, $y \in \partial B_1$:

$$
|D_x^\alpha G^1_1(x, y)| \leq C|x - y|^{m-n-k}(1 - |x|)^m \quad (11)
$$

for some constant $C > 0$, where $D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \partial y_2^{\alpha_2} \cdots \partial y_n^{\alpha_n}}$, $\alpha_1 + \alpha_2 + \cdots + \alpha_n = |\alpha|$. For the Green’s function $G_R$ on $B_R$ and $G^+_R$ on $B_R(P_R)$ the estimate (11) transforms as follows:

$$
|D_x^\alpha G_R(x, y)| \leq C|x - y|^{m-n-k}(R - |x|)^m \quad \text{if } x \in B_R, \ y \in \partial B_R.
$$
Likewise,
\[
|D_x^\alpha G_R^+(x, y)| \leq C|x - y|^{m-n-k}|x|^m, \quad (12)
\]
if \( x = (0, 0, \ldots, x_n) \in B_R(P_R) \) with \( x_n \in (0, R) \), \( y \in \partial B_R(P_R) \).

**Proof of Lemma 2.** The proof of Theorem 2.1 consists of three steps. Let us first consider the case where \( m \) is even. For \( x = (0, 0, \ldots, x_n) \in R^n_+ \) with \( x_n > 0 \) fixed. In the following, we consider \( R > 2x_n \). Since for \( |\alpha| = m - 1, m \geq 2 \),
\[
|D_x^\alpha u| = O(|y|^a), \quad \text{for large} \ |y|, \ \text{and for some} \ 0 < a < 1, \quad (13)
\]
by the Mean Value Theorem and the boundary conditions, for \( 0 \leq i \leq \frac{m}{2} - 1 \), it is easy to see
\[
\left| \frac{\partial}{\partial \nu} [\Delta^i u(y)] \right| = O(|y|^{a+m-2i-2}) \quad (14)
\]
and
\[
|\Delta^i u(y)| = O(|y|^{a+m-2i-1}). \quad (15)
\]
By (12), we have
\[
\left| \frac{\partial}{\partial \nu} [(-\Delta)^{m-1-i} G_R^+(x, y)] \right| \leq \frac{C}{|x - y|^{n+m-2i-1}} \quad (16)
\]
and
\[
|(-\Delta)^{m-1-i} G_R^+(x, y)| \leq \frac{C}{|x - y|^{n+m-2i-2}}. \quad (17)
\]
Combining (9) with (16)–(17), for \( x \in B_R(P_R) \), we obtain
\[
0 \leq Q_R := \left| \int_{B_R(P_R)} G_R^+(x, y)(-\Delta)^m u(y) \, dy - u(x) \right|
\]
\[
\leq \sum_{i=0}^{m/2-1} \int_{\partial B_R(P_R)} \left( |\Delta^i u| \left| \frac{\partial}{\partial \nu} [\Delta^{m-1-i} G_R^+(x, y)] \right| + |\Delta^{m-1-i} G_R^+(x, y)| \left| \frac{\partial}{\partial \nu} (\Delta^i u) \right| \right) \, dS_y
\]
\[
\leq C \sum_{i=0}^{m/2-1} \int_{\partial B_R(P_R)} \left( \frac{1}{|x - y|^{n+m-2i-1}} + \left| \frac{\partial}{\partial \nu} (\Delta^i u) \right| \frac{1}{|x - y|^{n+m-2i-2}} \right) \, dS_y \quad (18)
\]
\[
\leq C \sum_{i=0}^{m/2-1} \int_{\partial B_R(P_R)} \left( \frac{1}{|x - y|^n} \left| \Delta^i u \right| + \left| \frac{\partial}{\partial \nu} (\Delta^i u) \right| \right) \, dS_y. \quad (19)
\]
Step 1. We first estimate

\[ I = \int_{\partial B_R(P_R)} \frac{1}{|x-y|^n} dS_y. \]

We use spherical polar coordinates. Let \( \theta \) denote the angle between \( \overrightarrow{PR_y} \) and \( \overrightarrow{PR_x} \) \((0 \leq \theta \leq \pi)\). By cosine formula, we have

\[ |x-y|^2 = R^2 + (R - x_n)^2 - 2R(R - x_n) \cos \theta = 2R(R - x_n)(1 - \cos \theta) + x_n^2. \] (20)

Let \( |S^k_r| \) be the area of \( k \)-dimension sphere of radius \( r \). Since \( |S_{R \sin \theta}^{n-2}| = (R \sin \theta)^{n-2} \cdot |S_1^{n-2}| \), we can rewrite

\[
I = \int_{\partial B_R(P_R)} \frac{1}{|x-y|^n} dS_y
\]

\[
= \frac{\pi}{R} \left( R \sin \theta \right)^{n-2} |S_1^{n-2}| \cdot \frac{1}{[2R(R - x_n)(1 - \cos \theta) + x_n^2]^\frac{n}{2}} R \, d\theta
\]

\[
+ \int_{\frac{\pi}{2}}^\pi \left( R \sin \theta \right)^{n-2} |S_1^{n-2}| \cdot \frac{1}{[2R(R - x_n)(1 - \cos \theta) + x_n^2]^\frac{n}{2}} R \, d\theta
\]

\[ = I_1 + I_2. \] (21)

Let \( t = R - R \cos \theta \), then \( 0 \leq t \leq 2R \). We have

\[ \sin^2 \theta = 1 - \cos^2 \theta = \frac{2t}{R} - \left( \frac{t}{R} \right)^2 = \frac{t}{R} \left( 2 - \frac{t}{R} \right) \leq \frac{2t}{R}. \] (22)

By (20)–(22), for \( \delta > 0 \), we calculate

\[
I_1 = R^{n-1} |S_1^{n-2}| \int_0^{\frac{\pi}{2}} \frac{\sin^{n-3} \theta \cdot \sin \theta \, d\theta}{[2R(R - x_n)(1 - \cos \theta) + x_n^2]^\frac{n}{2}}
\]

\[
\leq C R^{n-1} \int_0^R \frac{(\frac{2t}{R})^\frac{n-3}{2} \cdot \frac{dt}{R}}{[2t(R - x_n) + x_n^2]^\frac{n}{2}}
\]

\[
= C \left( R^{\frac{n-1}{2}} \int_0^{\frac{\delta}{2}} t^{\frac{n-3}{2}} \, dt + R^{\frac{n-1}{2}} \int_{\frac{\delta}{2}}^R t^{\frac{n-3}{2}} \, dt \right)
\]

\[ = C(h_1 + h_2). \]
Let \( t = \frac{x_n^2}{R} z \). Since \( 2x_n \leq R \), we obtain

\[
\begin{align*}
  h_1 &= R^{\frac{n-1}{2}} \int_0^\delta \frac{t^{\frac{n-3}{2}} dt}{\left[ 2t(R - x_n) + x_n^2 \right]^\frac{n}{2}} \\
  &= \frac{1}{\sqrt{R}} \int_0^\delta \frac{t^{\frac{n-3}{2}} dt}{\left[ 2t(1 - \frac{x_n}{R}) + \frac{x_n^2}{R} \right]^\frac{n}{2}} \\
  &\leq \frac{1}{\sqrt{R}} \int_0^\delta \frac{t^{\frac{n-3}{2}} dt}{(t + \frac{x_n^2}{R})^\frac{n}{2}} \\
  &\leq \frac{1}{\sqrt{R}} \int_0^\delta \frac{dt}{t^{\frac{1}{2}}(t + \frac{x_n^2}{R})} \\
  &= \frac{1}{x_n} \int_0^{\frac{R}{x_n}} \frac{dz}{\sqrt{z(z+1)}} \\
  &\leq \frac{C}{x_n}.
\end{align*}
\]  

(23)

Here, for each large \( R \), we choose \( \delta = \frac{1}{R^b} \), for some \( 0 < b < 1 \). See Fig. 1.

