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Completion of L-Fuzzy Relations

UMBERTO CERRUTI

Istituto di Geometria, Università di Torino, 10123 Torino, Italy

Submitted by L. Zadeh

It is shown that the category $\tilde{\mathscr{S}}(L)$ of complete L-similarities on L-sets is a full reflective subcategory of $\mathscr{R}(L)$ (L-fuzzy graphs); $\tilde{\mathscr{S}}(L)$ is equivalent to $\mathscr{S}h(L)$ (sheaves on L). Connections with other known "fuzzy" categories are also studied.

I. PRELIMINARIES

In the following we shall make use of the chain of categories

$$\mathscr{U}(L) \underset{u_2}{\hookrightarrow} \mathscr{S}(L) \underset{u_1}{\hookrightarrow} \mathscr{R}(L), \tag{I.1}$$

where each arrow denotes a full embedding. These categories have been introduced and studied in [1]. For the reader's convenience we repeat the definitions here.

(I.2). DEFINITION OF $\mathscr{R}(L)$. An object in $\mathscr{R}(L)$ is a triplet composed of

(a) A set $|R|$;

(b) an L-fuzzy subset E_R of $|R|$, i.e., a function $E_R: |R| \to L$;

(c) an L-relation R on E_R, i.e., a function $R: |R| \times |R| \to L$ s.t. $R(x, y) \leqslant E_R(x) \wedge E_R(y)$. We shall denote the object by the same symbol used for the relation. A morphism $f: R \to S$ is a function $f: |R| \to |S|$ such that

(d) $R(x, y) \leqslant S(f(x), f(y))$;

(e) $E_R(x) = E_S(f(x))$.

Composition of morphisms is that of functions.

(I.3). DEFINITION OF $\mathscr{S}(L)$. We denote by $\mathscr{S}(L)$ the full subcategory of $\mathscr{R}(L)$ composed of objects $R \in |\mathscr{R}(L)|$ satisfying the following additional properties:

(a) $R(x, y) = R(y, x)$ (symmetry);

(b) $R(x, y) \wedge R(y, z) \leqslant R(x, z)$ (transitivity).

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We observe that (a) and (b) immediately imply

(c) $E_R(x) = R(x, x)$ (reflexivity).

To introduce $\mathscr{U}(L)$ we need

(I.4). DEFINITIONS. (a) Given $R \in |\mathscr{S}(L)|$, we call a *ball of center x and radius* $1/\alpha$ ($\alpha \in L$) the set $R_\alpha(x) = \{y \in |R| : R(x, y) \geqslant \alpha\}$.

(b) Given the ball $B = R_\alpha(x)$ we say that $y \in B$ is an *extremal point* of B if $E_R(y) = \alpha$.

(c) The set $R \in |\mathscr{S}(L)|$ is *spherically complete* if every chain of nonempty balls in R has nonempty intersection.

(I.5). *Remarks.* (a) The ball $R_\alpha(x)$ is nonempty iff $\alpha \leqslant E_R(x)$.

(b) Two balls of the same radius with nonempty intersection are equal.

(c) If L is totally ordered, any two balls with nonempty intersection are contained one within the other.

(d) If $y \in R_\alpha(x) = B$ is an extremal point of B, then E_R attains in y its minimum possible value on B.

(I.6). DEFINITION OF $\mathscr{U}(L)$. The category $\mathscr{U}(L)$ is the full subcategory of $\mathscr{S}(L)$ composed of objects $R \in |\mathscr{S}(L)|$ satisfying the following additional properties:

(a) every nonempty ball in R has a unique extremal point;

(b) the set R is sperically complete.

(I.7). PROPERTY. *Category $\mathscr{S}(L)$ is a full reflective subcategory of $\mathscr{R}(L)$.*

Proof. It is easy to see that the reflector is the transitive closure t: $tR =$ transitive closure of R, $tf = f$. Indeed if f is a morphism f: $R \to S$ with S transitive, the same f: $tR \to S$ is a morphism, and then we can choose as component ε_R of the natural transformation $1 \to^{\varepsilon} ut$ the identity map on $|R|$. ∎

In (13) Zadeh introduced the concept of "class" represented by an element x (with respect to a similarity R).

(I.8). DEFINITION OF CLASS. We say that an L-subset A of $|R|$ is a *class*—and precisely the class represented by $x \in |R|$—if

$$\forall y \quad A(y) = R(x, y).$$

We shall denote this class by $\tilde{x}$.

This concept is an extension of the concept of "crisp" class for a "crisp" equivalence relation. Now this concept can be extended in another way without using any fixed element of the support.

(I.9). DEFINITION OF TYPE. Given R in $\mathscr{S}(L)$, a type on R is an L-fuzy subset of $|R|$ such that

(a) $A(x) \wedge R(x, y) \leqslant A(y)$;

(b) $A(x) \wedge A(y) \leqslant R(x, y)$.

