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Blowup of solutions for improved Boussinesq type equation ☆

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Abstract

The paper studies the existence and uniqueness of local solutions and the blowup of solutions to the initial boundary value problem for improved Boussinesq type equation $u_{tt} - u_{xx} - u_{xxtt} = \sigma(u)_{xx}$. By a Galerkin approximation scheme combined with the continuation of solutions step by step and the Fourier transform method, it proves that under rather mild conditions on initial data, the above-mentioned problem admits a unique generalized solution $u \in W^{2,\infty}([0,T]; H^2(0,1))$ as long as $\sigma \in C^2(\mathbf{R})$. In particular, when $\sigma(s) = as^p$, where $a \neq 0$ is a real number and $p > 1$ is an integer, specially $a < 0$ if p is an odd number, the solution blows up in finite time. Moreover, two examples of blowup are obtained numerically.

Keywords: Local solution; Blowup of solutions; Initial boundary value problem; Improved Boussinesq equation

1. Introduction

We consider the following initial boundary value problem (IBVP) of the improved Boussinesq type equation

$$u_{tt} - u_{xx} - u_{xxtt} = \sigma(u)_{xx} \quad \text{on } (0,1) \times (0,\infty), \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \tag{1.2}$$

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$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\quad 0\leqslant x\leqslant 1, \tag{1.3}$$

where $\sigma(s)$ is a given nonlinear function. Equations of type (1.1) are a class of essential model equations appearing in physics and fluid mechanics. Especially when $\sigma(s)=s^2$, Eq. (1.1) becomes the improved Boussinesq (IBq) equation

$$u_{tt}-u_{xx}-u_{xxtt}=(u^2)_{xx}, \tag{1.4}$$

which can be obtained from the exact hydrodynamical set of equations and is used to describe wave propagation at right angles to the magnetic field, and also to approach the "bad" Boussinesq equation

$$u_{tt}-u_{xx}-u_{xxxx}=(u^2)_{xx}, \tag{1.5}$$

see Makhankov [8]. Eq. (1.5) is a well-known model equation derived by Boussinesq in 1872 to describe shallow-water waves, see [1,2]. And it also arises in a large range of physical phenomena including the propagation of ion-sound waves in a plasma and nonlinear lattice waves, see [3,6,8]. The study of the Boussinesq equation has recently attracted considerable attention of many mathematicians and physicists, see [1,3–5,7,8]. Especially, Levine and Sleeman [7] studied in detail the initial value Dirichlet problem for the equation of type (1.5) and proved the nonexistence of global positive solutions both weak and classical for a general class of initial data. When $\sigma(s)=as^3$, where and in the sequel a $(\neq 0)$ is a real number, Eq. (1.1) becomes the modification of the improved Boussinesq (IMBq) equation

$$u_{tt}-u_{xx}-u_{xxtt}=a(u^3)_{xx}, \tag{1.6}$$

which is used to study the properties of anharmonic lattice and the propagation of nonlinear Alfvén waves, see [8]. When the boundary condition (1.2) is substituted by

$$u(0,t)=u(1,t)=0,\quad t>0, \tag{1.7}$$

the author studied the existence and nonexistence of global solutions for problem (1.1), (1.7), (1.3) and especially obtained the global existence and uniqueness of generalized solution for IBVP (1.7), (1.3) of IMBq equation (1.6), with $a>0$, and the nonexistence of global generalized solutions for IBVP (1.7), (1.3) of IBq equation (1.4), see [13].

In this paper, by a Galerkin approximation scheme combined with the continuation of solutions step by step and the Fourier transform method, which are completely different from those used in [13], we first investigate the existence and uniqueness of generalized solution of problem (1.1)–(1.3). Second, for $\sigma(s)=as^p$, Eq. (1.1) becomes

$$u_{tt}-u_{xx}-u_{xxtt}=a(u^p)_{xx}, \tag{1.8}$$

where and in the sequel p (>1) is an integer and specially $a<0$ if p is an odd number, and we prove that the solution of problem (1.8), (1.2), (1.3) blows up in finite time under appropriate conditions on initial data. Moreover, for $p=2$ and $p=3$, by virtue of the ordinary difference scheme, two examples of blowup are obtained numerically.

The plan of the paper is as follows. The main results and some notations are stated in Section 2. The existence and uniqueness of solution of problem (1.1)–(1.3) are discussed in detail in Section 3. Two blowup theorems are proved and two numerical examples are given in Section 4.

2. Statement of main results

We first introduce the following abbreviations

$$L_p = L_p(0,1), \qquad H^k = H^k(0,1), \qquad \|\cdot\|_p = \|\cdot\|_{L^p}, \qquad \|\cdot\| = \|\cdot\|_{L_2},$$

where $1 \leqslant p \leqslant +\infty$, $k = 1, 2, \ldots$. The notation $(\cdot,\cdot)$ for the L_2-inner product will also be used for the notation of duality pairing between dual spaces. Define the Fourier transform $\hat{}: L_2 \to l^2$, for any $f \in L_2$, $\hat{f}(k) = 2\int_0^1 f(x)\cos k\pi x\,dx = f_k$, $k = 0, 1, \ldots$. Obviously $f_{-k} = f_k$ $(k = 0, 1, \ldots)$. Let $\tilde{f} = (f_0, \ldots, f_k, \ldots)$, then $\tilde{f} \in l^2$, $f(x) = f_0/2 + \sum_{k=1}^{\infty} f_k \cos k\pi x$ and $2\|f\|^2 = f_0^2/2 + \sum_{k=1}^{\infty} f_k^2$. The notation $\tilde{f} \geqslant 0$ (> 0) denotes $f_k \geqslant 0$ (> 0), $k = 0, 1, \ldots$, and a similar notation is used for $\tilde{f} \leqslant 0$ (< 0).

The above mentioned Fourier transform has the following properties.

Lemma 2.1 [12]. (I) *If $f \in H^1$, $f'(0) = f'(1) = 0$, then $(f'')_k = -(k\pi)^2 f_k$, $k = 0, 1, \ldots$.*

(II) *If $f^1, f^2, \ldots, f^p \in H^1$, then*

$$(f^1 \ldots f^p)_k = 2^{1-p} \sum_{r_1+\cdots+r_p=k} f^1_{r_1} \cdots f^p_{r_p},$$

where r_i $(i = 1, \ldots, p)$ are integers.

Let $A = \{v(x) \mid v \in H^2,\ v'(0) = v'(1) = 0\}$, then A is a Hilbert space under the norm $\|v\|_A = \|v\|_{H^2} = (\|v\|^2 + \|v_{xx}\|^2)^{1/2}$. The sequence $\{e_0 = 1/2,\ e_k = \cos k\pi x\}_{k=1}^{\infty}$ is an orthogonal basis in L_2 and at the same time in A. For any $v \in A$, $v = \sum_{k=0}^{\infty} v_k e_k$ in A, where $v_k = \hat{v}(k)$, and the corresponding $\tilde{v} = (v_0, v_1, \ldots, v_k, \ldots)$. Let $\tilde{A} = \{\tilde{v} \mid \tilde{v} \in l^2, (0, \pi^2 v_1, \ldots, (k\pi)^2 v_k, \ldots) \in l^2\}$ and $\tilde{A}$ be equipped with the norm

$$\|\tilde{v}\|_{\tilde{A}} = \|v\|_A = \left[\frac{1}{2}\left(v_0^2/2 + \sum_{k=1}^{\infty}\bigl(1 + (k\pi)^4\bigr)v_k^2\right)\right]^{1/2},$$

then, $\tilde{A}$ and A are isometrically isomorphic, so $\tilde{A}$ is also a Hilbert space. Let X^m and A^m be respectively the subspaces spanned by $\{e_0, e_1, \ldots, e_m\}$ in L_2 and in A, the operator $P_m : L_2 \to X^m$ be an orthogonal projection, i.e., for any $f \in L_2$, $P_m f = f^m = \sum_{k=0}^{m} f_k e_k$. Let $\tilde{f}^m = (f_0, \ldots, f_m, 0, \ldots)$, $\widetilde{X}^m = \{\tilde{f}^m \mid f \in L_2\}$, $\|\tilde{f}^m\|_{\widetilde{X}^m} = \|f^m\|_{X^m} = \|f^m\|$, and a similar notation is used for $\tilde{f}^m \geqslant 0$ (> 0) and $\tilde{f}^m \leqslant 0$ (< 0).

