Journal of Combinatorial Theory, Series B **75**, 100–109 (1999)
Article ID jctb.1998.1868, available online at http://www.idealibrary.com on IDEAL®

Two-Coloring the Edges of a Cubic Graph Such That Each Monochromatic Component Is a Path of Length at Most 5

Carsten Thomassen

Department of Mathematics, Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark

Received July 15, 1997

We prove the conjecture made by Bermond, Fouquet, Habib, and Péroche in

1. INTRODUCTION

Akiyama *et al.* [1] showed that every cubic graph has an edge-coloring in two colors such that each monochromatic component is a path. A short proof was given in [2]. Bermond *et al.* [4] conjectured that the coloring can be made such that all paths have length at most 5.

Jackson and Wormald [5] proved a weaker version with 18 instead of 5. Aldred and Wormald [3] proved the analogous result with 18 replaced by 9 (and 7 for cubic graphs of edge-chromatic number 3). We shall here prove the conjecture which is best possible in the sense that 5 cannot be replaced by 4. However, as pointed out in [5], the two cubic graphs on 6 vertices are the only known cubic connected graphs for which 5 cannot be replaced by 4.

2. TERMINOLOGY

If G is a multigraph, then $V(G)$ denotes its vertex set. An edge between vertices x and y is denoted xy and yx. A *graph* is a multigraph with no multiple edges. Multiple edges occur only at the beginning of the proof of Theorem 2. A *path* is a graph with distinct vertices $x_1, x_2, ..., x_m$ and edges x_1x_2, x_2x_3, ..., $x_{m-1}x_m$. We denote it by $x_1x_2 \cdots x_m$. If we add the edge x_mx_i we obtain the *cycle* $x_1x_2 \cdots x_mx_1$ (if $i=1$) or *lollipop* $x_1x_2 \cdots x_mx_i$.

0095-8956/99 $30.00

When we color the edges of a graph such that each monochromatic component is a path, then we may orient the edges in such a way that each monochromatic component $x_1 x_2 \cdots x_m$ becomes a *directed path.* If the direction is from x_1 to x_m then we denote it by $x_1 x_2 \cdots x_m$ (and not $x_m x_{m-1} \cdots x_1$). We say that it *starts* at x_1 and *terminates* at x_m.

If y is a vertex of degree 2, and the two edges incident with y have the same color, then we say that y is of *type* 0. We say that y is of *type* i ($i \geqslant 1$) if either y is of type 0 or if there is a monochromatic path of length at most i terminating at y. We abbreviate red and blue by r and b, respectively.

3. THE METHOD

We shall prove the conjecture by first selecting a cycle G_0 in the cubic graph and then successively extend it to $G_1, G_2, \ldots$, where G_{i+1} is obtained from G_i by adding an appropriate path or lollipop. We color the edges of G_i in red and blue and we orient the edges such that each monochromatic component is a directed path of length at most 5. We assume that the vertices of degree 2 are of small type so that we can extend the coloring of G_i to G_{i+1} without creating long monochromatic paths. Rather that focusing on the actual statements of the lemmas below it may be more convenient to think of the simple algorithms given in the proofs.

LEMMA 1. *Let x, y be vertices of degree 2 in a graph G whose edges are colored red and blue and oriented in such a way that each monochromatic component is a directed path of length of most 5. Assume that x is of type 2 and that y is of type 1. Let H be obtained from G by adding a path P: $v_0 v_1 v_2 \cdots v_k v_{k+1}$ where $k \geqslant 1$, $v_0 = x$, $v_{k+1} = y$, and none of $v_1, v_2, \ldots, v_k$ is in G. Then the coloring and orientation of G can be extended to H such that*

(i) *each monochromatic component of H is a directed path of length $\leqslant 5$,*

(ii) *v_k is of type 2,*

(iii) *each vertex in $\{v_1, v_2, \ldots, v_{k-1}\}$ (if any) is of type 1, and*

(iv) *if $k \geqslant 2$, then there is either a monochromatic directed path of length 1 starting at v_k or two monochromatic directed paths of length $\leqslant 4$ terminating at v_k (one of which has length $\leqslant 2$).*

If G, x, y are as above except that now x is of type 3, then (i), (ii), (iii) *and* (iv) *can still be fulfilled unless $k = 1$ or 2 and there is a monochromatic directed path of length 3 (respectively 1) terminating at x (respectively y)*

such that the two paths have the same color when $k=1$ *and distinct colors when* $k=2$.