It follows that

\[ l = O\left( R^{\frac{1-b}{2}} \right). \]

Set

\[
D_1 = \{ y = (y', y_n) \in R_+^n \mid 0 < y_n \leq 1, \, |y'| \leq l \},
\]

and

\[
D_\delta = \{ y = (y', y_n) \in R_+^n \mid 0 < y_n \leq \delta, \, |y'| \leq l \}.
\]

It is easy to see \( D_\delta \subseteq D_1 \).
By assumption (13), we have

$$|D^\alpha u(y)| = O\left(R^{(1-b)a_2}\right), \quad |\alpha| = m - 1, \quad y \in \mathcal{D}_1.$$  

By the boundary conditions and the Mean Value Theorem, we get

$$u(y) = O\left(R^{(1-b)a_2}\right), \quad y \in \mathcal{D}_1. \quad (24)$$

By the equation $$(-\Delta)^mu = u^p$$, we obtain

$$|\Delta^m u(y)| = u^p(y) = O\left(R^{(1-b)a_p}\right), \quad y \in \mathcal{D}_1. \quad (25)$$

It follows from (24), (25), and the Sobolev embedding, we have

$$|D^\gamma u(y)| = O\left(R^{(1-b)a_p}\right), \quad y \in \mathcal{D}_1, \quad |\gamma| = m.$$ 

Consequently, by using the boundary conditions and the Mean Value Theorem again, for $$b$$ sufficiently close to 1, we deduce, for $$|\alpha| = m - 1$$,

$$|D^\alpha u(y)| = |D^\gamma u(y)|O\left(\frac{1}{R^b}\right) = O\left(R^{(1-b)a_p-b}\right) \rightarrow 0, \quad \text{as } R \rightarrow \infty, \quad y \in \mathcal{D}_\delta. \quad (26)$$

$$|D^\beta u(y)| = O\left(R^{(1-b)a_p-(m-|\beta|)b}\right) \rightarrow 0, \quad \text{as } R \rightarrow \infty, \quad |\beta| = 0, 1, \ldots, m - 2, \quad y \in \mathcal{D}_\delta. \quad (27)$$

By (23), (26), and (27), we arrive at that $$\forall \epsilon > 0, \exists \delta > 0$$, such that

$$\sum_{i=0}^{m/2-1} \int_{\partial B_R(\mathcal{P}_R)^{0 \leq t<\delta}} \frac{1}{|x-y|^n}\left(|\Delta^i u| + \left|\frac{\partial}{\partial \nu} (\Delta^i u)\right|\right) dS_y < \epsilon, \quad \text{for } R \text{ sufficiently large.} \quad (28)$$

**Step 2.** On the other hand, by (18)–(19), (14)–(15) and (28), we obtain

$$0 \leq Q_R$$

$$\leq C \sum_{i=0}^{m/2-1} \int_{\partial B_R(\mathcal{P}_R)^{0 \leq t<\delta}} \left(|\Delta^i u| \frac{1}{|x-y|^{n+m-2i-1}} + \left|\frac{\partial}{\partial \nu} (\Delta^i u)\right| \frac{1}{|x-y|^{n+m-2i-2}}\right) dS_y$$

$$+ C \sum_{i=0}^{m/2-1} \int_{\partial B_R(\mathcal{P}_R)^{t \leq \delta<2R}} \left(|\Delta^i u| \frac{1}{|x-y|^{n+m-2i-1}} + \left|\frac{\partial}{\partial \nu} (\Delta^i u)\right| \frac{1}{|x-y|^{n+m-2i-2}}\right) dS_y$$

$$\leq C \sum_{i=0}^{m/2-1} \int_{\partial B_R(\mathcal{P}_R)^{0 \leq t<\delta}} \frac{1}{|x-y|^n}\left(|\Delta^i u| + \left|\frac{\partial}{\partial \nu} (\Delta^i u)\right|\right) dS_y$$
\[ + C \sum_{i=0}^{m/2-1} \int_{\partial B_R(P_R) \delta \leq t < 2R} \left( |\Delta^i u| \frac{1}{|x-y|^{n+m-2i-1}} + \left| \frac{\partial}{\partial \nu} (\Delta^i u) \right| \frac{1}{|x-y|^{n+m-2i-2}} \right) dS_y \]

\[ \leq o(1) + C \int_{\partial B_R(P_R) \delta \leq t < 2R} \frac{1}{|x-y|^{n-a}} dS_y, \quad \text{(29)} \]

We calculate

\[ \int_{\partial B_R(P_R) \delta \leq t < 2R} \frac{1}{|x-y|^{n-a}} dS_y \leq C R^{a-1} \int_{\delta}^{2R} \frac{t^{-\frac{a-3}{2}} dt}{[2t(R-x_n) + x_2^a]^{\frac{a-a}{2}}} \]

\[ = C R^{a-1} \int_{\delta}^{2R} \frac{t^{-\frac{a-3}{2}} dt}{[2t(1-x_n/R) + x_2^a]^{\frac{a-a}{2}}} \]

\[ \leq C R^{a-1} \int_{\delta}^{2R} \frac{t^{-\frac{a-3}{2}} dt}{(t + x_2^a)^{\frac{a-a}{2}}} \]

\[ \leq C R^{a-1} \int_{\delta}^{2R} \frac{t^{-\frac{a-3}{2}} dt}{(t + x_2^a)^{\frac{a-a}{2}}} \]

\[ \leq C R^{a-1} \int_{\delta}^{2R} t^{-\frac{a-3}{2}} dt \]

\[ \leq C (R\delta)^{a-1} - CR^{a-1} \]

\[ \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty. \quad \text{(30)} \]

Here we have used \(2x_n \leq R\) and \(0 < a < 1\).

By (29)–(30), we have

\[ Q_R \rightarrow 0, \quad \text{as} \quad R \rightarrow \infty. \quad \text{(31)} \]

Then by (9), (31) and Lemma 1 together with the Monotone Convergence Theorem we deduce

\[ u(x) = \lim_{R \rightarrow \infty} \int_{B_R(P_R)} G^+_R(x, y)u^p(y) \, dy = \int_{R^n_+} G^+_\infty(x, y)u^p(y) \, dy. \]

This completes the proof in the case where \(m\) is an even integer.

In the case where \(m\) is odd, only minor modifications are needed. We use (10) instead of (9) and the proof goes similarly.

This completes the proof of Theorem 2.1. \(\square\)
2.2. The proof of Theorem 2

Theorem 2.2. If $u$ is a positive smooth solution of integral equation

$$u(x) = \int_{\mathbb{R}_+^n} G_\infty^+(x, y) u^p(y) \, dy,$$

then a constant multiple of $u$ satisfies

$$\begin{cases} 
(-\Delta)^m u = u^p & \text{in } \mathbb{R}_+^n, \\
\frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{m-1} u}{\partial x_{n-1}^{m-1}} = 0 & \text{on } \partial \mathbb{R}_+^n. 
\end{cases}$$

Proof. Since

$$\begin{cases} 
(-\Delta)^m G_\infty^+(x, y) = \delta(x - y) & \text{in } \mathbb{R}_+^n, \\
\frac{\partial G_\infty^+}{\partial x_n} = \cdots = \frac{\partial^{m-1} G_\infty^+}{\partial x_{n-1}^{m-1}} = 0 & \text{on } \partial \mathbb{R}_+^n,
\end{cases}$$

we have

$$u = \frac{\partial u}{\partial x_n} = \frac{\partial^2 u}{\partial x_n^2} = \cdots = \frac{\partial^{m-1} u}{\partial x_{n-1}^{m-1}} = 0 \text{ on } \partial \mathbb{R}_+^n.$$ And also

$$(-\Delta)^m u(x) = \int_{\mathbb{R}_+^n} (-\Delta)^m G_\infty^+(x, y) u^p(y) \, dy = \int_{\mathbb{R}_+^n} \delta(x - y) u^p(y) \, dy = u^p(x).$$

This completes the proof of Theorem 2.2. $\Box$

2.3. The proof of Theorem 4

In this section, we prove

Theorem 2.3. Assume $2m < n$ and $1 < p \leq \frac{n+2m}{n-2m}$. If $u \in H_0^m(\mathbb{R}_+^n)$ is a weak solution of partial differential equations (3), then a constant multiple of $u$ satisfies integral equation (4).
Proof. Assume that \( u \in H^m_0(R^n_+) \) is a weak solution of

\[
\begin{cases}
(-\Delta)^m u = u^p & \text{in } R^n_+,
\end{cases}
\]

\[
\begin{align*}
u &= \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{m-1} u}{\partial x_n^{m-1}} = 0 & \text{on } \partial R^n_+.
\end{align*}
\]

(34)

Let \( D_R \) be the upper half ball of radius \( R \) centered at \( O = (0, 0, \ldots, 0) \). Denote the hemisphere part of its boundary by \( \Gamma_R \).