Now, we state the main results of the paper.

Theorem 2.1. *Assume that $\sigma \in C^2(\mathbf{R})$, $\varphi, \psi \in A$. Then problem* (1.1)–(1.3) *admits a unique generalized solution $u \in W^{2,\infty}([0,T]; A)$, where $0 < T < T^0$ and $[0, T^0)$ is the maximal time interval of existence of u. Moreover, if $\sup_{0 \leqslant t < T^0} \|u(t)\|_A < +\infty$, then $T^0 = +\infty$.*

For problem (1.8), (1.2), (1.3), we have the following blowup theorems.

Theorem 2.2. *Assume that*

(i) $\varphi \in A$, $\psi \in A$, $\tilde{\varphi} \leqslant 0$, $\tilde{\psi} \leqslant 0$, *specially* $\varphi_0 \leqslant -2(|a|p)^{-1/(p-1)}$;

(ii) *one of the following conditions holds*:

(H$_1$) $a > 0$, p $(\geqslant 4)$ *is an even number*, $\varphi_p < 0$ *and* $\psi_1 < 0$.

(H$_2$) $a > 0$, $p = 2$, $\varphi_1 \leqslant \frac{-1}{a\pi^2}\big[\frac{(1+\pi^2)(1+4\pi^2)}{8}\big]^{1/2}$, $\varphi_2 \leqslant -\big[\frac{2(1+\pi^2)^3}{1+4\pi^2}\big]^{1/4}\big(\frac{-\varphi_1}{a\pi^2}\big)^{1/2}$, $\psi_2 \leqslant \big[\frac{1+\pi^2}{1+4\pi^2}\big]^{1/2}\psi_1 < 0$.

(H$_3$) $a < 0$, p $(\geqslant 3)$ *is an odd number*, $\psi_1 < 0$.

Then the solution u of problem (1.8), (1.2), (1.3), *which exists on* $[0, T^0)$ *as Theorem* 2.1, *blows up in finite time* $\widetilde{T}$, *i.e.*,

$$u(0,t) \to -\infty, \quad \|u(t)\| \to +\infty \quad \text{as } t \to \widetilde{T}^-,$$

where and in the sequel $\widetilde{T}$ *is different for different problems.*

Theorem 2.3. *Assume that*

(i) $a < 0$, $\varphi \in A$, $\psi \in A$, $\tilde{\varphi} \geqslant 0$, $\tilde{\psi} \geqslant 0$, *specially* $\varphi_0 \geqslant 2(-ap)^{-1/(p-1)}$;

(ii) *one of the following conditions holds*:

(H$_4$) p $(\geqslant 4)$ *is an even number*, $\varphi_p > 0$ *and* $\psi_1 > 0$.

(H$_5$) $p = 2$, $\varphi_1 \geqslant \frac{-1}{a\pi^2}\big[\frac{(1+\pi^2)(1+4\pi^2)}{8}\big]^{1/2}$, $\varphi_2 \geqslant \big[\frac{2(1+\pi^2)^3}{1+4\pi^2}\big]^{1/4}\big(\frac{\varphi_1}{-a\pi^2}\big)^{1/2}$, $\psi_2 \geqslant \big[\frac{1+\pi^2}{1+4\pi^2}\big]^{1/2}\psi_1 > 0$.

(H$_6$) p $(\geqslant 3)$ *is an odd number*, $\psi_1 > 0$.

Then the solution u of problem (1.8), (1.2), (1.3), *which exists on* $[0, T^0)$ *as Theorem* 2.1, *blows up in finite time* $\widetilde{T}$, *i.e.*,

$$u(0,t) \to +\infty, \quad \|u(t)\| \to +\infty \quad \text{as } t \to \widetilde{T}^-.$$

3. Local existence of solutions

Proof of Theorem 2.1. We give the proof of Theorem 2.1 by five steps.

Step 1. The Galerkin approximation. We look for approximate solutions of problem (1.1)–(1.3) of the form

$$u^m(t) = \sum_{k=0}^{m} u_k^m(t) e_k,$$

where $\tilde{u}^m(t) = (u_0^m(t), \ldots, u_m^m(t), 0, \ldots)$ satisfy

$$\ddot{\tilde{u}}^m(t) = \tilde{f}\big(u^m(t)\big), \quad t > 0, \tag{3.1}$$

$$\tilde{u}^m(0) = \tilde{\varphi}^m \to \tilde{\varphi} \quad \text{in } \tilde{A}, \qquad \dot{\tilde{u}}^m(0) = \tilde{\psi}^m \to \tilde{\psi} \quad \text{in } \tilde{A}, \tag{3.2}$$

and $\cdot = d/dt$, $\tilde{\varphi}^m = (\varphi_0, \ldots, \varphi_m, 0, \ldots)$, $\tilde{\psi}^m = (\psi_0, \ldots, \psi_m, 0, \ldots)$, $\tilde{f}(u^m(t)) = f(u^m(t))_k = -k^2\pi^2/(1+k^2\pi^2)(u_k^m(t) + \sigma(u^m(t))_k)$, $k = 0, 1, \ldots, m$. By the Lipschitz continuity of $\sigma(s)$ and the Sobolev embedding theorem, for any $u, v \in A^m \subset X^m$,

$$\|\tilde{f}(u) - \tilde{f}(v)\|_{\widetilde{X}^m} \leqslant L\|u - v\|_{X^m} \leqslant L\|u - v\|_{A^m}, \tag{3.3}$$

where L is a local Lipschitz constant. Since $\widetilde{X}^m$ and $\mathbf{R}^{m+1}$ are isometrically isomorphic, it follows from o.d.e.'s theory in $\mathbf{R}^{m+1}$ [9] that for any m, problem (3.1), (3.2) admits a unique noncontinuable solution $\tilde{u}^m(t) = (u_0^m(t), \ldots, u_m^m(t), 0, \ldots)$ defined on the maximal interval J_m.

Step 2. A lemma of continuation of solutions. We consider the following initial value problem of o.d.e.'s

$$\ddot{\tilde{u}}(t) = \tilde{f}\big(u(t)\big), \quad t > 0, \qquad \tilde{u}(0) = \tilde{\varphi}, \qquad \dot{\tilde{u}}(0) = \tilde{\psi}, \tag{3.4}$$

where $\tilde{u}(t) = (u_0(t), \ldots, u_k(t), \ldots)$, $\tilde{f}(u(t)) = (f(u(t))_0, \ldots, f(u(t))_k, \ldots)$, $f(u(t))_k = [-k^2\pi^2/(1+k^2\pi^2)](u_k(t) + \sigma(u(t))_k)$, $k = 0, 1, \ldots$, $\tilde{\varphi} = (\varphi_0, \ldots, \varphi_k, \ldots)$, $\tilde{\psi} = (\psi_0, \ldots, \psi_k, \ldots)$.