Proof. We may assume that the two edges incident with y have color blue or that there is a red directed path of length 1 terminating at y.

Now we make P into a directed path $yv_kv_{k-1}...$ and we color its edges alternately $r, b, r, b, ...$. If x is of type 0, and x is incident with three edges of the same color, then we change the color of the edge v_1x and the direction of v_2v_1x. If x is not of type 0, then we change the direction of v_1x. If necessary, we also change the color of v_1x and the direction of v_2v_1. This method also works if x is of type 3 except that now (iv) is not necessarily satisfied if $k=2$, and also yv_k may become part of a red path of length 6 when $k=1$. ∎

LEMMA 2. *Let x be a vertex of degree 2 in a graph G whose edges are colored red and blue and oriented in such way that each monochromatic component is a directed path of length at most 5. Assume that x is of type 3. Let H be obtained from G by adding a path $v_0v_1v_2\cdots v_k$ and an edge v_kv_m where $m\geqslant 1$, $k\geqslant 3$, $v_0=x$, and none of $v_1, v_2, ..., v_k$ are in G. Then the coloring and orientation of G can be extended to H such that*

(i) *each monochromatic component of H is a directed path of length* $\leqslant 5$,

(ii) v_k *is of type* 3,

(iii) *each vertex in* $\{v_1, v_2, ..., v_{k-1}\}$ *is of type* 1,

(iv) *there is a directed path of length* 1 *starting at* v_k, *and*

(v) *if v_k is not of type 2, then $m\geqslant 2$, $k-m$ is odd, and there is another coloring of the lollipop such that* (i), (ii), (iii) *and* (iv) *hold and the colors of the lollipop $v_1v_2\cdots v_kv_m$ interchange.*

Proof. Assume that either x is of type 0 and the two edges incident with x are blue or there is a red directed path of length $\leqslant 3$ terminating at x. Now we color the edges of $v_0v_1v_2\cdots v_k$ alternately $r, b, r, b, ...$. If $k-m$ is even, we give v_kv_m the same color as v_mv_{m+1} and we make $v_{m+1}v_mv_k$ and $v_kv_{k-1}\cdots v_0$ into directed paths. If $k-m$ is odd and $m=1$, then we color $v_0v_1v_2\cdots v_k$ by the colors $r, r, b, r, b, r, ..., r$ and we color v_mv_k blue. We orient the edges as above. Finally, if $k-m$ is odd and $m\geqslant 2$, then we make $v_{m+2}v_{m+1}v_mv_k$ into a directed path of any prescribed color and color the directed paths $v_kv_{k-1}\cdots v_{m+2}$ and $v_mv_{m-1}\cdots v_1$ alternately. We color v_0v_1 red. ∎

4. THE CONJECTURE

In Theorems 1 and 2 below we apply Lemmas 1 and 2 together with some modifications. When we color an edge, we also orient it such that all monochromatic paths become directed paths. In most cases we leave it to the reader to orient (or change orientation of some edges) such that the monochromatic paths have the desired type.

THEOREM 1. *Let G be a 2-connected cubic graph and let H be a subdivision of G. (Possibly $H = G$.) Then there exists a 2-coloring and an orientation of the edge set of H such that*

(i) *each monochromatic component is a directed path of length at most* 5, *and*

(ii) *for each edge e of G, all vertices of degree 2 on the path in H corresponding to e are of type* 1 *except possibly one which is of type* 3.

Proof. We form a sequence of subgraphs $G_0, G_1, \ldots$ of G such that

$$G_0 \subset G_1 \subset \cdots$$

and, for each $i > 0$, G_i is obtained from G_{i-1} by adding a path (as in Lemma 1) or a lollipop (as in Lemma 2) or a vertex joined to three neighbors of G_{i-1}. H_i denotes the subgraph of H corresponding to G_i. Each edge of H_i will be colored red or blue and oriented such that each monochromatic component is a directed path of length at most 5 and such that

(A_i) each vertex, except possibly one, which has degree 2 in H_i but degree 3 in H is of type 1. If the exceptional vertex exists, then it is of type 3 and it is the end of the path or lollipop used when we obtain H_{i+1} from H_i, and

(B_i) part (ii) is satisfied with G_i and H_i instead of G and H.

