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Ballots and Plane Trees

JOHN RIORDAN

The Rockefeller University, New York, New York 10021

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The correspondence of certain plane trees and binary sequences reported by D. A. Klarner in [1], and a ballot interpretation of the latter, are used to make an independent evaluation of the number of classes of isomorphic, $(k+1)$-valent, planted plane trees with $kn+2$ points. This provides an interesting multivariable identity for binomial coefficients.

1. The binary sequences in [1] are those sequences $(b_1, ..., b_{kn-k})$ with $b_i = 0$ or 1, such that $b_1 + \cdots + b_{jk} \geqslant j$, $j = 1(1)\overline{n-2}$ and $b_1 + \cdots + b_{kn-k} = n-1$. Write

$$c_j = b_{jk-k+1} + \cdots + b_{jk}, \ j = 1(1)\overline{n-1}$$

and consider the sequences $(c_1, c_2, ..., c_{n-1})$ such that

$$c_1 + \cdots + c_j \geqslant j, j = 1(1)\overline{n-2} \qquad \text{and} \quad c_1 + \cdots + c_{n-1} = n-1.$$

Each element of the sequence is of course either zero or a positive integer. If any c_j has the value i, i of k binary elements are 1 and their positions may be chosen in $\binom{k}{i}$ ways. Each sequence $(c_1, ..., c_{n-1})$ replaces $\binom{k}{c_1} \cdots \binom{k}{c_n}$ binary sequences and the total number of binary sequences is the weighted sum of, for brevity, c-sequences, with the weight of a c-sequence the number of binary sequences it replaces. To illustrate, for $n = 4$, the c-sequences and their weights are

c_1	c_2	c_3	Weights
3	0	0	$\binom{k}{3}$
2	1	0	$\binom{k}{2}\binom{k}{1}$
2	0	1	$\binom{k}{2}\binom{k}{1}$
1	2	0	$\binom{k}{2}\binom{k}{1}$
1	1	1	k^3

The number of binary sequences for $n = 4$ is

$$\binom{k}{3} + 3\binom{k}{1}\binom{k}{2} + k^3 = \tfrac{1}{4}\binom{4k}{3}.$$

Interpreting c_j as the content of (number of objects contained in) cell j, the sequence $(c_1, c_2, ..., c_n)$ is a distribution of n like objects into n cells arrayed on a line, such that the total content of cells 1 to j is at least j. As noticed in my paper [3], this distribution problem is identical with the classification of weak lead ballot lattice paths ending on the diagonal by their horizontal segment sequence. If $D(n; k_1, ..., k_n)$ is the number of such paths ending at (n, n) with k_i horizontal segments of length i, so that $n = k_1 + 2k_2 + \cdots + nk_n$, the enumerator of paths by horizontal segment specification is defined by

$$D_n(x_1, ..., x_n) = \sum D(n; k_1, ..., k_n)\, x_1^{k_1} \cdots x_n^{k_n}$$

with summation over all partitions $1^{k_1} \cdots n^{k_n}$ of n. The evaluation of D_n given, a little disguised, in [3] is with $k = k_1 + k_2 + \cdots k_n$, $n = k_1 + \cdots + nk_n$.

$$D_n(x_1, ..., x_n) = \sum \frac{1}{n+1}\binom{n+1}{k}\frac{k!}{k_1! \cdots k_n!}\, x_1^{k_1} \cdots x_n^{k_n}. \tag{1}$$

In this notation, the number of binary sequences is $D_{n-1}(x_1, ..., x_{n-1})$, with $x_i = \binom{k}{i}$ and since this number is also $(1/n)\binom{kn}{n-1}$, the following identity appears

$$\frac{1}{n+1}\binom{kn+k}{n} = \sum \frac{1}{n+1}\binom{n+1}{k}\frac{k!}{k_1! \cdots k_n!}\binom{k}{1}^{k_1} \cdots \binom{k}{n}^{k_n}. \tag{2}$$

2. To determine* the enumerator $D_n \equiv D_n(x_1, ..., x_n)$, classify the paths by the first contact with the diagonal. Write $D_n^* \equiv D_n^*(x_1, ..., x_n)$ for the enumerator of paths with first diagonal contact at lattice point (n, n), the strict lead ballots. Then

$$D_n = \sum_{k=1}^{n} D_k^* D_{n-k} \quad (D_0 = 1). \tag{3}$$

* I owe the form of the derivation below to correspondence with Colin Mallows, my former colleague at Bell Telephone Laboratories, on what seemed at first, to both of us, a non-ballot problem.