Lemma 3.1. *Assume that*

(i) $\sigma \in C^2(\mathbf{R})$, $\varphi \in A$, $\psi \in A$.
(ii) *The solution $\tilde{u}(t)$ of problem* (3.4) *exists on an interval $J = [0, d]$ or $J = [0, d)$ $(d \geqslant 0)$, $\tilde{u}(t) \in \tilde{A}$, $t \in J$; and the corresponding $u \in W^{2,\infty}(J; A)$ is a solution of problem* (1.1)–(1.3) *on J. The solution $\tilde{u}^m(t)$ of problem* (3.1), (3.2) *exists on intervals $[0, d_n] \subset J_m$, $\{d_n\} \subset J$, $d_n \to d$ $(n \to \infty)$ and $\|\tilde{u}^m(t) - \tilde{u}(t)\|_{\tilde{A}} < \theta$, $t \in [0, d_n]$, where θ is a positive constant independent of d_n.*
(iii) *There exists an open sphere $Q \subset \mathbf{R} \times \tilde{A}$ such that the graph of $\tilde{u}(t)$ on J: $G = \{(t, \tilde{u}(t)) \mid t \in J\} \subset Q$, and the distance*

$$\rho(\partial Q, G) = \inf_{(s,\tilde{v})\in\partial Q,\ (t,\tilde{u}(t))\in G} \big\{|s - t| + \|\tilde{v} - \tilde{u}(t)\|_{\tilde{A}}\big\} \geqslant 3\theta,$$

where ∂Q is the boundary of Q.

Then there exists a positive constant d' $(> d)$ and a subsequence of $\{\tilde{u}^m\}$, still denoted by $\{\tilde{u}^m\}$, such that $\tilde{u}^m(t)$ and $\tilde{u}(t)$ are all continued to interval $[0, d']$ and

$$\|\tilde{u}^m(t) - \tilde{u}(t)\|_{\tilde{A}} < 3\theta, \quad t \in [0, d']. \tag{3.5}$$

Moreover,

$$\begin{aligned} &u^m \to u \quad \text{weak* in } W^{2,\infty}\big([0, d']; A\big), \\ &u^m \to u \quad \text{strongly in } C^1\big([0, d']; L^2\big) \end{aligned} \tag{3.6}$$

as $m \to \infty$, and the corresponding limit function $u \in W^{2,\infty}([0, d']; A)$ is a solution of problem (1.1)–(1.3) *on $[0, d']$.*

Proof. For any b_0: $0 \leqslant b_0 < d$ (if $d = 0$, take $b_0 = 0$), since $d_n \to d$ $(n \to \infty)$, without loss of generality we assume that $b_0 < d_n < d$ (if $d = 0$, take $d_n \equiv 0$) for any n. We consider the following initial value problem:

$$\ddot{\tilde{v}}^m(t) = \tilde{f}\big(v^m(t)\big), \quad t > 0, \qquad \tilde{v}^m(0) = \tilde{u}^m(b_0), \qquad \dot{\tilde{v}}^m(0) = \dot{\tilde{u}}^m(b_0). \tag{3.7}$$

We denote the neighborhood of the graph G in $\mathbf{R} \times \tilde{A}$ by

$$G(\delta) = \{(s, \tilde{w}) \in \mathbf{R} \times \tilde{A} \mid \rho((s, \tilde{w}), G) < \delta\},$$

the neighborhoods of $(b_0, \tilde{u}^m(b_0))$ in $\mathbf{R} \times \tilde{A}^m$ and $(b_0, u^m(b_0))$ in $\mathbf{R} \times A^m$ respectively by

$$\tilde{\mu}_m(\theta) = \{(s, \tilde{w}) \in \mathbf{R} \times \tilde{A}^m \mid |s - b_0| + \|\tilde{w} - \tilde{u}^m(b_0)\|_{\tilde{A}} < \theta\},$$
$$\mu_m(\theta) = \{(s, w) \in \mathbf{R} \times A^m \mid |s - b_0| + \|w - u^m(b_0)\|_A < \theta\}. \tag{3.8}$$

It follows from the isometrically isomorphism of $\tilde{A}$ and A that $(s, \tilde{w}) \in \tilde{\mu}_m(\theta)$ if and only if $(s, w) \in \mu_m(\theta)$. For any $(s, \tilde{w}) \in \tilde{\mu}_m(\theta)$, by (3.8) and assumption (ii),

$$\rho((s, \tilde{w}), (b_0, \tilde{u}(b_0))) \leqslant |s - b_0| + \|\tilde{w} - \tilde{u}^m(b_0)\|_{\tilde{A}} + \|\tilde{u}^m(b_0) - \tilde{u}(b_0)\|_{\tilde{A}} < 2\theta,$$

i.e., $\tilde{\mu}_m(\theta) \subset G(2\theta)$. Hence,

$$C_m(\theta) = \sup_{(s,w)\in\mu_m(\theta)} \|\tilde{f}(w)\|_{\tilde{X}^m} \leqslant C(\|w\|_A + 1) \leqslant M_1, \tag{3.9}$$

where and in the sequel C, M_j $(j = 1, 2, \ldots)$ and M denote positive constants depending only on θ. From (3.3) we know that $\tilde{f}(w): A^m \to \widetilde{X}^m$ is Lipschitz continuous on $\mu_m(\theta)$, and thus from the theorem of existence and uniqueness of solution of o.d.e.'s we deduce that for any m, problem (3.7) admits a unique solution $\tilde{v}^m(t)$ on $[0, h] \subset [0, h_m]$ and

$$(t, \tilde{v}^m(t)) \in \tilde{\mu}_m(\theta) \subset G(2\theta), \quad t \in [0, h], \tag{3.10}$$

where $h = \min\{\theta, \theta/M_1\}$, $h_m = \min\{\theta, \theta/C_m(\theta)\}$. Take $b_0 = \max\{0, d - h/2\}$, $d' = b_0 + h$ $(> d)$, and let

$$\tilde{u}^m(t) = \begin{cases} \tilde{u}^m(t), & 0 \leqslant t < b_0, \\ \tilde{v}^m(t - b_0), & b_0 \leqslant t \leqslant d'. \end{cases} \tag{3.11}$$

It follows from the uniqueness of solution of problem (3.1), (3.2) that the solution $\tilde{u}^m(t)$ is continued to $[0, d']$ and for each m

$$(t, \tilde{u}^m(t)) \in G(2\theta), \quad t \in [0, d']. \tag{3.12}$$

For $\tilde{A}$ and A are isometrically isomorphic, by (3.12) and (3.1),

$$\|u^m(t)\|_A \leqslant M, \qquad \|u^m_{tt}(t)\|_A = \|\tilde{f}(u^m(t))\|_{\tilde{A}} \leqslant C\|u^m(t)\|_A \leqslant M,$$
$$\|u^m_t(t)\|_A \leqslant \|\psi^m\|_A + \int_0^t \|\tilde{f}(u^m(\tau))\|_{\tilde{A}}\, d\tau \leqslant M, \quad t \in [0, d']. \tag{3.13}$$

By (3.13), we can choose a subsequence of $\{u^m\}$, still denoted by $\{u^m\}$, such that (3.6) holds. By the Lagrange mean value theorem, (3.13) and (3.6),

$$\|\sigma(u^m(t)) - \sigma(u(t))\| \leqslant M\|u^m(t) - u(t)\| \to 0 \tag{3.14}$$

uniformly on $[0, d']$ as $m \to \infty$. Letting $m \to \infty$ in (3.1), (3.2), we deduce from (3.6) and (3.14) that $\tilde{u}(t)$ is a solution of problem of (3.4) on $[0, d']$.