Multiplying both sides of the equation in (34) by

\[
G(x, y) = \frac{c_n}{|x - y|^{n-2m}} \int_0^{\Delta x_n y_n} \frac{z^{m-1}}{(1 + z)^{n/2}} \, dz
\]

and integrating by parts, we have, for each fixed \( x \in D_R \),

\[
\int_{D_R} G(x, y) u^p(y) \, dy = \int_{D_R} u (-\Delta)^m G(x, y) \, dy + \sum_{i=0}^{m-1} \left\{ \int_{\Gamma_R} (-\Delta)^i u \frac{\partial}{\partial \nu} [(-\Delta)^{m-1-i} G(x, y)] 
\right.
\]

\[
\left. - (-\Delta)^{m-1-i} G(x, y) \frac{\partial}{\partial \nu} [(-\Delta)^i u] \right\} \, dS_y \right\}.
\]

(35)

Let \( R \) be sufficiently large. For each fixed \( x \in D_R \),

\[
\frac{4x_n y_n}{|x - y|^2} \leq \frac{C}{|x - y|^2}
\]

(36)

then by (36),

\[
G(x, y) \leq \frac{c_n}{|x - y|^{n-2m}} \int_0^{\Delta x_n y_n} \frac{z^{m-1}}{(1 + z)^{n/2}} \, dz
\]

\[
\leq \frac{C}{|x - y|^{n-m}}.
\]

Also we can calculate

\[
\left| \frac{\partial G(x, y)}{\partial y_k} \right| \leq \frac{C}{|x - y|^{n-m+1}}, \quad k = 1, 2, \ldots, n.
\]

Similarly,

\[
\left| \frac{\partial}{\partial v} [(-\Delta)^{m-1-i} G(x, y)] \right| \leq \frac{C}{|x - y|^{n+m-2i-1}}, \quad i = 0, 1, \ldots, m - 1.
\]

(37)

\[
\left| (-\Delta)^{m-1-i} G(x, y) \right| \leq \frac{C}{|x - y|^{n+m-2i-2}}, \quad i = 0, 1, \ldots, m - 1.
\]

(38)
On the other hand, we recall the fact that if \( u \in L^p(R^n_+) \), there exists \( R_j \to \infty \), such that
\[
R_j \int_{\Gamma R_j} |u|^p dS \to 0.
\]

Then by Jensen’s inequality, we derive
\[
\frac{1}{R_j^{n-1-\frac{p}{n}}} \int_{\Gamma R_j} |u(y)| dS \to 0, \quad \text{as } R_j \to \infty. \tag{39}
\]

Since \( u \in H_0^m(R^n_+) \), we have \( u^p \in L^{\frac{2}{p}}(R^n_+) \), \( 1 < p \leq \frac{n+2m}{n-2m} \). By Eq. (34), it is easy to see \((-\Delta)^m u \in L^p(R^n_+)\). By Sobolev embedding, we have
\[
(-\Delta)^i u \in L^{\frac{2n}{np-4m+4i-2}}(R^n_+) \quad \text{and} \quad D^\gamma u \in L^{\frac{2n}{np-2(m-2i)-2}}(R^n_+), \quad |\gamma| = 2i + 1.
\]

Now we estimate (35). By (37) and (38), we have
\[
I_1 := \int_{\Gamma R} (-\Delta)^i u \frac{\partial}{\partial v} \left[ (-\Delta)^{m-1-i} G(x,y) - (-\Delta)^{m-1-i} G(x,y) \frac{\partial}{\partial v} [(-\Delta)^i u] \right] dS_y
\]
\[
\leq \frac{C}{R^{n+m-2i-1}} \int_{\Gamma R} (-\Delta)^i u \right] dS_y + \frac{C}{R^{n+m-2i-2}} \int_{\Gamma R} \left| \frac{\partial}{\partial v} [(-\Delta)^i u] \right| dS_y.
\]

Here we have used \(|x - y| \sim |y|\) as \( R \to \infty \). Since \( p > 1 \), it is easy to verify that
\[
n + m - 2i - 1 \geq n - 1 - \frac{np - 4(m - i)}{2},
\]
\[
n + m - 2i - 2 \geq n - 1 - \frac{np - 2(2m - 2i - 1)}{2}.
\]

Similarly to (39), we deduce
\[
I_1 \to 0, \quad \text{as } R \to \infty.
\]

Then by (35), let \( R \) go to infinity, we obtain
\[
u(x) = \int_{R^n_+} G(x,y) u^p(y) dy.
\]

This completes the proof of Theorem 2.3. \( \Box \)
3. The method of moving planes in integral forms and Kelvin transforms

In this section, we prove

**Theorem 3.1.** Assume $\frac{n}{n-2m} < p \leq \frac{n+2m}{n-2m}$. If $u \in L_{\text{loc}}^{\frac{n(p-1)}{2m}}(\mathbb{R}^n_+)$ is a non-negative solution of

$$u(x) = \int_{\mathbb{R}^n_+} G_\infty^+(x, y)u^p(y) \, dy.$$  \hfill (40)

Then $u(x) \equiv 0$.

### 3.1. Some lemmas

Let $\lambda$ be a positive real number and let the moving plane be $T_\lambda = \{x \in \mathbb{R}^n_+ \mid x_n = \lambda\}$. We denote $\Sigma_\lambda$ the region between the plane $x_n = 0$ and the plane $x_n = \lambda$. That is

$$\Sigma_\lambda = \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n_+ \mid 0 < x_n < \lambda\}.$$  

Let

$$x^\lambda = (x_1, \ldots, x_{n-1}, 2\lambda - x_n)$$

be the reflection of the point $x = (x_1, \ldots, x_{n-1}, x_n)$ about the plane $T_\lambda$,

$$\Sigma_\lambda^C = \mathbb{R}^n_+ \setminus \Sigma_\lambda,$$

the complement of $\Sigma_\lambda$,

$$u_\lambda(x) = u(x^\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x).$$

Before carrying on the method of moving planes, we state some properties of the Green’s function, which was established independently in [18] and [35].

**Lemma 3 (Fang–Zhang).**

(i) For any $x, y \in \Sigma_\lambda$, $x \neq y$, we have

$$G_\infty^+(x^\lambda, y^\lambda) > \max\{G_\infty^+(x^\lambda, y), G_\infty^+(x, y^\lambda)\}$$  \hfill (41)

and

$$G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y) > |G_\infty^+(x^\lambda, y) - G_\infty^+(x, y^\lambda)|.$$  \hfill (42)

(ii) For any $x \in \Sigma_\lambda$, $y \in \Sigma_\lambda^C$, it holds

$$G_\infty^+(x^\lambda, y) > G_\infty^+(x, y).$$  \hfill (43)
The following lemma is a key ingredient in our integral estimates.

**Lemma 4.** For any \( x \in \Sigma_\lambda \), it holds

\[
u(x) - u^\lambda(x) \leq \int_{\Sigma_\lambda} \left[ G^+_{\infty}(x^\lambda, y) - G^+_{\infty}(x, y^\lambda) \right][u^p(y) - u^p(x^\lambda)] dy.
\]

**Proof.** The proof is similar to that in [18] and [35], for the convenience of the readers, we sketch it here.

\[
u(x) = \int_{\Sigma_\lambda} G^+_{\infty}(x, y) u^p(y) dy + \int_{\Sigma_\lambda} G^+_{\infty}(x, y^\lambda) u^p_\lambda(y) dy + \int_{\Sigma^C_\lambda \setminus \Sigma_{\lambda^\lambda}} G^+_{\infty}(x, y) u^p(y) dy,
\]

\[
u(x^\lambda) = \int_{\Sigma_\lambda} G^+_{\infty}(x^\lambda, y) u^p(y) dy + \int_{\Sigma_\lambda} G^+_{\infty}(x^\lambda, y^\lambda) u^p_\lambda(y) dy + \int_{\Sigma^C_\lambda \setminus \Sigma_{\lambda^\lambda}} G^+_{\infty}(x^\lambda, y) u^p(y) dy,
\]

where \( \tilde{\Sigma}_\lambda = \{x^\lambda \mid x \in \Sigma_\lambda\} \). By Lemma 3, we arrive at

\[
u(x) - \nu(x^\lambda) = \int_{\Sigma_\lambda} \left[ G^+_{\infty}(x, y) - G^+_{\infty}(x^\lambda, y) \right]u^p(y) dy
\]

\[+ \int_{\tilde{\Sigma}_\lambda} \left[ G^+_{\infty}(x, y^\lambda) - G^+_{\infty}(x^\lambda, y^\lambda) \right]u^p_\lambda(y) dy
\]

\[+ \int_{\Sigma^C_\lambda \setminus \Sigma_{\lambda^\lambda}} \left[ G^+_{\infty}(x, y) - G^+_{\infty}(x^\lambda, y) \right]u^p(y) dy
\]

\[\leq \int_{\Sigma_\lambda} \left[ G^+_{\infty}(x, y) - G^+_{\infty}(x^\lambda, y) \right]u^p(y) dy - \int_{\tilde{\Sigma}_\lambda} \left[ G^+_{\infty}(x^\lambda, y^\lambda) - G^+_{\infty}(x^\lambda, y^\lambda) \right]u^p_\lambda(y) dy
\]

\[\leq \int_{\Sigma_\lambda} \left[ G^+_{\infty}(x^\lambda, y^\lambda) - G^+_{\infty}(x, y^\lambda) \right]u^p(y) dy - \int_{\Sigma_\lambda} \left[ G^+_{\infty}(x^\lambda, y^\lambda) - G^+_{\infty}(x, y^\lambda) \right]u^p_\lambda(y) dy
\]

\[= \int_{\Sigma_\lambda} \left[ G^+_{\infty}(x^\lambda, y^\lambda) - G^+_{\infty}(x, y^\lambda) \right][u^p(y) - u^p(x^\lambda)] dy. \]

We also need the following inequality.