We now explain how we construct $G_0, G_1, \ldots$. We let $C_0, C_1, \ldots$ denote the cycles we obtain after deleting a perfect matching of G. We may assume that each of $C_0, C_1, \ldots$ has length > 3. (This is easy to prove by induction: If G has a triangle, then contract it and apply the induction hypothesis.) If C_0 has no chord, then we put $G_0 = C_0$. If C_0 has a chord, then we let G_0 denote a chordless cycle which consists of a path of C_0 together with a chord of C_0. We color and orient the edges of H_0 such that all vertices are of type 1 except possibly one (if H_0 has odd length) which is of type 2. Let $C_0: v_1 v_2 \cdots v_q v_1$ and $G_0: v_1 v_2 \cdots v_\delta v_1$ where $3 \leqslant \delta \leqslant q$. If C_0 has only one chord, we let G_1 consist of C_0 and that chord. Otherwise, let s be the smallest number such that $s > \delta$, and v_s has a neighbor, v_t say, in

$\{v_1, v_2, ..., v_{s-2}\}$. We let G_1 be the union of G_0 and the path (or lollipop) $v_\delta v_{\delta+1} \cdots v_s v_t$. We assume that all vertices of H_0, except possibly v_δ, are of type 1 and we use Lemma 1 or 2 to color and orient H_1 such that the conclusion of Lemma 1 or 2 holds. We continue like this defining $G_3, G_4, ...$

Before we discuss the difficulties that may occur, we also explain how G_1 is defined when C_0 is chordless and $G_0 = C_0$. More generally, let us assume that i and j are natural numbers such that G_i consists of the subgraph of G induced by $V(C_0) \cup V(C_1) \cup \cdots \cup V(C_j)$. We assume that (A_i) and (B_i) are satisfied and that the exceptional vertex v mentioned in (A_i) is joined to a vertex u_1 in $C_{j+1} \cup C_{j+2} \cup \cdots$, say in C_{j+1}. (If the exceptional vertex in (A_i) does not exist, we let v be any vertex in G_i which has a neighbor in $C_{j+1} \cup C_{j+2} \cup \cdots$.) Let $C_{j+1}: u_1 u_2 \cdots u_p u_1$. We then let k be the smallest natural number $\geqslant 2$ such that u_k has a neighbor v' in $V(G_i) \cup \{u_1, u_2, ..., u_{k-2}\}$, and we let G_{i+1} be obtained from G_i by adding the path or lollipop $vu_1 u_2 \cdots u_k v'$. We use Lemma 1 or 2 to color and orient the edges of H_{i+1} not in H_i. We continue defining $H_{i+2}, H_{i+3}, ...$ in this way.

We now discuss the difficulties that may occur. When we add a lollipop and apply Lemma 2 we may create a vertex of type 3. Let us consider the above example where G_{i+1} is obtained from G_i by adding the lollipop $vu_1 u_2 \cdots u_k v'$ where $v' \in \{u_1, u_2, ..., u_{k-2}\}$. If the cycle in H_{i+1} corresponding to the cycle $v' u_k u_{k-1} ...$ has even length and the edge $v' u_k$ has length 1 in H_{i+1}, then Lemma 2 implies that u_k is of type 3. This is problematic if G_{i+2} is obtained from G_{i+1} by adding a path $u_k u_{k+1} v''$ which has also length 2 in H. Then we cannot apply Lemma 1 if there is a monochromatic path of length 1 terminating at v'' and that path has the same color as the monochromatic path of length 3 terminating at u_k. Applying (v) in Lemma 2 we can change the color of the directed path terminating at u_k. Then we can apply Lemma 1 unless the color shift also changes color of the directed path terminating at v'' which will happen iff $v'' \in \{u_2, u_3, ..., u_{k-1}\}$. (If $v'' = u_1$, we may assume that u_1 is of type 0 and then Lemma 1 applies.) Let $v' = u_\alpha$, $v'' = u_\beta$ where $2 \leqslant \alpha \leqslant k-2$, $2 \leqslant \beta \leqslant k-1$, $\alpha \neq \beta$. The above problem occurs when the color of $u_{\alpha+1} u_\alpha$ equals the color of $u_{\beta+1} u_\beta$, say they have the color red. If $\beta < \alpha$, then replace each color of the edge $u_\alpha u_k$ and the path $u_k u_{k-1} \cdots u_{\beta+1}$ by the opposite color. Then we color $u_\beta u_{k+1}$ red and $u_k u_{k+1}$ blue and we also orient them in that direction. If $\alpha < \beta$, then we replace each color of $u_{\beta+1} u_\beta \cdots u_{\alpha+1}$ by its opposite color, and we color $u_\beta u_{k+1}$ blue and $u_k u_{k+1}$ red, and we orient them in that direction. In either case u_{k+1} is of type 3, and there are two monochromatic directed paths of length $\leqslant 4$ terminating at u_{k+1}. Let us call the path $vu_1 \cdots u_{k+1}$ together with the two edges $u_\alpha u_k$, $u_\beta u_{k+1}$ an *extended lollipop*. If the extended lollipop is followed by a path of length 2, then we can apply the method of Lemma 1 because of the additional monochromatic path of length $\leqslant 4$ terminating at u_{k+1}.