Rewrite problem (3.4) as

$$\big(u_{tt} - u_{xx} - u_{xxtt} - \sigma(u)_{xx}, e_k\big) = 0 \quad \text{on } (0,\infty),$$
$$\big(u(0), e_k\big) = (\varphi, e_k), \quad \big(u_t(0), e_k\big) = (\psi, e_k), \quad k = 0, 1, \ldots. \tag{3.15}$$

Since $u_{tt} - u_{xx} - u_{xxtt} - \sigma(u)_{xx} \in L_2$, $t \in [0, d']$, and $\{e_k\}_{k=0}^{\infty}$ is dense in L_2 and A,

$$u_{tt} - u_{xx} - u_{xxtt} = \sigma(u)_{xx} \quad \text{in } L_2,\ t \in [0, d'],$$
$$u(0) = \varphi, \quad u_t(0) = \psi \quad \text{in } A,$$

i.e., $u \in W^{2,\infty}([0, d']; A)$ is a generalized solution of problem (1.1)–(1.3) on $[0, d']$.

By the sequential weak* lower semicontinuity of the norm in $L_\infty([b_0, d']; \tilde{A})$ and (3.10),

$$\begin{aligned}\big\|\tilde{u}(t) - \tilde{u}(b_0)\big\|_{\tilde{A}} &\leqslant \big\|\tilde{u}(t) - \tilde{u}(b_0)\big\|_{L_\infty([b_0,d'];\tilde{A})} \\ &\leqslant \lim_{m\to\infty} \inf \big\|\tilde{u}^m(t) - \tilde{u}^m(b_0)\big\|_{L_\infty([b_0,d'];\tilde{A})} \\ &= \lim_{m\to\infty} \inf \big\|\tilde{v}^m(t - b_0) - \tilde{u}^m(b_0)\big\|_{L_\infty([b_0,d'];\tilde{A})} < \theta, \quad t \in [b_0, d'].\end{aligned} \tag{3.16}$$

From (3.16), (3.11), (3.10) and assumption (ii) we deduce that

$$\begin{aligned}\big\|\tilde{u}^m(t) - \tilde{u}(t)\big\|_{\tilde{A}} &\leqslant \big\|\tilde{u}^m(t) - \tilde{u}^m(b_0)\big\|_{\tilde{A}} + \big\|\tilde{u}^m(b_0) - \tilde{u}(b_0)\big\|_{\tilde{A}} + \big\|\tilde{u}(b_0) - \tilde{u}(t)\big\|_{\tilde{A}} \\ &< 3\theta, \quad t \in [b_0, d'],\end{aligned} \tag{3.17}$$

and therefore (3.5) holds. The proof of Lemma 3.1 is completed. □

Step 3. Local existence of solutions. Take $d = 0$, $J = [0, 0]$ (a single point) and $d_n \equiv 0$ $(n = 1, 2, \ldots)$ in Lemma 3.1. Obviously problem (3.4) admits a solution $\tilde{u}(t) = \tilde{\varphi}$ $(\in \tilde{A})$, $t \in J$, satisfying $\dot{\tilde{u}}(0) = \tilde{\psi}$; and the corresponding $u(t) = \varphi$ $(\in A)$ is a solution of problem (1.1)–(1.3) on J, satisfying $u_t(0) = \psi$. For any m, problem (3.1), (3.2) admits a solution $\tilde{u}^m(t) = \tilde{\varphi}^m$, $t \in [0, d_n]$, satisfying $\dot{\tilde{u}}^m(0) = \tilde{\psi}^m$. Since $\|\tilde{\varphi}^m - \tilde{\varphi}\|_{\tilde{A}} \to 0$ $(m \to \infty)$, there exists a positive constant θ such that $\|\tilde{\varphi}^m - \tilde{\varphi}\|_{\tilde{A}} < \theta$. Take a bounded open sphere $Q_1 \subset \mathbf{R} \times \tilde{A}$ such that $(0, \tilde{\varphi}) \in Q_1$ and $\rho(\partial Q_1, (0, \tilde{\varphi})) \geqslant 3\theta$, then the conditions of Lemma 3.1 are satisfied. Therefore, there exists a positive constant b_1 depending only on θ and a subsequence $\{\tilde{u}^{1,m}\} \subset \{\tilde{u}^m\}$ such that $\tilde{u}^{1,m}(t), \tilde{u}(t)$ are all continued onto $[0, b_1]$, (3.5) and (3.6) hold (substituting u^m and d' there by $u^{1,m}$ and b_1 respectively) and the corresponding $u \in W^{2,\infty}([0, b_1]; A)$ is a solution of problem (1.1)–(1.3) on $[0, b_1]$.

Take a sequence $\{b_{1n}\} \subset [0, b_1]$, $b_{1n} \to b_1$ $(n \to \infty)$. By (3.5), for any n, m,

$$\big\|\tilde{u}^{1,m}(t) - \tilde{u}(t)\big\|_{\tilde{A}} < \theta_1\ (= 3\theta), \quad t \in [0, b_{1n}]. \tag{3.18}$$

Take a bounded open sphere $Q_2 \subset \mathbf{R} \times \tilde{A}$ such that the graph of $\tilde{u}(t)$ on $[0, b_1]$: $G_1 = \{(t, \tilde{u}(t)) \mid t \in [0, b_1]\} \subset Q_2$ and $\rho(\partial Q_2, G_1) \geqslant 3\theta_1$. Hence, by Lemma 3.1, there exists a positive constant b_2 $(> b_1)$ and a subsequence $\{\tilde{u}^{2,m}\} \subset \{\tilde{u}^{1,m}\}$ such that $\tilde{u}^{2,m}(t)$ and $\tilde{u}(t)$ are all continued onto $[0, b_2]$, (3.5) and (3.6) hold (substituting u^m, θ and d' there by $u^{2,m}$, θ_1 and b_2 respectively), and the corresponding $u \in W^{2,\infty}([0, b_2]; A)$ is a solution of problem (1.1)–(1.3) on $[0, b_2]$.

Repeating above process, we get a series of bounded open spheres Q_n: $Q_1 \subset Q_2 \subset \cdots \subset Q_n \subset \cdots$, the radius of Q_n tends to infinity as $n \to \infty$, a monotonically increasing sequence $\{b_n\}$ and a subsequence $\{\tilde{u}^{n,m}\}$: $\{\tilde{u}^{n,m}\} \subset \{\tilde{u}^{n-1,m}\} \subset \cdots \subset \{\tilde{u}^m\}$ such that $\tilde{u}^{n,m}(t)$ and $\tilde{u}(t)$ are all continued onto $[0, b_n]$, (3.5) and (3.6) hold (substituting u^m, θ and d' there by $u^{n,m}$, θ_n $(= 3^n\theta)$ and b_n respectively), and the corresponding $u \in W^{2,\infty}([0, b_n]; A)$ is a solution of problem (1.1)–(1.3) on $[0, b_n]$, where b_n are positive constants depending only on θ and n. Since $\{b_n\}$ is monotonically increasing, $\lim_{n\to\infty} b_n = T^0 \leqslant \infty$. By the standard diagonal process, we can choose a diagonal sequence $\{\tilde{u}^{m,m}\}$ such that for any compact subinterval $[0, T] \subset J^0 = [0, T^0)$, $\lim_{m\to\infty} \inf J_{mm} \supset [0, T]$ and

$$\begin{aligned} &u^{m,m} \to u \quad \text{weak}^* \text{ in } W^{2,\infty}\big([0, T]; A\big),\\ &u^{m,m} \to u \quad \text{strongly in } C^1\big([0, T]; L_2\big) \end{aligned} \tag{3.19}$$

as $m \to \infty$ and $u \in W^{2,\infty}([0, T]; A)$ is a solution of problem (1.1)–(1.3) on $[0, T]$.