**Lemma 5** (An equivalent form of the Hardy–Littlewood–Sobolev inequality). Let \( g \in L^{\frac{np}{n+2mp}}(\mathbb{R}^n) \) for \( \frac{n}{n-2m} < p < \infty \). Define

\[
Tg(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2m}} g(y) \, dy.
\]

Then

\[
\|Tg\|_{L^p} \leq C(n, p, m) \|g\|_{L^{\frac{np}{n+2mp}}}.
\]

This can be derived directly from the classical Hardy–Littlewood–Sobolev inequality, and the proof can be found in Chapter 1 of [11].

### 3.2. Kelvin transforms

Because there is no global integrability assumptions on the solutions \( u \), one is not able to carry on the method of moving planes directly on \( u \). To circumvent this difficulty, we resort to Kelvin type transforms.

For \( z^0 \in \partial \mathbb{R}^n_+ \), let

\[
\tilde{u}(x) = \frac{1}{|x-z^0|^{n-2m}} u \left( \frac{x-z^0}{|x-z^0|^2} + z^0 \right)
\]

be the Kelvin type transform of \( u \). We consider two possible cases.

**Case 1.** If there is a \( z^0 = (z^0_1, \ldots, z^0_{n-1}, 0) \in \partial \mathbb{R}^n_+ \) such that \( \tilde{u}(x) \) is not singular at \( z^0 \), then by (45), we obtain

\[
u(y) = \frac{1}{|y-z^0|^{n-2m}} \tilde{u} \left( \frac{y-z^0}{|y-z^0|^2} + z^0 \right).
\]

And we further deduce

\[
u(y) = O \left( \frac{1}{|y|^{n-2m}} \right), \quad \text{as } |y| \to \infty.
\]

Since \( u \in L^{\frac{n(p-1)}{2m}}_{loc}(\mathbb{R}^n_+) \), by (46), we have

\[
\int_{\mathbb{R}^n_+} u^{\frac{n(p-1)}{2m}}(y) \, dy < \infty.
\]
In this case, the proof consists of two steps. In the first step, we start from the very low end of our region \( R^n_+ \), i.e. near \( x_n = 0 \). We will show that for \( \lambda \) sufficiently small,

\[
w_{\lambda}(x) = u_{\lambda}(x) - u(x) \geq 0, \quad \text{a.e. } \forall x \in \Sigma_{\lambda}.
\]

In the second step, we will move our plane \( T_{\lambda} = \{x \in R^n_+ \mid x_n = \lambda \} \) along the positive \( x_n \) direction as long as inequality (48) holds.

Unlike traditional method of moving planes, here we do not have any differential equations and the corresponding maximum principles for \( w_{\lambda} \). Instead, we will exploit some global properties of the integral equation and estimate some \( L^q \) norm of \( w_{\lambda} \).

**Step 1.** Define

\[
\Sigma_{\lambda}^- = \{ x \in \Sigma_{\lambda} \mid w_{\lambda}(x) < 0 \}.
\]

We show that for \( \lambda \) sufficiently small, \( \Sigma_{\lambda}^- \) must be measure zero. In fact, for any \( x \in \Sigma_{\lambda}^- \), by the Mean Value Theorem, Lemma 3, and Lemma 4, we obtain

\[
0 < u(x) - u_{\lambda}(x) \\
\leq \int_{\Sigma_{\lambda}} \left[ G_\infty^+(x, y) - G_\infty^+(x, y) \right] [u^p(y) - u_{\lambda}^p(y)] \, dy \\
= \int_{\Sigma_{\lambda}^-} \left[ G_\infty^+(x, y) - G_\infty^+(x, y) \right] [u^p(y) - u_{\lambda}^p(y)] \, dy \\
+ \int_{\Sigma_{\lambda} \setminus \Sigma_{\lambda}^-} \left[ G_\infty^+(x, y) - G_\infty^+(x, y) \right] [u^p(y) - u_{\lambda}^p(y)] \, dy \\
\leq \int_{\Sigma_{\lambda}^-} \left[ G_\infty^+(x, y) - G_\infty^+(x, y) \right] [u^p(y) - u_{\lambda}^p(y)] \, dy \\
\leq \int_{\Sigma_{\lambda}^-} G_\infty^+(x, y) [u^p(y) - u_{\lambda}^p(y)] \, dy \\
= p \int_{\Sigma_{\lambda}^-} G_\infty^+(x, y) \psi_{\lambda}^{p-1}(y) [u(y) - u_{\lambda}(y)] \, dy \\
\leq p \int_{\Sigma_{\lambda}^-} G_\infty^+(x, y) u^{p-1}(y) [u(y) - u_{\lambda}(y)] \, dy,
\]

where since \( \psi_{\lambda}(y) \) is valued between \( u(y) \) and \( u_{\lambda}(y) \), and therefore on \( \Sigma_{\lambda}^- \), we have

\[
0 \leq u_{\lambda}(y) \leq \psi_{\lambda}(y) \leq u(y).
\]
Notice that $0 < 2m < n$, 

\[
G^+_n(x^\lambda, y^\lambda) = \frac{c_n}{|x^\lambda - y^\lambda|^{n-2m}} \int_0^{\frac{4(2\lambda - y_n)(2\lambda - y_n)}{|x^\lambda - y^\lambda|^2}} \frac{z^{m-1}}{(z+1)^{\frac{n}{2}}} \, dz
\]

\[
\leq \frac{C}{|x^\lambda - y^\lambda|^{n-2m}}
\]

\[
= \frac{C}{|x - y|^{n-2m}}.
\]

By (49), we get

\[
0 < u(x) - u_\lambda(x) \leq \int_{\Sigma_{\lambda}^-} \frac{C}{|x - y|^{n-2m}} |u^{p-1}(y)||u(y) - u_\lambda(y)| \, dy. \quad (50)
\]

We apply the Hardy–Littlewood–Sobolev inequality (44) and Hölder inequality to (50) to obtain, for any $q > \frac{n}{n-2m}$,

\[
\|w_\lambda\|_{L^q(\Sigma_{\lambda}^-)} \leq C \left\| u^{p-1} w_\lambda \right\|_{L_{\frac{2m}{n}}(\Sigma_{\lambda}^-)}^{\frac{2m}{n}} \left\| u^{p-1} \right\|_{L_{\frac{nq}{n}}(\Sigma_{\lambda}^-)} \|w_\lambda\|_{L^q(\Sigma_{\lambda}^-)}. \quad (51)
\]

By (47), we can choose sufficiently small positive $\lambda$ such that

\[
C \left\| u^{p-1} \right\|_{L_{\frac{2m}{n}}(\Sigma_{\lambda}^-)} = C \left\{ \int_{\Sigma_{\lambda}^-} u^{\frac{n(p-1)}{2m}}(y) \, dy \right\}^{\frac{2m}{n}} \leq \frac{1}{2}.
\]

Now inequality (51) implies

\[
\|w_\lambda\|_{L^q(\Sigma_{\lambda}^-)} = 0,
\]

and therefore $\Sigma_{\lambda}^-$ must be measure zero.

**Step 2.** Inequality (48) provides a starting point to move the plane $T_\lambda = \{ x \in \mathbb{R}^n_+ | x_n = \lambda \}$. Now we start from the neighborhood of $x_n = 0$ and move the plane up as long as (48) holds.