Condition (a) is the translation of "if x belongs to A and x is equivalent to y, then y belongs to A," i.e., "A is a union of classes." Condition (b) is the translation of "if x and y belong to A, then x is equivalent to y," i.e., "A is contained in a class."

We remark that a type is exactly a "singleton" in the definition by Fourman and Scott (4).

In (11) and (18) Sanchez has studied and applied the concept of "eigenset."

(I.10). DEFINITION OF EIGENSET. Given an L-fuzzy relation R, an L-fuzzy subset A of $|R|$ is an *eigenset* if

(a) $R \circ A = A$, i.e.,

(b) $\bigvee_{y \in |R|} (R(x, y) \wedge A(y)) = A(x)$.

(I.11). PROPERTY. *If $R \in |\mathscr{S}(L)|$, then a class on R is a type, and a type is an eigenset, but not conversely.*

Proof. (1) Let us take $A = \tilde{x}$, with $x \in |R|$; then

$$A(y) \wedge R(y, z) = R(x, y) \wedge R(y, z) \leqslant R(x, z),$$

$$A(y) \wedge A(z) = R(x, y) \wedge R(x, z) \leqslant R(y, z)$$

(by symmetry and transitivity of R); so we obtain I.9(a) and I.9(b).

(2) Let A be a type on R. Then from I.9(a)

$$\forall y \quad A(y) \wedge R(y, x) \leqslant A(x)$$

and then $\bigvee_{y \in |R|} (R(y, x) \wedge A(y)) \leqslant A(x)$. On the other hand, I.9(b) implies $A(x) \leqslant R(x, x)$; then $A(x) = A(x) \wedge R(x, x)$ and $A(x) \leqslant \bigvee_{y \in |R|} (R(x, y) \wedge A(y))$. Thus A is an eigenset.

(3) We choose $L = \{0, 1\}$ and $R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, corresponding to the partition of $|R| = \{x, y\}$, $\{\{x\}, \{y\}\}$. Then $(0, 0)$ is a type but not a class, and $(1, 1)$ is an eigenset but not a type. ∎

In this paper we direct our attention to the middle class, that of types.

II. Completion of Similarity Relations

In general there are many types on R which cannot be represented by an element of $|R|$; on the contrary they can be represented—as a class—by many different elements. Using the method introduced in (4) we can "complete" R. Definition II.1 and the proof of II.2 can be found in (4) (with a different notation).

(II.1). Definition of completeness. The relation $R \in |\mathscr{S}(L)|$ is said to be complete if for every type A on R there exists a unique $x \in |R|$ such that $\forall y$, $A(y) = R(x, y)$, i.e., $A = \tilde{x}$.

(II.2). Property. *Given R in $\mathscr{S}(L)$ we construct $\tilde{R}$ in this way:*

$$|\tilde{R}| = \{A \in L^{|R|}: A \text{ is a type on } R\},$$

$$\tilde{R}(A, B) = \bigvee_{x \in |R|} (A(x) \wedge B(x)).$$

Then $\tilde{R} \in |\mathscr{S}(L)|$ and $\tilde{R}$ is complere; indeed if F is a type on $\tilde{R}$, $A: |R| \to L$, $A: x \mapsto F(\tilde{x})$, is the unique element of $|\tilde{R}|$ s.t. $F = \tilde{A}$ $(\forall B \in |\tilde{R}|$, $F(B) = \tilde{R}(A, B))$. Furthermore, $\tilde{R}(\tilde{x}, \tilde{y}) = R(x, y)$.

(II.3). Definition of $\tilde{\mathscr{S}}(L)$. We call $\tilde{\mathscr{S}}(L)$ the full subcategory of $\mathscr{S}(L)$ formed by complete $R \in |\mathscr{S}(L)|$.

(II.4). Definition of p. Given $R \in |\mathscr{S}(L)|$ we define $p(R) = \tilde{R} \in |\tilde{\mathscr{S}}(L)|$. Given f: $R \to S$ with R, $S \in |\mathscr{S}(L)|$ we define

$$\forall A \in |\tilde{R}| \quad \forall y \in |S| \qquad \langle \tilde{f}(A), y\rangle = \bigvee_{x \in |R|} (A(x) \wedge S(f(x), y)).$$

Proof that p is a functor. We must prove that $\tilde{f}(A) \in |\tilde{S}|$.