Step 4. $J^0 = [0, T^0)$ *is the maximal interval of existence of* $\tilde{u}(t)$*, and thus is that of* $u(t)$*.* If $T^0 = +\infty$, obviously the claim is valid.

If $T^0 < +\infty$, while $\tilde{u}(t)$ could be continued past to the right of T^0, then

$$\sup_{0\leqslant t<T^0} \big\|\tilde{u}(t)\big\|_{\tilde{A}} = \sup_{0\leqslant t<T^0} \big\|u(t)\big\|_A < +\infty. \tag{3.20}$$

Take a sequence of number $\{d_n\} \subset [0, T^0)$, $d_n \to T^0$ $(n \to \infty)$, then there must be a positive constant ν such that when m is sufficiently large, for any n

$$\big\|\tilde{u}^{m,m}(t) - \tilde{u}(t)\big\|_{\tilde{A}} < \nu, \quad t \in [0, d_n]. \tag{3.21}$$

In fact, since $A = A^*$ (the dual space of A), for any $\eta \in A$, $\|\eta\|_A = 1$, we deduce from (3.19) that $(u^{m,m}(t), \eta) \to (u(t), \eta)$ $(m \to \infty)$, $t \in [0, T^0)$. Hence when m is sufficiently large,

$$\begin{aligned} &\big|\big(u^{m,m}(t), \eta\big)\big| \leqslant \big|\big(u(t), \eta\big)\big| + 1 \leqslant \big\|u(t)\big\|_A + 1, \quad t \in [0, T^0),\\ &\sup_{0\leqslant t<T^0} \big\|\tilde{u}^{m,m}(t) - \tilde{u}(t)\big\|_{\tilde{A}} \leqslant \sup_{0\leqslant t<T^0} \big\|\tilde{u}^{m,m}(t)\big\|_{\tilde{A}} + \sup_{0\leqslant t<T^0} \big\|\tilde{u}(t)\big\|_{\tilde{A}}\\ &\qquad \leqslant 2 \sup_{0\leqslant t<T^0} \big\|\tilde{u}(t)\big\|_{\tilde{A}} + 1 < \nu. \end{aligned} \tag{3.22}$$

Therefore (3.21) holds.

By (3.21), we can choose a bounded open sphere Q_{n_0} from the above-mentioned open sphere sequence such that the graph of $\tilde{u}(t)$ over J^0: $G_{T^0} = \{(t, \tilde{u}(t)) \mid t \in J^0\} \subset Q_{n_0}$ and $\rho(\partial Q_{n_0}, G_{T_0}) \geqslant 3\nu$. Therefore, we deduce from Lemma 3.1 that there is a positive constant b_{n_0} $(> T^0)$ and a subsequence of $\{\tilde{u}^{m,m}\}$, still denoted by $\{\tilde{u}^{m,m}\}$, such that $\tilde{u}^{m,m}(t)$ and $\tilde{u}(t)$ are all continued onto $[0, b_{n_0}]$, (3.5), (3.6) and the other conclusions of Lemma 3.1 hold (substituting u^m, θ and d' there by $u^{m,m}$, ν and b_{n_0}, respectively). This contradicts the fact that $T^0 = \sup\{b_m\}$. Therefore, $J^0 = [0, T^0)$ is the maximal interval of existence of $\tilde{u}(t)$ and $u(t)$.

From above proving process we see that if $\sup_{0\leqslant t<T^0}\|u(t)\|_A<+\infty$, there must be $T^0=+\infty$. In fact, if $T^0<+\infty$, repeating above arguments one gets that there exists a positive constant $b_{n_0}>T^0$ such that $\tilde{u}(t)$ and $u(t)$ are all continued onto $[0,b_{n_0}]$, which contradicts the fact that $[0,T^0)$ is the maximal interval of existence of $\tilde{u}(t)$ and $u(t)$.

Step 5. The uniqueness of solution of problem (1.1)–(1.3). Assume that $u_1,u_2\in W^{2,\infty}([0,T];A)$ $(0<T<T^0)$ are two solutions of problem (1.1)–(1.3). Let $w=u_1-u_2$, then w satisfies

$$w_{tt}-w_{xx}-w_{xxtt}=\sigma(u_1)_{xx}-\sigma(u_2)_{xx}\quad \text{on } (0,1)\times(0,T], \tag{3.23}$$

$$w_x(0,t)=w_x(1,t)=0,\quad 0\leqslant t\leqslant T,$$

$$w(x,0)=0,\quad w_t(x,0)=0,\quad 0\leqslant x\leqslant 1. \tag{3.24}$$

Multiplying (3.23) by w_t, integrating the resulting expression over $(0,t)$, and making use of the Sobolev embedding theorem and the Cauchy inequality gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|w_t(t)\|^2+\|w_x(t)\|^2+\|w_{xt}(t)\|^2\right)&=-\left(\sigma'(u_1)u_{1x}-\sigma'(u_2)u_{2x},w_{xt}\right)\\ &\leqslant\|w_{xt}(t)\|^2+\left\|\sigma'\left(u_1(t)\right)\right\|_\infty^2\|w_x(t)\|^2+\left\|\left(u_2\sigma''(u_1+\delta u_2)\right)(t)\right\|_\infty^2\|w(t)\|^2\\ &\leqslant\|w_{xt}(t)\|^2+C(T)\left(\|w_x(t)\|^2+\|w(t)\|^2\right),\quad 0<t\leqslant T,\end{aligned} \tag{3.25}$$

where $0<\delta<1$, $C(T)$ is a positive constant depending only on T. Applying the Gronwall inequality to (3.25) one gets

$$\|w_t(t)\|=\|w_x(t)\|=\|w_{xt}(t)\|=0,\quad 0\leqslant t<T^0. \tag{3.26}$$

Therefore $w(t)\equiv 0$, i.e., $u_1(t)\equiv u_2(t)$, $t\in[0,T^0)$. Theorem 2.1 is proved. □

4. Blowup of solutions

In order to prove Theorems 2.2 and 2.3, we first give two lemmas.