We add paths or lollipops or extended lollipops and apply Lemmas 1 and 2 together with the modifications described above until we reach u_p (the last vertex of C_{j+1}). We also note that the exceptional vertex mentioned in (A_i) is of type 2 except after the addition of a lollipop satisfying Lemma 2(v) or an extended lollipop or perhaps when $V(G_i)$ is the union of some of the cycles $C_1, C_2, \ldots$.

Actually, we stop when the last added path (or lollipop) before u_p has length $\geqslant 3$ in H. (We give a more precise explanation below.) For notational convenience, let us assume that this happens already with the above path (or lollipop) $vu_1u_2 \cdots u_kv'$ (which clearly is not a path of length 2 in G). For each vertex u_r on C_{j+1} we let z_r denote the neighbor in G of u_r distinct from u_{r-1}, u_{r+1}. Now, the precise criterion for stopping and modifying the procedure described above is that each of the vertices z_{k+1}, $z_{k+2}, \ldots, z_p$ is in G_{i+1} and, furthermore, each edge incident with a vertex in $\{u_{k+1}, u_{k+2}, \ldots, u_p\}$ has length 1 in H. (So, if $z_p \in V(C_{j+2}) \cup V(C_{j+3}) \cup \cdots$ this situation does not happen for C_{j+1} and therefore we do not stop at C_{j+1} in order to modify the method. However, the situation will occur when we consider the last vertex of G, if not before. Note that possibly $u_k = u_p$.)

Assume first that $u_k = u_p$. If u_pu_1 has length $\geqslant 2$ in H, then we apply Lemma 1 or 2. So assume that u_pu_1 has length 1 in H. Now we use the fact that u_1 is of type 1. If u_1 is even of type 0, then we color u_1u_p by the opposite color of the two colored edges incident with u_1. Otherwise we color u_1u_p by the color of the monochromatic path of length 1 terminating at u_1. If the monochromatic path (of length at most 3) terminating at u_p has the same color, then u_1u_p is part of a monochromatic path of length $\leqslant 5$. Assume therefore that this is not the case. Then we apply (if possible) (iv) in Lemma 1 or 2. If there is a monochromatic path of length 1 starting at u_k, then we give u_1u_p the color of that path. If u_ku_{k-1} has length 1 in H, then u_{k-1} becomes of type 2 or 3 and will be the first vertex of the first path intersecting $V(C_{j+2}) \cup V(C_{j+3}) \cup \cdots$. If there is no monochromatic path of length 1 starting at u_k, then $vu_1u_2 \cdots u_kv'$ is a path of length 3 such that vu_1u_2 has length 2 in H and has the same color as the monochromatic directed path in G_i of length $\leqslant 3$ terminating at v. (In that case we should think of v as a vertex in C_{j+1} rather than a vertex in C_j.) But then we change the color of u_2u_1 so that $u_1 = u_{k-1}$ becomes of type 3 as above.

Suppose therefore that $p \geqslant k+1$. If u_k is not of type 2 (i.e., $vu_1u_2 \cdots u_kv'$ is a lollipop with an even cycle) and $u_ku_{k+1}z_{k+1}$ is not part of an extended lollipop, then by Lemma 2(v), we color the lollipop such that the monochromatic directed path of length 3 terminating at u_k does not have the same color as the monochromatic path of length 1 terminating at z_{k+1} (if z_{k+1} is of type 1 but not 0).