(a) We have

$$\begin{aligned}\langle \tilde{f}(A), y\rangle \wedge S(y, z) &= \bigvee_x (A(x) \wedge S(f(x), y)) \wedge S(y, z)\\ &= \bigvee_x (A(x) \wedge S(f(x), y) \wedge S(y, z))\\ &\leqslant \bigvee_x (A(x) \wedge S(f(x), z)) = \langle \tilde{f}(A), z\rangle.\end{aligned}$$

(b) We have

$$\langle \tilde{f}(A), y\rangle \wedge \langle \tilde{f}(A), z\rangle = \bigvee_x (A(x) \wedge S(f(x), y)) \wedge \bigvee_t (A(t) \wedge S(f(t), z))$$

$$= \bigvee_{x,t} (A(x) \wedge S(f(x), y) \wedge A(t) \wedge S(f(t), z));$$

now, $\forall x\, \forall t$:

$$\begin{aligned} &A(x) \wedge S(f(x), y) \wedge A(t) \wedge S(f(t), z) \\ &\quad \leqslant R(x, t) \wedge S(f(x), y) \wedge S(f(t), z) \\ &\quad \leqslant S(f(x), f(t)) \wedge S(f(x), y) \wedge S(f(t), z) \leqslant S(y, z). \end{aligned}$$

So $\tilde{f}$ is a function $\tilde{f}$: $\tilde{R} \to \tilde{S}$. We prove now that $\tilde{f}$ is a morphism.

(c) We have

$$\begin{aligned} &\tilde{S}(\tilde{f}(A), \tilde{f}(B)) \\ &= \bigvee_{y \in |S|} (\langle \tilde{f}(A), y\rangle \wedge \langle \tilde{f}(B), y\rangle) \\ &= \bigvee_{y \in |S|} \left[\bigvee_{x \in |R|} (A(x) \wedge S(f(x), y)) \wedge \bigvee_{h \in |R|} (B(h) \wedge S(f(h), y))\right] \\ &\geqslant \bigvee_{t \in |R|} \left[\bigvee_{x \in |R|} (A(x) \wedge S(f(x), f(t))) \wedge \bigvee_{h \in |R|} (B(h) \wedge S(f(h), f(t)))\right] \\ &\geqslant \bigvee_{t \in |R|} \left[\bigvee_{x \in |R|} (A(x) \wedge R(x, t)) \wedge \bigvee_{h \in |R|} (B(h) \wedge R(h, t))\right] \\ &= \bigvee_{t \in |R|} (A(t) \wedge B(t)) = \tilde{R}(A, B). \end{aligned}$$

(d) We have

$$\tilde{S}(\tilde{f}(A), \tilde{f}(A)) = \bigvee_{y \in |S|} \langle \tilde{f}(A), y\rangle = \bigvee_{y \in |S|} \left(\bigvee_{x \in |R|} (A(x) \wedge S(f(x), y))\right);$$

but, $\forall y \in |S|$

$$\bigvee_{x \in |R|} (A(x) \wedge S(f(x), y)) \leqslant \bigvee_{x \in |R|} A(x) = \tilde{R}(A, A);$$

and then $\tilde{S}(\tilde{f}(A), \tilde{f}(A)) \leqslant \tilde{R}(A, A)$.

So we can define $pf = \tilde{f}$. To be a functor p must satisfy $p1_R = \tilde{1}_R = 1_{\tilde{R}}$ and $p(f \circ g) = pf \circ pg$. We shall prove these.

(e) We have

$$\langle \tilde{1}_R(A), x \rangle = \bigvee_{t \in |R|} (A(t) \wedge R(1_R(t), x)) = \bigvee_{t \in |R|} (A(t) \wedge R(t, x)) = A(x);$$

then $\forall A \in |\tilde{R}|\ \tilde{1}_R(A) = A$ and $\tilde{1}_R = 1_{\tilde{R}}$.

(f) Given $R \to_f S \to_g T$ in $\mathscr{F}(L)$ we have $\tilde{R} \to_{\tilde{f}} \tilde{S} \to_{\tilde{g}} \tilde{T}$ in $\tilde{\mathscr{F}}(L)$. We must prove that $\forall A \in |\tilde{R}|\ \forall u \in |T|\ \langle \tilde{g}\tilde{f}(A), u \rangle = \langle \widetilde{gf}(A), u \rangle$.

$$\begin{aligned}\langle \tilde{g}\tilde{f}(A), u \rangle &= \bigvee_{y \in |S|} (\langle \tilde{f}(A), y \rangle \wedge T(g(y), u)) \\ &= \bigvee_{y \in |S|} \left(\bigvee_{x \in |R|} (A(x) \wedge S(f(x), y)) \wedge T(g(y), u) \right).\end{aligned}$$

Now $\forall y \in |S|$, we have

$$\begin{aligned}&\bigvee_{x \in |R|} (A(x) \wedge S(f(x), y)) \wedge T(g(y), u) \\ &= \bigvee_{x \in |R|} (A(x) \wedge S(f(x), y) \wedge T(g(y), u)) \\ &\leqslant \bigvee_{x \in |R|} (A(x) \wedge T(gf(x), g(y)) \wedge T(g(y), u)) \\ &\leqslant \bigvee_{x \in |R|} (A(x) \wedge T(gf(x), u)) = \langle \widetilde{gf}(A), u \rangle.\end{aligned}$$