Lemma 4.1. *Assume that $\eta(t)$ satisfies*

$$\ddot{\eta}(t)+\alpha\eta(t)\geqslant c\eta^r(t),\quad t>0,\qquad \eta(0)=\eta_0,\qquad \dot{\eta}(0)=\eta_1, \tag{4.1}$$

where α,c,r are real numbers, $c>0$, $r>1$, and $\eta_0\geqslant(\alpha/c)^{1/(r-1)}$ if $\alpha>0$, $\eta_0\geqslant 0$ if $\alpha\leqslant 0$, $\eta_1>0$. Then there exists a finite constant $\widetilde{T}$ such that $\eta(t)\to+\infty$ as $t\to\widetilde{T}^-$, where

$$\widetilde{T}=\int_{\eta_0}^{+\infty}\left[2c\left(\eta^{r+1}-\eta_0^{r+1}\right)/(r+1)-\alpha\left(\eta^2-\eta_0^2\right)+\eta_1^2\right]^{-1/2}d\eta<+\infty. \tag{4.2}$$

Proof. By assumptions of Lemma 4.1, we claim that

$$\eta(t)>\eta_0,\quad \dot{\eta}(t)>0,\quad t>0. \tag{4.3}$$

In fact, if there exists a t_0 (> 0) such that $\eta(t) > \eta_0$, $t \in [0, t_0)$, while $\eta(t_0) = \eta_0$, note that $c\eta^{r-1} - \alpha > c\eta_0^{r-1} - \alpha \geqslant 0$, then it follows from (4.1) that

$$\dot{\eta}(t) \geqslant \eta_1 + \int_0^t \eta(\tau)\big(c\eta^{r-1}(\tau) - \alpha\big)\, d\tau > \eta_1 > 0, \quad t \in (0, t_0), \tag{4.4}$$

i.e., $\eta(t)$ is monotonically increasing on $[0, t_0]$, $\eta(t_0) > \eta_0 \geqslant 0$, which contradicts the assumption. Hence $\eta(t) > \eta_0$, $t > 0$. Applying this fact to (4.4) gives $\dot{\eta}(t) > 0$, $t > 0$.

Multiplying inequality in (4.1) by $2\dot{\eta}(t)$ and integrating the resulting expression over $(0, t)$ one gets

$$\dot{\eta}^2(t) \geqslant \frac{2c}{r+1}\big(\eta^{r+1}(t) - \eta_0^{r+1}\big) - \alpha\big(\eta^2(t) - \eta_0^2\big) + \eta_1^2 = h(t), \quad t > 0. \tag{4.5}$$

Since $\dot{h}(t) = 2\eta(t)\dot{\eta}(t)(c\eta^{r-1}(t) - \alpha) \geqslant 2\eta(t)\dot{\eta}(t)(c\eta_0^{r-1} - \alpha) \geqslant 0$, $t \geqslant 0$, $h(t) \geqslant h(0) = \eta_1^2 > 0$, $t \geqslant 0$. Hence (4.5) yields

$$\dot{\eta}(t) \Big/ \left[\frac{2c}{r+1}\big(\eta^{r+1}(t) - \eta_0^{r+1}\big) - \alpha\big(\eta^2(t) - \eta_0^2\big) + \eta_1^2\right]^{1/2} \geqslant 1, \quad t \geqslant 0. \tag{4.6}$$

Integrating (4.6) over $[0, T]$ gives the conclusion of Lemma 4.1. □

Lemma 4.2 [10,11]. *Assume that f is a quasimonotone increasing function on $I \times D(f) \to \mathbf{R}^N$, where $I = [0, T]$ and $D(f)$ is a closed convex set in $\mathbf{R}^N$ containing the set $u^{\geqslant} = \{x \in \mathbf{R}^N \mid x \geqslant u(t)$ for some $t \in I\}$.*

If the functions $u(t), v(t) \in C(I; \mathbf{R}^N)$ satisfy the following conditions:

(a) $u(0) \leqslant v(0)$,
(b) $\dot{u}(t) - f(t, u) \leqslant \dot{v}(t) - f(t, v)$ *for* $t \in I$,
(c) f *is locally Lipschitz continuous on both* t *and* x *in* $I \times D(f)$,

then $u(t) \leqslant v(t)$, $t \in I$.

Proof of Theorem 2.2. Under the assumptions of Theorem 2.2, from Theorem 2.1 we deduce that problem (1.8), (1.2), (1.3) admits a unique generalized solution $u \in W^{2,\infty}([0, T]; A)$, $0 < T < T^0$, and $\tilde{u}(t) = (u_0(t), \ldots, u_k(t), \ldots)$, defined on $[0, T^0)$, is a unique noncontinuable solution of problem (3.4) and

$$\tilde{u}(t) \leqslant 0, \quad \dot{\tilde{u}}(t) \leqslant 0, \quad t \in [0, T^0). \tag{4.7}$$

In fact, we consider the auxiliary problem of (3.1), (3.2),

$$\begin{aligned} &(1 + k^2\pi^2)\ddot{u}_k^m(t) + k^2\pi^2 u_k^m(t) = -k^2\pi^2\sigma\big(u^m(t)\big)_k - \varepsilon, \quad t > 0, \\ &u_k^m(0) = \varphi_k, \quad \dot{u}_k^m(0) = \psi_k, \quad k = 0, 1, \ldots, m, \end{aligned} \tag{4.8}$$

where $\sigma(u^m(t))_k = 2^{1-p} a \sum_{r_1+\cdots+r_p=k} u_{r_1}^m(t) \ldots u_{r_p}^m(t)$ and $\varepsilon > 0$ is a constant. By o.d.e.'s theory in $\mathbf{R}^{m+1}$, for any compact subinterval $J_m^* \subset J_m$, when ε is sufficiently small, the solution $\tilde{u}^m(t; \varepsilon) = (u_0^m(t; \varepsilon), \ldots, u_m^m(t; \varepsilon), 0, \ldots)$ of problem (4.8) exists on J_m^*, and

$$\dot{\tilde{u}}^m(t; \varepsilon) < 0, \quad \tilde{u}^m(t; \varepsilon) < 0, \quad t \in J_m^* \text{ and } t > 0. \tag{4.9}$$

1. In fact, if there is a k_0: $0 \leqslant k_0 \leqslant m$ such that $\dot{u}_{k_0}^m(0;\varepsilon) = \psi_{k_0} = 0$, then taking $k = k_0$, $r_i = k_0$, $r_j = 0$, $j \neq i$ $(i, j = 1, \ldots, p)$ respectively and letting $t \to 0^+$ in (4.8) gives

$$\left(1 + k_0^2\pi^2\right)\ddot{u}_{k_0}^m(0;\varepsilon) \leqslant -k_0^2\pi^2\varphi_{k_0}\left[1 + ap(\varphi_0/2)^{p-1}\right] - \varepsilon \leqslant -\varepsilon < 0. \tag{4.10}$$

From (4.10) and the continuity of $\ddot{\tilde{u}}^m(t,\varepsilon)$ we deduce that there exists a right neighborhood of 0: $(0,\delta)$ such that $\dot{u}_{k_0}^m(t;\varepsilon) < \psi_{k_0} = 0$ and thus $u_{k_0}^m(t;\varepsilon) < \varphi_{k_0} \leqslant 0$, $t \in (0,\delta)$.

2. If there is a k_1: $0 \leqslant k_1 \leqslant m$, $t_0 \in J_m^*$ and $t_0 > 0$ such that $\dot{\tilde{u}}^m(t;\varepsilon) < 0$, $t \in [0, t_0)$, while $\dot{u}_{k_1}^m(t_0;\varepsilon) = 0$, then $\tilde{u}^m(t;\varepsilon) < 0$, $t \in (0, t_0]$. Taking $t = t_0$, $k = k_1$ and $r_i = k_1$, $r_j = 0$, $j \neq i$ $(i, j = 1, \ldots, p)$ in (4.8) gives

$$\begin{aligned}\left(1 + k_1^2\pi^2\right)\ddot{u}_{k_1}^m(t_0;\varepsilon) &\leqslant k_1^2\pi^2 u_{k_1}^m(t_0;\varepsilon)\left[1 + ap\left(u_0^m(t_0;\varepsilon)/2\right)^{p-1}\right] - \varepsilon \\ &\leqslant -k_1^2\pi^2 u_{k_1}^m(t_0;\varepsilon)\left[1 + ap(\varphi_0/2)^{p-1}\right] - \varepsilon < 0.\end{aligned} \tag{4.11}$$

(4.11) implies that there is a left neighborhood of t_0: $(t_0 - \delta, t_0)$ such that $\dot{u}_{k_1}^m(t;\varepsilon) > 0$, $t \in (t_0 - \delta, t_0)$, which contradicts the assumption.