Define

\[
\lambda_0 = \sup \{ \lambda \mid w_\rho(x) \geq 0, \rho \leq \lambda, \forall x \in \Sigma_{\rho} \}. \quad (52)
\]

We will prove

\[
\lambda_0 = +\infty.
\]
Suppose in the contrary that \( \lambda_0 < \infty \), we will show that \( u(x) \) is symmetric about the plane \( T_{\lambda_0} \), i.e.

\[
w_{\lambda_0} \equiv 0, \quad \text{a.e. } \forall x \in \Sigma_{\lambda_0}. \tag{53}
\]

Otherwise, for such a \( \lambda_0 \), we have \( w_{\lambda_0} \geq 0 \), but \( w_{\lambda_0} \not\equiv 0 \) a.e. on \( \Sigma_{\lambda_0} \). We show that the plane can be moved further to the above. More precisely, there exists an \( \epsilon > 0 \) such that for all \( \lambda \) in \( [\lambda_0, \lambda_0 + \epsilon) \)

\[
u(x) \leq u_{\lambda}(x) \quad \text{a.e. } \Sigma_{\lambda}.
\]

By inequality (51), we have

\[
\|w_{\lambda}\|_{L^q(\Sigma_\lambda^-)} \leq C \left\{ \int_{\Sigma_\lambda^-} u^{n(p-1)/2m} (y) dy \right\}^{2m/n} \|w_{\lambda}\|_{L^q(\Sigma_\lambda^-)}. \tag{54}
\]

By condition (47), we can choose \( \epsilon \) sufficiently small so that for all \( \lambda \) in \( [\lambda_0, \lambda_0 + \epsilon) \),

\[
C \left\{ \int_{\Sigma_\lambda^-} u^{n(p-1)/2m} (y) dy \right\}^{2m/n} \leq \frac{1}{2}. \tag{55}
\]

We postpone the proof of (55) for a moment. Now by (54) and (55), we have \( \|w_{\lambda}\|_{L^q(\Sigma_\lambda^-)} = 0 \), and therefore \( \Sigma_\lambda^- \) must be measure zero. Hence, for these values of \( \lambda > \lambda_0 \), we have

\[
w_{\lambda}(x) \geq 0, \quad \text{a.e. } \forall x \in \Sigma_{\lambda}.
\]

This contradicts with the definition of \( \lambda_0 \). Therefore (53) must hold.

By (53), we derive that the plane \( x_n = 2\lambda_0 \) is the symmetric image of the boundary \( \partial R^n_+ \) with respect to the plane \( T_{\lambda_0} \), and hence \( u(x) = 0 \) when \( x \) is on the plane \( x_n = 2\lambda_0 \). This contradicts with our assumption \( u(x) > 0 \). Therefore, (52) must be valid.

Now we prove inequality (55). For any small \( \eta > 0 \), we can choose \( R \) sufficiently large so that

\[
\left( \int_{R^n_+ \setminus B_R(0)} u^{n(p-1)/2m} (y) dy \right)^{2m/n} < \eta. \tag{56}
\]

We fix this \( R \) and then show that the measure of \( \Sigma_{\lambda}^- \cap B_R(0) \) is sufficiently small for \( \lambda \) close to \( \lambda_0 \). First, we have

\[
w_{\lambda_0}(x) > 0 \tag{57}
\]

in the interior of \( \Sigma_{\lambda_0} \).
Indeed, by the first two expressions in the proof of Lemma 4 and Lemma 3, we have

\[ u_\lambda(x) - u(x) \geq \int_{\Sigma_\lambda} \left[ G_\infty^+(x^\lambda, y^\lambda) - G_\infty^+(x, y^\lambda) \right] u_\lambda^p(y) - u^p(y) \, dy 
+ \int_{\Sigma_\lambda^C \setminus \tilde{\Sigma}_\lambda} \left[ G_\infty^+(x^\lambda, y) - G_\infty^+(x, y) \right] u_\lambda^p(y) \, dy 
\geq \int_{\Sigma_\lambda^C \setminus \tilde{\Sigma}_\lambda} \left[ G_\infty^+(x^\lambda, y) - G_\infty^+(x, y) \right] u_\lambda^p(y) \, dy. \]  

(58)

If (57) is violated, there exists some point \( x_0 \in \Sigma_{\lambda_0} \) such that \( u(x_0) = u_{\lambda_0}(x_0) \). And then by (43) and (58), we obtain

\[ u(y) \equiv 0, \quad \forall y \in \Sigma_{\lambda_0}^C \setminus \tilde{\Sigma}_{\lambda_0}. \]  

(59)

This is a contradiction with our assumption that \( u > 0 \). Therefore (57) must be true.

For any \( \gamma > 0 \), let

\[ E_\gamma = \{ x \in \Sigma_{\lambda_0} \cap B_R(0) \mid w_{\lambda_0}(x) > \gamma \}, \quad F_\gamma = (\Sigma_{\lambda_0} \cap B_R(0)) \setminus E_\gamma. \]  

(60)

It is obviously that

\[ \lim_{\gamma \to 0} \mu(F_\gamma) = 0. \]

For \( \lambda > \lambda_0 \), let

\[ D_\lambda = (\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \cap B_R(0). \]

Then it is easy to see that

\[ (\Sigma_\lambda^- \cap B_R(0)) \subset (\Sigma_\lambda^- \cap E_\gamma) \cup F_\gamma \cup D_\lambda. \]  

(61)

Apparently, the measure of \( D_\lambda \) is small for \( \lambda \) close to \( \lambda_0 \). We show that the measure of \( \Sigma_\lambda^- \cap E_\gamma \) can be sufficiently small as \( \lambda \) close to \( \lambda_0 \). In fact, for any \( x \in \Sigma_\lambda^- \cap E_\gamma \), we have

\[ w_\lambda(x) = u_\lambda(x) - u(x) = u_\lambda(x) - u_{\lambda_0}(x) + u_{\lambda_0}(x) - u(x) < 0. \]

Hence

\[ u_{\lambda_0}(x) - u_\lambda(x) > w_{\lambda_0}(x) > \gamma. \]

It follows that

\[ (\Sigma_\lambda^- \cap E_\gamma) \subset G_\gamma \equiv \{ x \in B_R(0) \mid u_{\lambda_0}(x) - u_\lambda(x) > \gamma \}. \]  

(62)
By the well-known Chebyshev inequality, we have

$$\mu(G_{\gamma}) \leq \frac{1}{\gamma^{p+1}} \int_{G_{\gamma}} |u_{\lambda_0}(x) - u_\lambda(x)|^{p+1} \, dx \leq \frac{1}{\gamma^{p+1}} \int_{B_R(0)} |u_{\lambda_0}(x) - u_\lambda(x)|^{p+1} \, dx.$$  \hspace{1cm} (63)

For each fixed $\gamma$, as $\lambda$ close to $\lambda_0$, the right hand side of the above inequality can be made as small as we wish. Therefore by (61) and (62), the measure of $\Sigma_\lambda^- \cap B_R(0)$ can also be made sufficiently small. Combining this with (56), we obtain (55).

Now from (52), $u$ is monotone increasing with respect to $x_n$. This contradicts with (46). Hence Case 1 is impossible, and what remains is the following.

**Case 2.** For all $z^0 = (z_1^0, \ldots, z_{n-1}^0, 0) \in \partial R^n_+$, $\tilde{u}(x)$ is singular at $z^0$. Here we will prove that $\tilde{u}(x)$ is rotationally symmetric about the line passing through $z^0$ and parallel to the $x_n$-axis. We calculate

$$\tilde{u}(x) = \frac{1}{|x - z^0|^{n-2m}} u \left( \frac{x - z^0}{|x - z^0|^2} + z^0 \right)$$

$$= \frac{1}{|x - z^0|^{n-2m}} \int_{R^n_+} G_\infty^+ \left( \frac{x - z^0}{|x - z^0|^2} + z^0, y \right) u^p(y) \, dy$$

$$= \frac{1}{|x - z^0|^{n-2m}} \int_{R^n_+} G_\infty^+ \left( \frac{x - z^0}{|x - z^0|^2} + z^0, \frac{\bar{y} - z^0}{|\bar{y} - z^0|^2} + z^0 \right) u^p \left( \frac{\bar{y} - z^0}{|\bar{y} - z^0|^2} + z^0 \right) d\bar{y}$$

$$= \int_{R^n_+} G_\infty^+ (x, y) \frac{\bar{u}^p(y)}{|y - z^0|^\beta} \, dy, \quad \text{for } x \in R^n_+ \setminus B_\epsilon(z^0), \forall \epsilon > 0,$$

where $\frac{n-2m}{n-2m} < p \leq \tau$, $\beta = (n-2m)(\tau - p) \geq 0$, $\tau = \frac{n+2m}{n-2m}$.

(i) For $p = \tau = \frac{n+2m}{n-2m}$, then if $u(x)$ is a solution of

$$u(x) = \int_{R^n_+} G_\infty^+(x, y) u^\tau(y) \, dy,$$

then $\tilde{u}$ is also a solution of (65). Since $u \in L_{loc}^{\frac{2n}{n-2m}}(R^n_+)$, for any domain $\Omega$ that is a positive distance away from $z^0$, we have

$$\int_\Omega \tilde{u}^{\frac{n}{2m}}(y) \, dy < \infty.$$  \hspace{1cm} (66)

From now on, we only need to deal with $\tilde{u}$. For simplicity, we still denote it by $u$.  