Now we color the paths $u_1u_pz_p$, $u_pu_{p-1}z_{p-1}$... using the method of Lemma 1. If Lemma 1 forces us to make $u_p, u_{p-1}, \ldots$ of type 0, we do so. (Note that the new monochromatic directed paths we create in this way all have length 2 or 3 except perhaps the one containing u_1u_p which may have length 4.)

We consider first the case where the mid-vertex of each of the paths $u_1u_pz_p, u_pu_{p-1}z_{p-1}, \ldots, u_{k+2}u_{k+1}z_{k+1}$ is forced to be of type 0. Then we color $u_{k+1}u_k$ by the color distinct from the color of $u_{k+2}u_{k+1}z_{k+1}$. We may assume that the color of $u_{k+1}u_k$ is distinct from the color of the monochromatic path of length $\leqslant 3$ terminating at u_k. Then either $u_{k+1}u_k$ is part of a monochromatic directed path of length 2 or else u_ku_{k+1} is part of a monochromatic directed path of length $\leqslant 6$ by (iv) in Lemma 2 or (iv) in Lemma 1 or the last part of Lemma 1. In the former case u_{k-1} becomes of type 2 if u_ku_{k-1} has length 1 in H. (Possibly u_ku_{k-1} is part of one of the monochromatic paths containing $u_1u_pz_p, u_pu_{p-1}z_{p-1}, \ldots$ but all of these have length $\leqslant 4$ so they may be extended by one edge.) If the above path of length $\leqslant 6$ has length 6, then $vu_1u_2 \cdots u_kv'$ is a path of length 3 such that vu_1u_2 has length 2 in H. Moreover, v is of type 3 in a lollipop or extended lollipop preceding the path vu_1u_2v' (and v is a vertex in C_{j+1} rather than in C_j) or else v is of type 3 in C_j.

If v is in C_{j+1}, then we change the color of u_2u_1 so that u_1 becomes of type 3 and will be the first vertex of the first path or lollipop intersecting $C_{j+2} \cup C_{j+3} \cup \cdots$, if possible. If this is not possible, then there is a chord of C_{j+1} which joins u_1 to one of $u_p, u_{p-1}, \ldots, u_{k+1}$, and u_2u_1 with its new color becomes part of a monochromatic path of length $\leqslant 6$. If that path has length 6, the above chord joins u_1 to u_p. (Because of the oversimplified notation, there is also an edge of C_{j+1} which joins the first u_1 to u_p.) Now change all colors of $u_1u_pu_{p-1} \cdots u_{k+1}$ and give $u_{k+1}u_k$ the same color as $u_{k+2}u_{k+1}$. So assume that v is in C_j. Since the above coloring of the paths $u_1u_pz_p, u_pu_{p-1}z_{p-1}, \ldots$ colors all edges of C_{j+1} except one, it follows that all edges of C_{j+1} and all edges of G incident with C_{j+1} have length 1 in H. Moreover, p (the length of C_{j+1}) is odd, and the edges $u_1v, u_pz_p, u_{p-1}z_{p-1}, \ldots u_3z_3$ are colored alternately, say $r, b, r, b, \ldots, b$. z_2u_2 is the last edge of a blue path of length $\leqslant 2$, and u_2u_1v' is the first part of a red path of length 5. Now we color the edges of $u_2u_1u_pu_{p-1} \cdots$ alternately $b, r, b, r, \ldots$ except that both u_4u_3 and u_3u_2 are colored red. (Here we are using the assumption that C_{j+1} has length >3.)

So when we color those paths $u_1u_pz_p, u_pu_{p-1}z_{p-1}, \ldots$ for which the mid-vertex is forced to be of type 0 we may assume that this process stops at u_{q+1} where $q \geqslant k+1$. Then we color and orient the edges $u_{q+1}u_q$, z_qu_q such that there are two monochromatic paths of length $\leqslant 2$ terminating at u_q. Assume that the monochromatic path of length $\leqslant 3$ terminating at u_k is red. Then color $u_ku_{k+1} \cdots u_q$ alternately $r, b, r, b, \ldots$. The edges