So (4.9) is valid.

By the continuous dependence of solutions of o.d.e.'s for the parameter, letting $\varepsilon \to 0$ in (4.9) gives

$$\dot{\tilde{u}}^m(t) \leqslant 0, \quad \tilde{u}^m(t) \leqslant 0, \quad t \in J_m, \tag{4.12}$$

where $\tilde{u}^m(t)$ is a solution of problem (3.1), (3.2). By the arguments of the proof of Theorem 2.1, we can choose a subsequence $\{\tilde{u}^{m,m}\}$ from $\{\tilde{u}^m\}$ such that $\lim_{m\to\infty}\inf J_{m,m} \supset J^0$, and for any compact subinterval $\tilde{J}^0 \subset J^0$,

$$\|\tilde{u}^{m,m} - \tilde{u}\|_{C^1(\tilde{J}^0;l^2)} \to 0 \tag{4.13}$$

as $m \to \infty$. (4.13) implies that (4.7) holds.

Rewrite problem (3.4) as (where $\sigma(u) = au^p$)

$$(1 + k^2\pi^2)\ddot{u}_k(t) = -k^2\pi^2\left(u_k(t) + 2^{1-p}a\sum_{r_1+\cdots+r_p=k} u_{r_1}(t)\ldots u_{r_p}(t)\right), \quad t > 0, \tag{4.14}$$

$$u_k(0) = \varphi_k, \quad \dot{u}_k(0) = \psi_k, \quad k = 0, 1, \ldots. \tag{4.15}$$

1. If assumption (H_1) holds, then taking $k = 1$, $r_i = p$, $r_j = -1$ and $r_i = 1$, $r_j = 0$, $j \neq i$ $(i, j = 1, \ldots, p)$ respectively in (4.14) gives

$$(1 + \pi^2)\ddot{\eta}(t) + \nu_1\eta(t) \geqslant 2^{1-p}ap\pi^2\eta^{p-1}(t)z(t), \quad t > 0, \tag{4.16}$$

where $\eta(t) = -u_1(t)$ $(\geqslant 0)$, $z(t) = -u_p(t)$ $(\geqslant 0)$, $\nu_1 = \pi^2[1 + (\varphi_0/2)^{p-1}ap] \leqslant 0$, and the fact $u_0(t) = \psi_0 t + \varphi_0 \leqslant \varphi_0$ $(t > 0)$ has been used. Note that $\dot{z}(t) \geqslant 0$ and $z(t) \geqslant -\varphi_p > 0$, it follows from (4.16) that

$$\ddot{\eta}(t) + \nu\eta(t) \geqslant c\eta^{p-1}(t), \quad t > 0, \qquad \eta(0) = -\varphi_1, \qquad \dot{\eta}(0) = -\psi_1, \tag{4.17}$$

where $\nu = \nu_1/(1 + \pi^2)$, $c = -2^{1-p}ap\pi^2\varphi_p/(1 + \pi^2)$. Applying Lemma 4.1 to (4.17) one gets that there exists a finite constant $\widetilde{T}$ such that $\eta(t) \to +\infty$ as $t \to \widetilde{T}^-$, where

$$\widetilde{T} = \int_{-\varphi_1}^{+\infty}\left[2c\left(\eta^p - \varphi_1^p\right)/p - \nu\left(\eta^2 - \varphi_1^2\right) + \psi_1^2\right]^{-1/2} d\eta < +\infty. \tag{4.18}$$

Since $u(0,t) = u_0(t)/2 + \sum_{k=1}^{\infty} u_k(t) \leqslant u_1(t)$, $\|u(t)\|^2 \geqslant u_1^2(t) = \eta^2(t)$,

$$u(0,t) \to -\infty, \quad \|u(t)\| \to +\infty \quad \text{as } t \to \widetilde{T}^-.$$

2. If assumption (H$_2$) holds, then taking $k = 1$, $r_i = 0$, $r_j = 1$ and $r_i = -1$, $r_j = 2$, $j \neq i$ $(i, j = 1, 2)$, after that taking $k = 2$, $r_i = 0$, $r_j = 2$ and $r_i = r_j = 1$, $j \neq i$ $(i, j = 1, 2)$ respectively in (4.14) gives

$$\begin{aligned} &(1+\pi^2)\ddot{\bar{\eta}}(t) + \pi^2(1+a\varphi_0)\bar{\eta}(t) \geqslant a\pi^2\bar{\eta}(t)\bar{z}(t), \\ &(1+4\pi^2)\ddot{\bar{z}}(t) + 4\pi^2(1+a\varphi_0)\bar{z}(t) \geqslant 2a\pi^2\bar{\eta}^2, \quad t > 0, \end{aligned} \tag{4.19}$$

where $\bar{\eta}(t) = -u_1(t)$, $\bar{z}(t) = -u_2(t)$. Let $\bar{\eta}(t) = [2(1+\pi^2)(1+4\pi^2)]^{1/2}\eta^*(t)/2a\pi^2$, $\bar{z}(t) = (1+\pi^2)z^*(t)/a\pi^2$, then we have

$$\begin{aligned} &\ddot{\eta}^*(t) \geqslant \alpha\eta^*(t) + \eta^*(t)z^*(t), \quad \ddot{z}^*(t) \geqslant \beta z^*(t) + \eta^{*2}(t), \quad t > 0, \\ &\eta^*(0) = \eta_0, \qquad \dot{\eta}^*(0) = \eta_1, \qquad z^*(0) = z_0, \qquad \dot{z}^*(0) = z_1, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} &0 \leqslant \alpha = -\pi^2(1+a\varphi_0)/(1+\pi^2) \leqslant \beta = -4\pi^2(1+a\varphi_0)/(1+4\pi^2), \\ &\eta_0 = -2a\pi^2\varphi_1 / \left[2(1+\pi^2)(1+4\pi^2)\right]^{1/2}, \\ &\eta_1 = -2a\pi^2\psi_1 / \left[2(1+\pi^2)(1+4\pi^2)\right]^{1/2}, \\ &z_0 = -a\pi^2\varphi_2/(1+\pi^2), \qquad z_1 = -a\pi^2\psi_2/(1+\pi^2). \end{aligned}$$

We consider the following initial value problem

$$\begin{aligned} &\ddot{\eta}(t) = \alpha\eta(t) + \eta(t)z(t), \quad \ddot{z}(t) = \beta z(t) + \eta^2(t), \quad t > 0, \\ &\eta(0) = \eta_0, \qquad \dot{\eta}(0) = \eta_1, \qquad z(0) = z_0, \qquad \dot{z}(0) = z_1, \end{aligned} \tag{4.21}$$

where $\eta(t) \geqslant \eta_0$, $z(t) \geqslant z_0$, $\dot{\eta}(t) \geqslant 0$, $\dot{z}(t) \geqslant 0$, $t > 0$. Obviously, (4.21) is equivalent to the problem

$$\begin{aligned} &\dot{\eta}(t) = \eta_1 + \int_0^t \big(\eta(\tau)z(\tau) + \alpha\eta(\tau)\big)\,d\tau, \\ &\dot{z}(t) = z_1 + \int_0^t \big(\eta^2(\tau) + \beta z(\tau)\big)\,d\tau, \quad t > 0, \\ &\eta(0) = \eta_0, \qquad z(0) = z_0. \end{aligned} \tag{4.22}$$