In this case, we need to redefine $\Sigma_\lambda$. For a given real number $\lambda$, define
\[
\hat{\Sigma}_\lambda = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ \mid x_1 < \lambda \}
\]
and let
\[
x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n).
\]
For $x, y \in \hat{\Sigma}_\lambda$, $x \neq y$, we have
\[
G_\infty^+(x, y) = G_\infty^+(x^\lambda, y^\lambda),
\]
\[
G_\infty^+(x^\lambda, y) = G_\infty^+(x, y^\lambda), \quad \text{and}
\]
\[
G_\infty^+(x^\lambda, y^\lambda) > G_\infty^+(x, y^\lambda).
\]
(67)

Obviously, we have
\[
u(x) = \int_{\hat{\Sigma}_\lambda} G_\infty^+(x, y) u^\tau(y) \, dy + \int_{\hat{\Sigma}_\lambda} G_\infty^+(x, y^\lambda) u^\tau_\lambda(y) \, dy,
\]
\[
u(x^\lambda) = \int_{\hat{\Sigma}_\lambda} G_\infty^+(x^\lambda, y) u^\tau(y) \, dy + \int_{\hat{\Sigma}_\lambda} G_\infty^+(x^\lambda, y^\lambda) u^\tau_\lambda(y) \, dy.
\]
By (67), it is easy to see
\[
u(x) - \nu(x^\lambda) = \int_{\hat{\Sigma}_\lambda} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] u^\tau(y) \, dy
\]
\[+ \int_{\hat{\Sigma}_\lambda} \left[ G_\infty^+(x, y^\lambda) - G_\infty^+(x^\lambda, y^\lambda) \right] u^\tau_\lambda(y) \, dy
\]
\[= \int_{\hat{\Sigma}_\lambda} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \left[ u^\tau(y) - u^\tau_\lambda(y) \right] \, dy.
\]
(68)

The proof consists of two steps. In the first step, we will show that for $\lambda$ sufficiently negative,
\[
w^\lambda(x) = u^\lambda(x) - u(x) \geq 0, \quad a.e. \forall x \in \hat{\Sigma}_\lambda.
\]
(69)

In the second step, we deduce that $\hat{T}$ can be moved to the right all the way to $z^0$. And furthermore, we derive $w^\lambda_0 \equiv 0, \forall x \in \hat{\Sigma}_\lambda^0$. 
Step 1. Define

\[ \hat{\Sigma}_\lambda^- = \{ x \in \hat{\Sigma}_\lambda \setminus B_\epsilon (z^0) \mid w_\lambda (x) < 0 \}, \]

where \((z^0)^\lambda\) is the reflection of \(z^0\) about the plane \(\hat{T}_\lambda = \{ x \in \mathbb{R}^n_+ \mid x_1 = \lambda \}\). We show that for \(\lambda\) sufficiently negative, \(\hat{\Sigma}_\lambda^-\) must be measure zero. In fact, by the Mean Value Theorem, we obtain, for \(x \in \hat{\Sigma}_\lambda^-\),

\[
0 < u(x) - u_\lambda(x) = \int_{\hat{\Sigma}_\lambda^-} \left[ G^+_{\infty}(x, y) - G^+_{\infty}(x, y^\lambda) \right] [u^\tau(y) - u_\lambda^\tau(y)] \, dy + \int_{\hat{\Sigma}_\lambda \setminus \hat{\Sigma}_\lambda^-} \left[ G^+_{\infty}(x, y) - G^+_{\infty}(x, y^\lambda) \right] [u^\tau(y) - u_\lambda^\tau(y)] \, dy \\
\leq \int_{\hat{\Sigma}_\lambda^-} \left[ G^+_{\infty}(x, y) - G^+_{\infty}(x, y^\lambda) \right] [u^\tau(y) - u_\lambda^\tau(y)] \, dy \\
\leq \int G^+_{\infty}(x, y) [u^\tau(y) - u_\lambda^\tau(y)] \, dy \\
= \tau \int \frac{C}{|x - y|^{n-2m}} [u^{\tau-1}(y) - u_\lambda(u(y)) \, dy \\
\leq \tau \int G^+_{\infty}(x, y) u^{\tau-1}(y) [u(y) - u_\lambda(y)] \, dy \\
\leq \int \frac{C}{|x - y|^{n-2m}} |u^{\tau-1}(y)| |u(y) - u_\lambda(y)| \, dy. \tag{70}
\]

We apply the Hardy–Littlewood–Sobolev inequality (44) and Hölder inequality to (70) to obtain, for any \(q > \frac{n}{n-2m}\),

\[
\| w_\lambda \|_{L^q(\hat{\Sigma}_\lambda^-)} \leq C \left\| u^{\tau-1} w_\lambda \right\|_{L^\frac{n}{n+2m}(\hat{\Sigma}_\lambda^-)} \leq C \left\| u^{\tau-1} \right\|_{L^\infty(\hat{\Sigma}_\lambda^-)} \| w_\lambda \|_{L^q(\hat{\Sigma}_\lambda^-)}. \tag{71}
\]

By (66), we can choose \(N\) sufficiently large such that for \(\lambda \leq -N\),

\[
C \left\| u^{\tau-1} \right\|_{L^\frac{n}{n+2m}(\hat{\Sigma}_\lambda^-)} = C \left\{ \int_{\hat{\Sigma}_\lambda^-} u^{\frac{2n}{n-2m}}(y) \, dy \right\}^{\frac{2m}{n}} \leq \frac{1}{2}.
\]
Now inequality (71) implies
\[ \| w_\lambda \|_{L^q(\hat{\Sigma}_\lambda)} = 0, \]
and therefore \( \hat{\Sigma}_\lambda^- \) must be measure zero.

**Step 2.** (Move the plane to the limiting position to derive symmetry.)

Inequality (69) provides a starting point to move the plane \( \hat{T}_\lambda = \{ x \in \mathbb{R}^n_+ \mid x_1 = \lambda \} \). Now we start from the neighborhood of \( x_1 = -\infty \) and move the plane to the right as long as (69) holds to the limiting position. Define
\[ \lambda_0 = \sup \{ \lambda \leq z_1^0 \mid w_\rho(x) \geq 0, \rho \leq \lambda, \forall x \in \hat{\Sigma}_\rho \} . \]

We prove that \( \lambda_0 = z_1^0 \). On the contrary, suppose that \( \lambda_0 < z_1^0 \). We will show that \( u(x) \) is symmetric about the plane \( T_{\lambda_0} \), i.e.
\[ w_{\lambda_0} \equiv 0, \quad a.e. \forall x \in \hat{\Sigma}_{\lambda_0} \setminus B_{\epsilon}(z_1^0)^{\lambda_0} \]. (72)

Suppose that for such a \( \lambda_0 \), we have \( w_{\lambda_0} \geq 0 \), but \( w_{\lambda_0} \not\equiv 0 \) a.e. on \( \hat{\Sigma}_{\lambda_0} \setminus B_{\epsilon}(z_1^0)^{\lambda_0} \). We show that the plane can be moved further to the right. More precisely, there exists a \( \zeta > 0 \) such that for all \( \lambda \in [\lambda_0, \lambda_0 + \zeta) \)
\[ u(x) \leq u_\lambda(x) \quad a.e \text{ on } \hat{\Sigma}_\lambda \setminus B_{\epsilon}(z_1^0)^{\lambda} \].