Furthermore,

$$\begin{aligned}\langle \tilde{g}\tilde{f}(A), u \rangle &= \bigvee_{y \in |S|} (\langle \tilde{f}(A), y \rangle \wedge T(g(y), u)) \\ &\geqslant \bigvee_{h \in |R|} (\langle \tilde{f}(A), f(h) \rangle \wedge T(gf(h), u)) \\ &= \bigvee_{h \in |R|} \left(\bigvee_{x \in |R|} (A(x) \wedge S(f(x), f(h))) \wedge T(gf(h), u) \right) \\ &\geqslant \bigvee_{h \in |R|} \left(\bigvee_{x \in |R|} (A(x) \wedge R(x, h)) \wedge T(gf(h), u) \right) \\ &= \bigvee_{h \in |R|} (A(h) \wedge T(gf(h), u) = \langle \widetilde{gf}(A), u \rangle\end{aligned}$$

and then we obtain the thesis. Then p is a functor $p: \mathscr{F}(L) \to \tilde{\mathscr{F}}(L)$. ∎

(II.5). LEMMA. (a) *Given* $R \in |\mathcal{F}(L)|$ *and* $x, y \in |R|$

$$R(x, y) = R(x, x) = R(y, y) \Leftrightarrow \tilde{x} = \tilde{y}.$$

(b) *Given* $S \in |\tilde{\mathcal{F}}(L)|$ *and* $x, y \in |S|$

$$S(x, y) = S(x, x) = S(y, y) \Leftrightarrow x = y.$$

Proof. (a) To show the $\Rightarrow$ part, let us take $z \in |R|$.

$$R(x, z) = R(x, x) \wedge R(x, z) = R(x, y) \wedge R(x, z) \leqslant R(y, z);$$

analogously $R(y, z) \leqslant R(x, z)$; then $\tilde{x} = \tilde{y}$.

To show the $\Leftarrow$ part, we have that, if $\tilde{x} = \tilde{y}$, then $R(x, x) = R(y, x)$ and $R(x, y) = R(y, y)$.

(b) This is an obvious consequence of part (a) and of the definition of completeness. ▮

(II.6). LEMMA. *Given* $R \in |\mathcal{F}(L)|$, *the map* $\eta_R : R \to \tilde{R}$, $\eta_R : x \to \tilde{x}$ *is a morphism.*

Proof. Indeed, from II.2, we have $\tilde{R}(\tilde{x}, \tilde{y}) = R(x, y)$. ▮

(II.7). LEMMA. *Given* $R \in |\mathcal{F}(L)|$ *and* $S \in |\tilde{\mathcal{F}}(L)|$, *if the diagram*

$$R \xrightarrow[\eta_R]{} \tilde{R} \overset{g}{\underset{h}{\rightrightarrows}} S$$

commutes, then $g = h$.

Proof. We have, $\forall x \in |R|$ and $A \in |\tilde{R}|$,

$$S(h(A), h(\tilde{x})) \geqslant \tilde{R}(A, \tilde{x}) = \bigvee_y (A(y) \wedge R(x, y)) = A(x);$$

in the same way, $S(g(A), g(\tilde{x})) \geqslant A(x)$.

From the hypothesis we have $\forall x$, $h(\tilde{x}) = g(\tilde{x})$ and then

$$S(h(A), h(\tilde{x})) \wedge S(g(A), g(\tilde{x})) \leqslant S(h(A), g(A)).$$

So $\forall x \in |R|$, $\forall A \in |\tilde{R}|$, $A(x) \leqslant S(h(A), g(A))$ and $\tilde{R}(A, A) = \bigvee_x A(x) \leqslant S(h(A), g(A))$; but h and g are morphisms and then

$$S(h(A), h(A)) = S(g(A), g(A)) = \tilde{R}(A, A);$$

thus $\forall A \in |\tilde{R}|$. $S(h(A), h(A)) = S(g(A), g(A)) = S(h(A), g(A))$, and from Lemma II.5, $\forall A \in |\tilde{R}|$, $h(A) = g(A)$. ▮

(II.8). PROPERTY. *If $S \in |\tilde{\mathscr{F}}(L)|$, $R \in |\mathscr{F}(L)|$, and $f\colon R \to S$ is any morphism, then there exists a unique $g\colon \tilde{R} \to S$ s.t. the following diagram commutes*:

$$\begin{array}{ccc} R & \xrightarrow{\eta_R} & \tilde{R} \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & S \end{array} \tag{II.8.1}$$

In other words, $\tilde{\mathscr{F}}(L)$ is a reflective full subcategory of $\mathscr{F}(L)$.

Proof. Given $A \in |\tilde{R}|$, we know that $\tilde{f}(A) \in |\tilde{S}|$. Since S is complete, there exists a unique $y \in |S|$ s.t. $\forall z \in |S|$, $\langle \tilde{f}(A), z\rangle = S(y, z)$; then we define $g\colon |\tilde{R}| \to |S|$ in this way: $g\colon A \mapsto y$.