To determine $D_n{}^*$, classify the weak lead ballots by length of initial segment, that is, by

$$D_n = \sum_{k=1}^{n} x_k D(n; k).$$

Then, by the obvious relation of weak lead and strict lead ballots,

$$D_n{}^* = \sum_{k=1}^{n+1} x_{k+1} D(n-1, k), \qquad n = 2, 3, \ldots, \tag{4}$$

while $D_1{}^* = x_1 = D_1$. By (3) and (4)

$$D_n = x_1 D_{n-1} + x_2 \sum_{j=1}^{n-1} D(j, 1)\, D_{n-j-1} + \cdots + x_{k+1} \sum_{j=k}^{n-1} D(j, k)\, D_{n-j-1}$$

$$+ \cdots + x_n D(n-1, n-1). \tag{5}$$

Hence $D(n; 1) = D_{n-1}$, $D(n; 2) = D_{n-2} + D_{n-3}D_1 + \cdots + D_0 D_{n-2} = D_{n-2}(2)$, the convolution of the sequence $(D_0, \ldots, D_{n-2})$ with itself. By induction it is found that $D(n; k) = D_{n-k}(k)$, a k-th convolution. Thus (5) implies

$$D_n = \sum_{k=1}^{n} x_k D_{n-k}(k), \tag{6}$$

and, if $D = D(x, y) = \sum y^n D_n(x_1, \ldots, x_n)$, it follows that

$$D = 1 + x_1 yD + x_2 (yD)^2 + \cdots + x_k (yD)^k + \cdots, \tag{7}$$

which effectively determines the D_k.

3. To find Eq. (1), write $\delta = yD$, so that (7) may be rewritten

$$\begin{aligned} y &= \delta(1 + x_1\delta + x_2\delta^2 + \cdots)^{-1} \\ &= \delta(1 - A_1\delta - A_2\delta^2 \cdots), \end{aligned} \tag{8}$$

with

$$A_n \equiv A_n(x_1, \ldots, x_n) = \sum (-1)^{k+1} \frac{k!}{k_1! \cdots k_n!} x_1^{k_1} \cdots x_n^{k_n},$$

which follows from the relation of elementary symmetric functions with homogeneous product sums (cf. [4, p. 188]). Since

$$\delta = y(1 + D_1 y + D_2 y^2 + \cdots)$$

it follows from the formula for reversion of series ([4, p. 149]) that

$$D_n = \sum \frac{1}{n+1} \binom{n+k}{k} \frac{k!}{k_1! \cdots k_n!} A_1^{k_1} \cdots A_n^{k_n}. \tag{9}$$

Setting $x = x_1 = x_2 = \cdots$, and noting that $A_n(x) = x(1-x)^{n-1}$,

$$\begin{aligned} D_n(x,\ldots,x) = d_n(x) &= \sum_{k=1}^{n} \frac{1}{n+1} \binom{n+k}{k} \binom{n-1}{k-1} x^k (1-x)^{n-k} \\ &= (1-x)^n H_n(x(1-x)^{-1}), \end{aligned} \tag{10}$$

with

$$H_n(x) = \sum_{k=1}^{n} \frac{1}{n+1} \binom{n+k}{k} \binom{n-1}{k-1} x^k.$$

From (10) it follows that

$$d_n(x) = \sum \frac{1}{n+1} \binom{n+1}{k} \binom{n-1}{k-1} x^k,$$

which is (1) with $x = x_1 = x_2 = \cdots$.

4. Returning to plane trees, write $x_i = \binom{k}{i}$ in (7) so that

$$\begin{aligned} D &= 1 + \binom{k}{1} yD + \cdots + \binom{k}{i} (yD)^i + \cdots + \binom{k}{k} (yD)^k \\ &= (1 + yD)^k \end{aligned} \tag{11}$$

or if $w = 1 + yD$

$$1 - w + yw^k = 0, \tag{11a}$$

the solution of which (cf. [2]) is

$$w = 1 + yD = 1 + \sum_{n=1} \frac{1}{n} \binom{kn}{n-1} y^n, \tag{12}$$

which proves the identity (2).

References

1. D. A. Klarner, Correspondences between plane trees and binary sequences, *J. Combinatorial Theory* **9** (1970), 401–411.
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