By inequality (71), we have
\[ \| w_\lambda \|_{L^q(\hat{\Sigma}_\lambda^-)} \leq C \left\{ \int_{\hat{\Sigma}_\lambda^-} u^{\frac{2n}{n-2m}}(y) \, dy \right\}^{\frac{2m}{n}} \| w_\lambda \|_{L^q(\hat{\Sigma}_\lambda^-)} \]. (73)

By condition (66), similar to the proof of (55), we can choose \( \zeta \) sufficiently small so that for all \( \lambda \in [\lambda_0, \lambda_0 + \zeta) \),
\[ C \left\{ \int_{\hat{\Sigma}_\lambda^-} u^{\frac{2n}{n-2m}}(y) \, dy \right\}^{\frac{2m}{n}} \leq \frac{1}{2} . \] (74)

We postpone the proof for a moment. Now by (73) and (74), we have \( \| w_\lambda \|_{L^q(\hat{\Sigma}_\lambda^-)} = 0 \), and therefore \( \hat{\Sigma}_\lambda^- \) must be measure zero. Hence, for these values of \( \lambda > \lambda_0 \), we have
\[ w_\lambda(x) \geq 0, \quad a.e. \forall x \in \hat{\Sigma}_\lambda \setminus B_{\epsilon}(z_1^0)^{\lambda}, \forall \epsilon > 0 . \]

This contradicts with the definition of \( \lambda_0 \). Therefore (72) must hold. That is, if \( \lambda_0 < z_1^0 \), for any \( \epsilon > 0 \),
\[ \tilde{u}(x) \equiv \tilde{u}_{\lambda_0}(x), \quad a.e. \forall x \in \hat{\Sigma}_{\lambda_0} \setminus B_{\epsilon}(z_1^0)^{\lambda_0} . \]
Since $\bar{u}$ is singular at $z^0$, $\bar{u}$ must also be singular at $(z^0)_{\lambda}$. This is impossible. So we deduce

$$\lambda_0 = z_1^0, \quad w_{\lambda_0}(x) \geq 0, \quad a.e. \ \forall x \in \hat{\Sigma}_{\lambda_0}.$$  

Entirely similarly, we can move the plane from near $x_1 = \infty$ to the left and derive that $w_{\lambda_0}(x) \leq 0$. Therefore we have

$$w_{\lambda_0}(x) = 0, \quad a.e. \ \forall x \in \hat{\Sigma}_{\lambda_0}, \quad \lambda_0 = z_1^0.$$  

Now we prove inequality (74). For any small $\eta > 0$, $\forall \epsilon > 0$, we can choose $R$ sufficiently large so that

$$\left( \int_{(R^n_+ \setminus B_\epsilon(z^0)) \setminus B_R(0)} u^{\frac{2n}{n-2m}}(y) \, dy \right)^{\frac{2m}{n}} < \eta. \quad (75)$$

We fix this $R$ and then show that the measure of $\hat{\Sigma}_{\lambda} - B_R(0)$ is sufficiently small for $\lambda$ close to $\lambda_0$. By (68), we have

$$w_{\lambda_0}(x) > 0 \quad (76)$$

in the interior of $\hat{\Sigma}_{\lambda_0} \setminus B_\epsilon((z^0)_{\lambda_0})$.

The rest is similar to the proof of (55) in Case 1. We only need to use $\hat{\Sigma}_{\lambda} \setminus B_\epsilon((z^0)_{\lambda})$ instead of $\Sigma_{\lambda}$ and $\hat{\Sigma}_{\lambda_0} \setminus B_\epsilon((z^0)_{\lambda_0})$ instead of $\Sigma_{\lambda_0}$.

(ii) For $\frac{n}{n-2m} < p < \frac{n+2m}{n-2m}$, since $u \in L_{loc}^{\frac{n(p-1)}{2m}}(R^n_+)$, for any domain $\Omega$ that is a positive distance away from $z^0$, we have

$$\int_{\Omega} \left[ \frac{\bar{u}^{p-1}(y)}{|y - z^0|^\beta} \right]^{\frac{n}{2m}} dy < \infty, \quad (77)$$

where $\beta = (n - 2m)(\tau - p) > 0$, $\tau = \frac{n+2m}{n-2m}$.

By (64), we have

$$\bar{u}(x) = \int_{\hat{\Sigma}_{\lambda}} G^+_\infty(x, y) \frac{\bar{u}^p(y)}{|y - z^0|^{\beta}} \, dy + \int_{\hat{\Sigma}_{\lambda}} G^+_\infty(x, y^{\lambda}) \frac{\bar{u}^p(y^{\lambda})}{|y^{\lambda} - z^0|^{\beta}} \, dy,$$

$$\bar{u}(x^{\lambda}) = \int_{\hat{\Sigma}_{\lambda}} G^+_\infty(x^{\lambda}, y) \frac{\bar{u}^p(y)}{|y - z^0|^{\beta}} \, dy + \int_{\hat{\Sigma}_{\lambda}} G^+_\infty(x^{\lambda}, y^{\lambda}) \frac{\bar{u}^p(y^{\lambda})}{|y^{\lambda} - z^0|^{\beta}} \, dy.$$  

By (67), we calculate
\[
\tilde{u}(x) - \tilde{u}_\lambda(x) = \int_{\tilde{\Sigma}_\lambda} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \frac{\tilde{u}^p(y)}{|y - z_0|^\beta} \, dy \\
+ \int_{\tilde{\Sigma}_\lambda} \left[ G_\infty^+(x, y^\lambda) - G_\infty^+(x^\lambda, y^\lambda) \right] \frac{\tilde{u}_\lambda^p(y)}{|y^\lambda - z_0|^\beta} \, dy \\
- \int_{\tilde{\Sigma}_\lambda} \left[ G_\infty^+(x, y^\lambda) - G_\infty^+(x^\lambda, y^\lambda) \right] \left[ \frac{\tilde{u}^p(y)}{|y - z_0|^\beta} - \frac{\tilde{u}_\lambda^p(y)}{|y^\lambda - z_0|^\beta} \right] \, dy.
\] (78)

The proof also consists of two steps.

**Step 1.** For any \( \epsilon > 0 \), define

\[
\hat{\Sigma}_\lambda^- = \{ x \in \hat{\Sigma}_\lambda \setminus B_\epsilon((z_0)^\lambda) \mid w_\lambda(x) = u_\lambda(x) - u(x) < 0 \}.
\]

We show that for \( \lambda \) sufficiently negative, \( \hat{\Sigma}_\lambda^- \) must be measure zero.

By the Mean Value Theorem, we obtain, for sufficiently negative values of \( \lambda \) and \( x \in \hat{\Sigma}_\lambda^- \),

\[
0 < \tilde{u}(x) - \tilde{u}_\lambda(x)
\]

\[
= \int_{\hat{\Sigma}_\lambda^-} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \left[ \frac{\tilde{u}^p(y)}{|y - z_0|^\beta} - \frac{\tilde{u}_\lambda^p(y)}{|y^\lambda - z_0|^\beta} \right] \, dy
\]

\[
= \int_{\hat{\Sigma}_\lambda^-} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \left[ \frac{\tilde{u}^p(y)}{|y - z_0|^\beta} - \frac{\tilde{u}_\lambda^p(y)}{|y^\lambda - z_0|^\beta} \right] \, dy
\]

\[
+ \int_{\hat{\Sigma}_\lambda \setminus \hat{\Sigma}_\lambda^-} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \left[ \frac{\tilde{u}^p(y)}{|y - z_0|^\beta} - \frac{\tilde{u}_\lambda^p(y)}{|y^\lambda - z_0|^\beta} \right] \, dy
\]

\[
\leq \int_{\hat{\Sigma}_\lambda^-} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \left[ \frac{\tilde{u}^p(y)}{|y - z_0|^\beta} - \frac{\tilde{u}_\lambda^p(y)}{|y^\lambda - z_0|^\beta} \right] \, dy
\]

\[
= \int_{\hat{\Sigma}_\lambda^-} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \left[ \frac{\tilde{u}^p(y)}{|y - z_0|^\beta} - \frac{\tilde{u}_\lambda^p(y)}{|y^\lambda - z_0|^\beta} \right] \, dy
\]

\[
= \int_{\hat{\Sigma}_\lambda^-} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \left[ \frac{\tilde{u}^p(y) - \tilde{u}_\lambda^p(y)}{|y - z_0|^\beta} + \tilde{u}_\lambda^p(y) \left[ \frac{1}{|y - z_0|^\beta} - \frac{1}{|y^\lambda - z_0|^\beta} \right] \right] \, dy
\]

\[
\leq \int_{\hat{\Sigma}_\lambda^-} \left[ G_\infty^+(x, y) - G_\infty^+(x^\lambda, y) \right] \frac{\tilde{u}^p(y)}{|y - z_0|^\beta} \, dy
\]
\[ \begin{align*}
\leq & p \int_{\hat{\Sigma}_{\lambda}^-} G_{\infty}^{+}(x,y) \frac{\bar{u}^{p-1}(y)}{|y-z^0|^\beta} \left[ \bar{u}(y) - \bar{u}_\lambda(y) \right] dy \\
\leq & \int_{\hat{\Sigma}_{\lambda}^-} \frac{C}{|x-y|^{n-2m}} \frac{\bar{u}^{p-1}(y)}{|y-z^0|^\beta} \left| \bar{u}(y) - \bar{u}_\lambda(y) \right| dy. 
\end{align*}\]