Let us suppose that $g(A_1) = y_1$ and $g(A_2) = y_2$. Then

$$\begin{aligned} S(g(A_1), g(A_2)) = S(y_1, y_2) &= \bigvee_{z \in |S|} (S(y_1, z) \wedge S(y_2, z)) \\ &= \bigvee_{z \in |S|} (\langle \tilde{f}(A_1), z\rangle \wedge \langle \tilde{f}(A_2), z\rangle) \\ &= \tilde{S}(\tilde{f}(A_1), \tilde{f}(A_2)) \geqslant \tilde{R}(A_1, A_2) \end{aligned}$$

since: $\tilde{f}\colon \tilde{R} \to \tilde{S}$ is a morphism. Furthermore,

$$\begin{aligned} S(g(A), g(A)) = S(y, y) &= \bigvee_{z \in |S|} S(y, z) \\ &= \bigvee_{z \in |S|} \langle \tilde{f}(A), z\rangle = \tilde{S}(\tilde{f}(A), \tilde{f}(A)) = \tilde{R}(A, A). \end{aligned}$$

Then g is a morphism $g\colon \tilde{R} \to S$.

Now we must prove that $\forall x \in |R|$, $g(\eta_R(x)) = g(\tilde{x}) = f(x)$. Since S is complete, it is sufficient to prove that

$$\forall z \in |S| \qquad \langle \tilde{f}(\tilde{x}), z\rangle = S(f(x), z).$$

Now

$$\begin{aligned} \langle \tilde{f}(\tilde{x}), z\rangle &= \bigvee_{y \in |R|} (\tilde{x}(y) \wedge S(f(y), z)) = \bigvee_{y \in |R|} (R(x, y) \wedge S(f(y), z)) \\ &\leqslant \bigvee_{y \in |R|} (S(f(x), f(y)) \wedge S(f(y), z)) \leqslant S(f(x), z) \end{aligned}$$

and

$$S(f(x), z) = S(f(x), f(x)) \wedge S(f(x), z) = R(x, x) \wedge S(f(x), z)$$

$$= \tilde{x}(x) \wedge S(f(x), z) \leqslant \bigvee_{y \in |R|} (\tilde{x}(y) \wedge S(f(y), z)) = \langle \tilde{f}(\tilde{x}), z \rangle.$$

Then diagram II.8.1 commutes. Unicity of g follows from Lemma II.7. ∎

We shall see now that the category $\mathscr{U}(L)$ (Definition I.6) is "between" $\tilde{\mathscr{T}}(L)$ and $\mathscr{T}(L)$.

(II.9). PROPERTY. *The category $\tilde{\mathscr{T}}(L)$ is a full subcategory of $\mathscr{U}(L)$.*

Proof. Let us take R in $|\tilde{\mathscr{T}}(L)|$, α in L, and a nonempty ball $R_\alpha(x)$; then $R(x, x) \geqslant \alpha$. If we define A: $|R| \to L$, $A(y) = R(x, y) \wedge \alpha$, it is easy to see that A is a type on R. But R is complete and thus there exists $x_0 \in |R|$ such that $\forall y \in |R|$, $R(x_0, y) = R(x, y) \wedge \alpha$. So $R(x_0, x) = R(x, x) \wedge \alpha = \alpha$ and $R(x_0, x_0) = R(x, x_0) \wedge \alpha = \alpha \wedge \alpha = \alpha$. Thus x_0 is an extremal point of $R_\alpha(x_0) = R_\alpha(x)$.

If y_0 is an extremal point on $R_\alpha(x_0)$, we have $R(x_0, x_0) = R(y_0, y_0) = R(x_0, y_0) = \alpha$ and then $x_0 = y_0$ by Lemma II.5(b).

We have seen that every ball in R has a unique extremal point. Now let $\mathscr{C}$ be a chain of nonempty balls in R,

$$\mathscr{C} = \{R_\alpha(x_\alpha)\}, \qquad \alpha \in I \subseteq L.$$

For what we have proved we can suppose, without any restriction, that for every α, x_α is the (unique) extremal point of $R_\alpha(x_\alpha)$. Then if $R_\alpha(x_\alpha) = R_\beta(x_\beta)$, it follows that $\alpha = \beta$ and $x_\alpha = x_\beta$. So the following definition of A: $|R| \to L$ is meaningful:

$$A(y) = \bigvee_{\alpha \in I} (R(x_\alpha, y) \wedge \alpha).$$

We can prove that A is a type:

$$A(y) \wedge R(x, y) = \bigvee_{\alpha \in I} (R(x_\alpha, y) \wedge \alpha) \wedge R(x, y)$$

$$= \bigvee_{\alpha \in I} (R(x_\alpha, y) \wedge \alpha \wedge R(x, y))$$

$$\leqslant \bigvee_{\alpha} (R(x_\alpha, x) \wedge \alpha) = A(x).$$

(b) We have

$$A(y) \wedge A(x) = \bigvee_{\alpha \in I} (R(x_\alpha, y) \wedge \alpha) \wedge \bigvee_{\beta \in I} (R(x_\beta, x) \wedge \beta)$$

$$= \bigvee_{\alpha, \beta \in I} (R(x_\alpha, y) \wedge R(x_\beta, x) \wedge \alpha \wedge \beta).$$