$z_{k+1}u_{k+1}, z_{k+2}u_{k+2}, ..., z_{q-1}u_{q-1}$ are oriented as indicated and colored in accordance with the type of $z_{k+1}, ..., z_{q-1}$. This coloring completes the proof except in the case where u_k is of type 3, and $u_k u_{k+1}$ is part of a red path of length 6. Then $vu_1u_2 \cdots u_k v'$ is a lollipop with an even cycle. Let $v' = u_\alpha$, $2 \leqslant \alpha \leqslant k-2$. We may assume that the edge preceding (respectively succeeding) u_α on $vu_1u_2...$ is blue (respectively red). We now change the color of the red edge incident with u_α on $u_1u_2...$ to blue. This reduces the length of the above red path of length 6 and creates a problem only if $u_{\alpha-1}u_\alpha$ has length 1 in H, and $u_{\alpha-1} = z_\beta$ for some β, where $k+2 \leqslant \beta \leqslant q$, and either $u_\alpha u_{\alpha-1} u_\beta u_{\beta-1} z_{\beta-1}$ or $u_\alpha u_{\alpha-1} u_\beta u_{\beta+1} z_{\beta+1}$ is part of a blue path of length 6. Assume therefore that such a blue path of length 6 is created if we change the color of the red edge incident with u_α on $u_1u_2...$ to blue. Let us therefore not change the color of that edge. Instead we change the coloring of the path $u_k u_{k+1} \cdots u_q$ to $b, r, b, r, ...$. This works unless there exists a γ, where $k+2 \leqslant \gamma \leqslant q$, such that $u_{k-1} = z_\gamma$, and $u_k u_{k-1}$ has length 1 in H, and $u_{k+1} u_k u_{k-1} u_\beta$ is part of a blue path of length 6. Let us assume that $\gamma < \beta$. (The case $\beta < \gamma$ is treated analogously.) If $\beta = \gamma + 1$ and $u_{k+1} u_k u_{k-1} u_\gamma u_\beta u_{\alpha-1} u_\alpha$ is blue, then we complete the proof by recoloring the red edge on $u_1u_2...$ incident with u_α, and also recoloring the path $u_k u_{k+1} \cdots u_\gamma u_\beta$ by $r, b, r, b, ..., b, r$. So if $\beta = \gamma + 1$, then we may assume that the coloring $b, r, b, r, ...$ of $u_k u_{k+1}...$ makes $u_\gamma u_\beta$ red, and hence $u_{k+1} u_k u_{k-1} u_\gamma u_{\gamma-1}$ is blue. Now the proof is completed in the case $\beta = \gamma + 1$ when we change the color of $u_\gamma u_{\gamma-1}$ to red. (Note that we cannot have $\beta = q$ and $u_q u_{q+1}$ red since we assume that recoloring the red edge of $u_1u_2...$ incident with u_α creates a blue path of length 6.) Assume therefore that $\beta \geqslant \gamma + 2$. If both $u_{\gamma-1} z_{\gamma-1}$ and $u_{\gamma+1} z_{\gamma+1}$ are blue we complete the proof by recoloring one of $u_\gamma u_{\gamma-1}$ and $u_\gamma u_{\gamma+1}$ so that they both become red. So assume that precisely one of $u_{\gamma-1} z_{\gamma-1}$ and $u_{\gamma+1} z_{\gamma+1}$ is blue. Now we complete the proof by recoloring $u_k u_{k+1} \cdots u_q$ almost alternately $b, r, b, r...$ except that either $u_{\gamma-2} u_{\gamma-1} u_\gamma$ is red (when $\gamma - k$ is odd) or $u_{\gamma-1} u_\gamma u_{\gamma+1}$ is red (when $\gamma - k$ is even).

This completes the proof. ∎

THEOREM 2. *Let H be a graph of maximum degree at most* 3. *Then the edge set of H has a coloring in two colors and an orientation of the edges such that each monochromatic component is a directed path of length at most* 5. *Moreover, if v is any vertex of degree* 2, *then the coloring can be chosen such that v is of type* 1 *and, for each path P in H whose vertices have degree* 2 *in H, all vertices of P are of type* 3, *and all vertices of P except possibly one, are of type* 1.

Proof (by induction on $|V(H)|$). We may assume that H is connected. Consider first the case where H is not 2-connected. Let $x_1 x_2 \cdots x_m$ be a maximal path in H such that each of x_1, x_m has degree 1 or 3 in H, each of $x_2, x_3, ..., x_{m-1}$ has degree 2 in H, and $H-\{x_2, x_3, ..., x_{m-1}\}$ (or $H-x_1x_2$ if $m=2$) has two components H_1 and H_2 where $x_1 \in V(H_1)$, $x_m \in V(H_2)$. We may assume that $v \notin V(H_2)$. Now we apply the induction hypothesis to each of H_1 and H_2 such that x_m is of type 1 in H_2. If $v \notin V(H_1)$, then we may assume that x_1 is of type 1 in H_1. If $m \geqslant 4$, then we may assume that $v \neq v_{m-1}$. We now complete the proof by applying Lemma 1. Note that we may interchange the colors of H_2 if necessary. So assume that H is 2-connected.