We apply the Hardy–Littlewood–Sobolev inequality (44) and Hölder inequality to (79) to obtain, for any \( q > \frac{n}{n-2m} \),
\[ \| w_\lambda \|_{L^q(\hat{\Sigma}_{\lambda}^-)} \leq C \left\| \frac{\bar{u}^{p-1}}{|y-z^0|^\beta} w_\lambda \right\|_{L^{\frac{nq}{n-2m}}(\hat{\Sigma}_{\lambda}^-)} \leq C \left\| \frac{\bar{u}^{p-1}}{|y-z^0|^\beta} \right\|_{L^{\frac{n}{2m}}(\hat{\Sigma}_{\lambda}^-)} \| w_\lambda \|_{L^q(\hat{\Sigma}_{\lambda}^-)}. \]  

By (77), we can choose \( N \) sufficiently large, such that for \( \lambda \leq -N \),
\[ C \left\{ \int_{\hat{\Sigma}_{\lambda}^-} \left[ \frac{\bar{u}^{p-1}}{|y-z^0|^\beta} \right] \frac{n}{2m} dy \right\}^{\frac{2m}{n}} \leq \frac{1}{2}. \]

Now inequality (80) implies
\[ \| w_\lambda \|_{L^q(\hat{\Sigma}_{\lambda}^-)} = 0, \]
and therefore \( \hat{\Sigma}_{\lambda}^- \) must be measure zero. Then we get
\[ w_\lambda(x) \geq 0, \quad \text{a.e. } x \in \hat{\Sigma}_{\lambda}. \]  

**Step 2.** (Move the plane to the limiting position to derive symmetry.)

Inequality (81) provides a starting point to move the plane \( \hat{T}_{\lambda} = \{ x \in R_n^+ \mid x_1 = \lambda \} \). Now we start from the neighborhood of \( x_1 = -\infty \) and move the plane to the right as long as (81) holds to the limiting position. Define
\[ \lambda_0 = \sup \{ \lambda \leq z_1^0 \mid w_\rho(x) \geq 0, \rho \leq \lambda, \forall x \in \hat{\Sigma}_\rho \}. \]

The rest is entirely similarly to the case \( p = \frac{n+2m}{n-2m} \). We only need to use \( \int \left[ \frac{\bar{u}^{p-1}}{|y-z^0|^\beta} \right] \frac{n}{2m} dy \) instead of \( \int u \frac{n}{n-2m} (y) dy \). We also conclude
\[ w_{\lambda_0}(x) \equiv 0, \quad \text{a.e. } x \in \hat{\Sigma}_{\lambda_0}, \lambda_0 = z_1^0. \]

In Case 2, for \( \frac{n}{n-2m} < p \leq \frac{n+2m}{n-2m} \), since we can choose any direction that is perpendicular to the \( x_n \)-axis as the \( x_1 \) direction, we have actually shown that the solution \( \bar{u}(x) \) is rotationally symmetric about the line parallel to \( x_n \)-axis and passing through \( z^0 \). Now, for any two
points $X^1$ and $X^2$, with $X^i = (x^i, x_n) \in \mathbb{R}^{n-1} \times [0, \infty)$, $i = 1, 2$. Let $z^0$ be the projection of $\bar{X} = \frac{X^1 + X^2}{2}$ on $\partial \mathbb{R}^n_+$. Set $Y^i = \frac{x^i}{|x^i-z^0|} + z^0$, $i = 1, 2$. From the above arguments, it is easy to see $\tilde{u}(Y^1) = \tilde{u}(Y^2)$, hence $u(X^1) = u(X^2)$. This implies that $u$ is independent of $(x_1, \ldots, x_{n-1})$.

Next, we prove that $u(x)$ is monotone and then $u \equiv 0$. For $x = (x', x_n), y = (y', y_n) \in \mathbb{R}^{n-1} \times [0, +\infty)$, we assume that $u(x) = u(x_n)$ is a solution of

$$u(x) = \int_{\mathbb{R}^n_+} G(x, y) u^p(y) \, dy, \tag{82}$$

where

$$G(x, y) = c_n \frac{|x-y|^{n-2m}}{|x-y'|^{n-2m}} \int_0^{4a y_n |x-y'|^2} \frac{z^{m-1}}{(z+1)^{n/2}} \, dz$$

on $\mathbb{R}^n_+$.

For each fixed $x \in \mathbb{R}^n_+$, set $|x_n - y_n|^2 = a^2$, $|y'|^2 = r^2$. By elementary calculations, we have

$$+\infty > u(x) = u(x_n)$$

$$= \int_{\mathbb{R}^n_+} \frac{c_n}{|x-y|^{n-2m}} \int_0^{4a y_n |x-y'|^2} \frac{z^{m-1}}{(z+1)^{n/2}} \, dz \, u^p(y) \, dy$$

$$\sim C \int_{\mathbb{R}^n_+} \frac{1}{|x-y|^{n-2m}} \int_0^{4a y_n |x-y'|^2} \frac{z^{m-1}}{d} \, dz \, u^p(y) \, dy$$

$$= C \int_0^\infty u^p(y_n) y_n^m \int_{\mathbb{R}^{n-1}} \frac{1}{(|x'-y'|^2 + |x_n - y_n|^2)^{\frac{n}{2}}} \, dy' \, dy_n$$

$$= C \int_0^\infty u^p(y_n) y_n^m \int_0^\infty \frac{\rho^{n-2}}{(r^2 + a^2)^{\frac{n}{2}}} \, d\rho \, dy_n$$

$$= C \int_0^\infty u^p(y_n) y_n^m \int_0^\infty \frac{a^{n-2} \tau^{n-2}}{a^n (\tau^2 + 1)^{\frac{n}{2}}} \, a \, d\tau \, dy_n$$

$$= C \int_0^\infty u^p(y_n) y_n^m \frac{dy_n}{|x_n - y_n|} \, dy_n. \tag{83}$$

It follows that there exists a sequence $\{y_n^i\}$ such that

$$u^p(y_n^i)^m \to 0, \quad \text{as } y_n^i \to \infty.$$
Absolutely, we have
\[ u(y_n^i) \to 0, \quad \text{as } y_n^i \to \infty. \tag{84} \]

For simplicity, we set \( u(x) = u(x_n) = u(t) \). Suppose otherwise that \( u \not\equiv 0 \). Then there is a \( t_0 > 0 \) such that \( u(t_0) > 0 \). By (82), we have \( u(t) > 0 \) in \((0, \infty)\).

For \( m = 2k, \ k = 1, 2, \ldots \), we have
\[ u^{(2m)}(t) = (-\Delta)^m u(x) = \int_{\mathbb{R}^n_+} (-\Delta)^m G(x, y) u^p(y) \, dy = u^p(x) > 0, \tag{85} \]

(85) implies
\[ u^{(2m-1)}(t) \text{ is monotone increasing.} \tag{86} \]

We show that
\[ u^{(2m-1)}(t) \leq 0. \tag{87} \]

If not, there is a \( t_0 > 0 \) such that \( u^{(2m-1)}(t_0) > 0 \). By (86), we have
\[ u^{(2m-1)}(t) \geq u^{(2m-1)}(t_0) > 0, \quad \text{for } t \geq t_0 > 0. \]

Integrating several times, and let \( t \to \infty \), we have \( u(t) \to \infty \). This is a contradiction with (84).

Now (87) implies
\[ u^{(2m-2)}(t) \text{ is nonincreasing.} \tag{88} \]

Consequently,
\[ u^{(2m-2)}(t) \geq 0. \]

Otherwise, there is a \( t_0 > 0 \), such that \( u^{(2m-2)}(t_0) < 0 \). By (88), we have
\[ u^{(2m-2)}(t) \leq u^{(2m-2)}(t_0) < 0, \quad \text{for } t \geq t_0 > 0. \]

Integrating several times, and let \( t \to \infty \), we have \( u(t) \to -\infty \). This is a contradiction with \( u(x) > 0 \). Continuing this way, we derive that
\[ u(t) \text{ is nonincreasing.} \tag{89} \]

Since \( u(x) \) is a non-negative solution and \( u(0) = 0 \), by (89), it is easy to see \( u(x) \equiv 0 \).

Similarly, for \( m = 2k + 1, \ k = 1, \ldots \), we deduce that
\[ u(t) \text{ is nondecreasing.} \]
\[ +\infty > u(x) = u(x_n) \]
\[ \sim C \int_0^\infty \frac{u^p(y_n)y_n^m}{|x_n - y_n|} \, dy_n \]
\[ \geq C \int_1^\infty \frac{u^p(y_n)y_n^m}{|x_n - y_n|} \, dy_n \]
\[ \geq C u^p(1) \int_1^\infty \frac{y_n^m}{|x_n - y_n|} \, dy_n = +\infty. \]

This is a contradiction, hence we must have \( u \equiv 0 \). This completes the proof of Theorem 3.1.

References