Now, given a pair $\alpha, \beta \in I$, we have—for example—$R_\alpha(x_\alpha) \subseteq R_\beta(x_\beta)$, since $\mathscr{C}$ is a chain. Then $\alpha = R(x_\alpha, x_\alpha) \geqslant R(x_\alpha, x_\beta) \geqslant \beta$, since $x_\alpha \in R_\beta(x_\beta)$ and x_α is the extremal point of $R_\alpha(x_\alpha)$ (this implies that $I \subseteq L$ is a chain). But $R(x_\alpha, x_\beta) \geqslant R(x_\beta, x_\beta) = \beta$ and thus $R(x_\alpha, x_\beta) = \beta = \alpha \wedge \beta$. So

$$\bigvee_{\alpha, \beta \in I} (R(x_\alpha, y) \wedge R(x_\beta, x) \wedge \alpha \wedge \beta)$$

$$= \bigvee_{\alpha, \beta \in I} (R(x_\alpha, y) \wedge R(x_\beta, x) \wedge R(x_\alpha, x_\beta)) \leqslant R(x, y).$$

Then A is a type. Since R is complete there exists $x_0 \in |R|$ such that $\forall y$. $A(y) = R(x_0, y)$. So

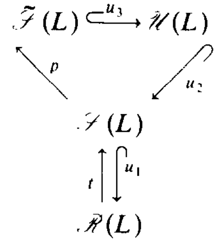
$$R(x_0, y) = \bigvee_\alpha (R(x_\alpha, y) \wedge \alpha).$$

and

$$R(x_0, x_\beta) = \bigvee_\alpha (R(x_\alpha, x_\beta) \wedge \alpha) = \bigvee_\alpha (\alpha \wedge \beta \wedge \alpha) = \beta$$

and $\forall \beta$, $x_0 \in R_\beta(x_\beta)$. So $\bigcap \mathscr{C}$ is nonempty and $R \in |\mathscr{U}(L)|$. ∎

If we call u_3 the full embedding $\tilde{\mathscr{F}}(L) \hookrightarrow \mathscr{U}(L)$, we see that diagram (I.1) can be enriched in this way:



where t and p are left adjoints, respectively, of u_1 and $u_2 \circ u_3$.

(II.10). EXAMPLE OF COMPLETION. Let L be $L = \{0, 1, 2, 3\}$ with usual order, $|R| = \{a, b, c, d, e, f\}$, and

	a	b	b	d	e	f
a	3	0	0	1	0	1
b	0	3	0	0	2	0
$R \equiv c$	3	0	3	1	0	1
d	1	0	1	3	0	2
e	0	2	0	0	3	0
f	1	0	1	2	0	3.

Then we have eleven types on R,

	a	b	c	d	e	f
$\tilde{a} = \tilde{c}$	3	0	3	1	0	1
$\tilde{b}$	0	3	0	0	2	0
$\tilde{d}$	1	0	1	3	0	2
$\tilde{e}$	0	2	0	0	3	0
$\tilde{f}$	1	0	1	2	0	3
g	1	0	1	1	0	1
h	2	0	2	1	0	1
i	0	1	0	0	1	0
l	0	2	0	0	2	0
m	1	0	1	2	0	2
n	0	0	0	0	0	0

and this is $\tilde{R}$:

	a	b	d	e	f	g	h	i	l	m	n
a	3	0	1	0	1	1	2	0	0	1	0
b	0	3	0	2	0	0	0	1	2	0	0
d	1	0	3	0	2	1	1	0	0	2	0
e	0	2	0	3	0	0	0	1	2	0	0
f	1	0	2	0	3	1	1	0	0	2	0
g	1	0	1	0	1	1	1	0	0	1	0
h	2	0	1	0	1	1	2	0	0	1	0
i	0	1	0	1	0	0	0	1	1	0	0
l	0	2	0	2	0	0	0	1	2	0	0
m	1	0	2	0	2	1	1	0	0	2	0
n	0	0	0	0	0	0	0	0	0	0	0

III. Complete Fuzzy Graphs and Categorical Logic

The categories of Heyting-valued-sets—considered as sheaves—were introduced by Higgs in 1973; for applications to the interpretation of first- and higher-order logic see (4) and (7). We recall some definitions.

(III.1). Definition of $\mathscr{H}(L)$. (a) Objects are the same as in $\mathscr{S}(L)$;

(b) a morphism $F\colon R \to S$ is a function $F\colon |R| \times |S| \to L$ s.t.