Let G be the multigraph such that H is a subdivision of G. If G has only two vertices and three edges, the proof is easily completed. So assume that G has at least three vertices.

Consider now the case where G has a double edge. Then H has a path $x_1x_2 \cdots x_m$ and a path $y_1y_2 \cdots y_k$ where $y_1=x_i$, $y_k=x_j$, $1<i<j<m$, x_1, x_m, y_1, y_k have degree 3 in H, and all other vertices in $\{x_1, x_2, ..., x_m, y_1, y_2, ..., y_k\}$ have degree 2 in H. We choose the paths such that $y_1y_2 \cdots y_k$ is at least as long as $x_ix_{i+1} \cdots x_j$. If they have the same length we may assume that $v \notin \{y_2, y_3, ..., y_{k-1}\}$. Now we apply the induction hypothesis to $H'=H-\{y_2, y_3, ..., y_{k-1}\}$. We may assume that x_j is of type 1 in H'. If $k \geqslant 4$, then we color the edges of $y_ky_{k-1}...$ alternately (as in Lemma 1) except that the last two edges may get the same color. This results in the desired coloring unless $v=y_{k-1}$ is of type 2. But then we may assume that x_i is of type 1 in H' (by letting x_i play the role of v when we use induction) and we apply Lemma 1 to $y_1y_2 \cdots y_k$ (rather than $y_ky_{k-1} \cdots y_1$). So assume that $k=3$. Hence $j-i \leqslant 2$, and if equality holds, then $y_2 \neq v$. The method in the case $k \geqslant 4$ also works for $k=3$ unless y_1y_2, y_2y_3, receive the same color and become part of a monochromatic, say red, path of length 6, or if $v=y_2$ is of type 2. Consider the case where $v \neq y_2$. There is also a blue edge incident with x_j. If that edge is also incident with x_{j-1}, then we color y_1y_2 red, and y_2y_3 blue. On the other hand, if x_jx_{j+1} is blue, then $j-i=2$ and also x_ix_{i+1} is blue. Then we change x_ix_{i-1} to blue, and we color $y_1y_2y_3x_{i+1}y_1$ by b, r, r, r. (If $v=x_{i-1}$, then we also change the color of $x_{i-2}x_{i-1}$).

We are left with the case where $k=3$, $v=y_2$, and $j=i+1$. This also means that G has at most one double edge since otherwise, we apply the above reasoning to another double edge.

If v exists, let e be the edge of G on which v is. If G has a multiple edge, then e is part of it. Let e' be the other edge of that multiple edge. If e' exists, then e has length 2 in H, and e' has length 1 in H. Now let M be a perfect matching of $G-e$ such that $G-M$ has no cycle of length 3. If e' exists we may assume that M contains e'. Let $C_1, C_2, ...$ be the cycles of $G-M$ such that C_1 contains e. Now we repeat the proof of Theorem 1 assuming that e is in G_0. We color H_0 (almost) alternately such that v is of type 1 and continues to be so throughout the proof. We may assume that in G_0 there is a red (respectively blue) directed path of length 1 terminating (respectively starting) at v. If e' exists, then we color it blue. ∎

REFERENCES

1. J. Akiyama, G. Exoo, and F. Harary, Covering and packing in graphs. II. Cyclic and acyclic invariants, *Math. Slovaca* **30** (1980), 405–417.
2. J. Akiyama and V. Chvátal, A short proof of the linear arboricity for cubic graphs, *Bull. Liber. Arts Sci. NMS* **2** (1981).
3. R. E. L. Aldred and N. C. Wormald, to appear.
4. J. C. Bermond, J. L. Fouquet, M. Habib, and B. Péroche, On linear k-arboricity, *Discrete Math.* **52** (1984), 123–132.
5. B. Jackson and N. C. Wormald, On the linear k-arboricity of cubic graphs, *Discrete Math.* **162** (1996), 293–297.