(b.1) $F(x, y) \wedge R(x, x') \leqslant F(x', y)$;

(b.2) $F(x, y) \wedge S(y, y') \leqslant F(x, y')$;

(b.3) $F(x, y) \wedge F(x, y') \leqslant S(y, y')$;

(b.4) $R(x, x) = \bigvee_{y \in |S|} F(x, y)$;

(c) Given $R \to_F S \to_G T$, composition is defined by

$$G \circ F(x, z) = \bigvee_{y \in |S|} (F(x, y) \wedge G(y, z));$$

the identity map on R is $R\colon |R| \times |R| \to L$.

(III.2). Definition of $\tilde{\mathscr{H}}(L)$. Category $\tilde{\mathscr{H}}(L)$ is the full subcategory of $\mathscr{H}(L)$ composed of complete R's.

(III.3). Property (Higgs). *Category $\mathscr{H}(L)$ is equivalent to the Grothendieck topos $\mathscr{S}h(L)$ of sheaves on L with canonical topology.*

(III.4). Property. [4]. *Category $\mathscr{H}(L)$ is equivalent to $\tilde{\mathscr{H}}(L)$.*

(III.5). Property. *Category $\tilde{\mathscr{S}}(L)$ is isomorphic to $\tilde{\mathscr{H}}(L)$.*

Proof. Given $f\colon R \to S$ in $\tilde{\mathscr{S}}(L)$ we define $F\colon |R| \times |S| \to L$ by

(a) $F(x, y) = S(f(x), y)$;

then

(b) $R(x, x') \wedge S(f(x), y) \leqslant S(f(x), f(x')) \wedge S(f(x), y) \leqslant S(f(x'), y)$.

(c) $S(y, y') \wedge F(x, y) = S(y, y') \wedge S(f(x), y) \leqslant S(f(x), y') = F(x, y')$.

(d) $F(x, y) \wedge F(x, y) = S(f(x), y) \wedge S(f(x), y') \leqslant S(y, y')$.

(e) $\bigvee_{y \in |S|} F(x, y) = \bigvee_{y \in |S|} S(f(x), y)$.

Now $\forall y$, $S(f(x), y) \leqslant S(f(x), f(x))$ and $f(x) \in |S|$; then

$$\bigvee_{y \in |S|} S(f(x), y) = S(f(x), f(x)) = R(x, x).$$

Properties (b)–(e) prove that conditions III.1(b)–(b.4) are verified. So F: $R \to S$ is in $\tilde{\mathscr{F}}(L)$. We put $\nu(f) = F$. Now we take $R \to^f S \to^g T$, $R \to^{\nu f} S \to^{\nu g} T$. Let be $\nu f = F$ and $\nu g = G$. We want to prove that $\nu(gf) = GF$, i.e.,

(f) $\forall x \in |R|, \forall z \in |T|, T(gf(x), z) = \bigvee_{y \in |S|} (S(f(x), y) \wedge T(g(y), z))$.

Now $T(gf(x), z) \leqslant T(gf(x), gf(x)) = S(f(x), f(x))$: then $T(gf(x), z) = S(f(x), f(x)) \wedge T(gf(x), z)$ and $T(gf(x), z) \leqslant \bigvee_{y \in |S|} (S(f(x), y) \wedge T(g(y), z))$.

On the other hand, given any $y \in |S|$,

$$S(f(x), y) \wedge T(g(y), z) \leqslant T(gf(x), g(y)) \wedge T(g(y), z) \leqslant T(gf(x), z):$$

it follows that

$$\bigvee_{y \in |S|} (S(f(x), y) \wedge T(g(y), z)) \leqslant T(gf(x), z)$$

and (f) is proved. Furthermore, $\nu(id_{|R|}) = R$.

So we have defined a functor ν: $\mathscr{F}(L) \to \tilde{\mathscr{F}}(L)$ which is the identity on the objects.

(III.5.1). PROPERTY. *Functor ν is faithful.*

Given $R \rightrightarrows^f_g S$, we suppose that $F = \nu f = \nu g = G$. Then $\forall x \in |R|$ and $\forall y \in |S|$, $F(x, y) = S(f(x), y) = S(g(x), y) = G(x, y)$, and—from the completeness of S—we obtain $f(x) = g(x)$.

(III.5.2). PROPERTY. *Functor ν is representative.*

Given $R \to^F S$ in $\tilde{\mathscr{F}}(L)$, we recall that—for fixed $x \in |R|$—B: $|S| \to L$, B: $y \longmapsto F(x, y)$ is a type on S. Relation S is complete and so there exists a unique element in S that we call $f(x)$ such that

(g) $\forall y \in |S|, F(x, y) = S(f(x), y)$.

In this way we have defined a function f: $|R| \to |S|$. We shall see now that f is a morphism f: $R \to S$ in $\mathscr{F}(L)$. Given x and x' in $|R|$ we find—as above—$f(x)$ and $f(x')$. We have $\forall y \in |S|$,

$$\begin{aligned} S(f(x), f(x')) &\geqslant S(f(x), y) \wedge S(f(x'), y) \\ &= F(x, y) \wedge F(x', y) \geqslant F(x, y) \wedge R(x, x') \end{aligned}$$

(recall III.1(b1)); then

$$S(f(x), f(x')) \geqslant \bigvee_{y \in |S|} (F(x, y) \wedge R(x, x'))$$

$$= R(x, x') \wedge \bigvee_{y \in |S|} F(x, y) = R(x, x') \wedge R(x, x) = R(x, x').$$

We know that $R(x, x) \geqslant F(x, y)$ $\forall y \in |S|$; thus $\forall y \in |S|$,

$$R(x, x) \geqslant S(f(x), y) \qquad \text{and} \qquad R(x, x) \geqslant S(f(x), f(x)).$$

So f is a morphism. Obviously $\nu f = F$.

We have proved at this point that *$\tilde{\mathscr{F}}(L)$ and $\tilde{\mathscr{H}}(L)$ are equivalent.* But now we see

(III.5.3). PROPERTY. *Functor ν is an isomorphism.*

Indeed, by III.5.2, we can construct an application μ: $\tilde{\mathscr{H}}(L) \to \tilde{\mathscr{F}}(L)$ —which is the identity on the objects—sending F: $R \to S$ into f: $R \to S$. The proof of $\mu(G \circ F) = \mu G \circ \mu F$ can be done as for (f). Furthermore, $\nu \circ \mu = id_{\tilde{\mathscr{H}}(L)}$ and $\mu \circ \nu = id_{\tilde{\mathscr{F}}(L)}$. ∎

(III.6.1). *Remark.* The same ν is obviously a functor ν: $\mathscr{F}(L) \to \mathscr{H}(L)$; but it is neither faithful nor representative.

(III.6.2). *Remark.* Category $\mathscr{S}(L)$ is not equivalent to $\tilde{\mathscr{F}}(L)$.

The category SET(L) of L-fuzzy sets has been introduced and studied in [5, 6]. Categorical characterizations of SET(L) can be found [2, 5] (in the context of fibre complete categories, see [8]). We repeat here some definitions.

(III.7). DEFINITION OF SET(L). (a) Objects of SET(L) are pairs (X, A), where X is a set and A is a function $A: X \to L$ (A is an L-fuzzy subset of X).

(b) Morphisms of SET(L) f: $(X, A) \to (Y, B)$ are functions f: $X \to Y$ s.t. $\forall x \in X$, $A(x) \leqslant Bf(x)$.

Another interesting category connected with our study is FUZ(L), defined in [3].

(III.8). DEFINITION OF FUZ(L). (a) $|\text{FUZ}(L)| = |\text{SET}(L)|$;

(b) Morphism of FUZ(L) f: $(X, A) \to (Y, B)$ are functions f: $X \times Y \to L$ s.t.

(b.1) $f(x, y) \leqslant A(x) \wedge B(y)$;

(b.2) $A(x) = \bigvee_{y \in Y} f(x, y)$;

(b.3) $f(x, y) \wedge f(x, y') \leqslant e(y, y')$, where

$$e(y, y') = 0, \quad \text{if} \quad y \neq y',$$
$$= 1, \quad \text{if} \quad y = y'$$

(1 and 0 are, of course, the maximum and minimum of L).

(c) Given $(X, A) \to^f (Y, B) \to^g (Z, C)$, gf: $(X, A) \to (Z, C)$ is defined by $g(x, z) = \bigvee_{y \in Y} f(x, y) \wedge g(y, z)$.

The identity map on (X, A) is $i(x, x') = A(x) \wedge e(x, x')$.

We have

(III.9). PROPERTY [3]. *The category* SET(L) *is a nonfull subcategory of* FUZ(L).

(III.10). PROPERTY [3]. *The category* FUZ(L) *is equivalent to* $\mathscr{W}(L)$.

(III.11). COROLLARY. *The category* FUZ(L) *is equivalent to* $\tilde{\mathscr{F}}(L)$.

IV. CONCLUDING REMARKS

From III.5 and III.11 it is clear that $\tilde{\mathscr{F}}(L)$ can be considered as a model for Heyting-algebra-valued set theory. The category $\tilde{\mathscr{F}}(L)$ has—with respect to the other ones—the advantage of being a category of structured sets.

For a survey in this area and other connections between topoi, logical categories, and fuzzy-sets seen as "variable sets," the interested reader can consult—apart from the already quoted [1, 3]—[9, 10] and the papers quoted therein. Deeper insights on completions, connections with categories of graphs, and tentative interpretations are given in our following paper "Graphs and Fuzzy Graphs."

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