By assumption (H$_2$), $z_0 \geqslant \sqrt{\eta_0} \geqslant 1/\sqrt{2}$, $z_1 \geqslant \eta_1/\sqrt{2} > 0$, and thus

$$z(t) \geqslant \sqrt{\eta(t)}, \quad t > 0. \tag{4.23}$$

In fact, if there exists a $t_0 > 0$ such that $z(t) > \sqrt{\eta(t)}$, $t \in (0, t_0)$, while $z(t_0) = \sqrt{\eta(t_0)}$, then it follows from the second equation in (4.22) that

$$2\sqrt{\eta(t_0)}\dot z(t_0) = 2z_1 z(t_0) + 2z(t_0)\int_0^{t_0}\big(\eta^2(\tau)+\beta z(\tau)\big)\,d\tau$$
$$> 2z_1 z_0 + 2\int_0^{t_0}\big(\eta^2(\tau)z(\tau)+\beta\eta(\tau)\big)\,d\tau. \tag{4.24}$$

By (4.24) and the first equation in (4.22),

$$2\sqrt{\eta(t_0)}\dot z(t_0) - \dot\eta(t_0) > 2z_1 z_0 - \eta_1 + \int_0^{t_0}\eta(\tau)z(\tau)\big(2\eta(\tau)-1\big)\,d\tau$$
$$+\int_0^{t_0}(2\beta-\alpha)\eta(\tau)\,d\tau \geqslant 0. \tag{4.25}$$

Therefore, when $t = t_0$,

$$\frac{d}{dt}\big[z(t)-\sqrt{\eta(t)}\big] = \big(2\sqrt{\eta(t)}\dot z(t) - \dot\eta(t)\big)\big/2\sqrt{\eta(t)} > 0. \tag{4.26}$$

(4.26) implies that there is a $\delta > 0$ such that $z(t) - \sqrt{\eta(t)} < z(t_0) - \sqrt{\eta(t_0)} = 0$, $t \in (t_0-\delta, t_0)$, which contradicts the assumption. Hence (4.23) is valid.

Substituting (4.23) into the first equation in (4.21) gives

$$\ddot\eta(t) \geqslant \alpha\eta(t) + \eta^{3/2}(t), \quad t > 0, \qquad \eta(0) = \eta_0, \qquad \dot\eta(0) = \eta_1. \tag{4.27}$$

Applying Lemma 4.1 to (4.27) gives that there exists a finite constant $\widetilde T$ such that $\eta(t) \to +\infty$ and thus $z(t) \to +\infty$ $(t \to \widetilde T^-)$.

Let $X(t) = (\eta(t), z(t), v(t), w(t))^T$, where $v(t) = \dot\eta(t)$, $w(t) = \dot z(t)$, $X_0 = (\eta_0, z_0, \eta_1, z_1)^T$, $f(t,X) = (v(t), w(t), \alpha\eta(t) + \eta(t)z(t), \beta z(t) + \eta^2(t))^T$. Rewrite problem (4.21) as

$$\dot X(t) = f(t,X), \quad t > 0, \qquad X(0) = X_0. \tag{4.28}$$

Rewrite problem (4.20) as

$$\dot X^*(t) \geqslant f(t,X^*), \quad t > 0, \qquad X^*(0) = X_0, \tag{4.29}$$

where $X^*(t) = (\eta^*(t), z^*(t), v^*(t), w^*(t))^T$, $v^*(t) = \dot\eta^*(t)$, $w^*(t) = \dot z^*(t)$. A simple verification shows that for any $T\colon\ 0 < T < T^0$, $f(t,X)\colon I\times D(f) \to \mathbf{R}^4$ is quasimonotone increasing and locally Lipschitz continuous on both t and X in $I\times D(f)$, where $I = [0,T]$ and $D(f) = \{X = (\eta,z,v,w)^T \mid \eta \geqslant \eta_0,\ z \geqslant z_0,\ v \geqslant 0,\ w \geqslant 0\} \subset \mathbf{R}^4$ is a closed convex set and $D(f) \supset X^{\geqslant} = \{Y \in \mathbf{R}^4 \mid Y \geqslant X(t)$ for some $t \in I\}$. So by Lemma 4.2 and the arbitrariness of $T\colon\ 0 < T < T^0$,

$$X^*(t) \geqslant X(t), \quad t \in J^0 = [0, T^0). \tag{4.30}$$

Therefore, $\eta^*(t) \geqslant \eta(t) \to +\infty$, $z^*(t) \geqslant z(t) \to +\infty$ as $t \to \widetilde T^-$, and thus $u(0,t) \to -\infty$ and $\|u(t)\| \to +\infty$ as $t \to \widetilde T^-$.

3. If the assumption (H_3) holds, then in (4.14) taking $k = 1$, taking 1 for $(p+1)/2$ times and -1 for $(p-1)/2$ times respectively in $r_1, \dots, r_p$, and taking $r_i = 1$, $r_j = 0$, $j \neq i$ $(i, j = 1, \dots, p)$ respectively gives

$$(1+\pi^2)\ddot{\eta}(t)+\nu\eta(t)=-2^{1-p}a\pi^2\eta^p(t),\quad t>0,$$
$$\eta(0)=-\varphi_1,\qquad \dot{\eta}(0)=-\psi_1, \tag{4.31}$$

where $\eta(t)=-u_1(t)$, $\nu=\pi^2(1+ap(\varphi_0/2)^{p-1})\leqslant 0$. Applying Lemma 4.1 to problem (4.31) one gets that there exists a finite constant $\widetilde{T}$: $0<\widetilde{T}<+\infty$ such that $\eta(t)\to+\infty$ as $t\to\widetilde{T}^-$ and thus $u(0,t)\to-\infty$ and $\|u(t)\|\to+\infty$ as $t\to\widetilde{T}^-$. Theorem 2.2 is proved. □

Proof of Theorem 2.3. Since $\varphi,\psi\in A$, from Theorem 2.1 we deduce that problem (1.8), (1.2), (1.3) admits a unique generalized solution $u\in W^{2,\infty}([0,T];A)$, $0<T<T^0$.

(i) If assumption (H_4) or (H_5) holds, let $v=-u$, then v satisfies

$$v_{tt}-v_{xx}-v_{xxtt}=-a(v^p)_{xx}\quad \text{on } (0,1)\times(0,T^0), \tag{4.32}$$
$$v_x(0,t)=v_x(1,t)=0,\quad t\in[0,T^0),$$
$$v(x,0)=-\varphi(x),\quad v_t(x,0)=-\psi(x),\quad 0\leqslant x\leqslant 1. \tag{4.33}$$

Applying Theorem 2.2 to problem (4.32), (4.33) gives the conclusion of Theorem 2.3.

(ii) If assumption (H_6) holds, still let $v=-u$, then v satisfies

$$v_{tt}-v_{xx}-v_{xxtt}=a(v^p)_{xx}\quad \text{on } (0,1)\times(0,T^0), \tag{4.34}$$

and conditions (4.33). Applying Theorem 2.2 to problem (4.34), (4.33) gives the conclusion of Theorem 2.3. Theorem 2.3 is proved. □

Example 1. For initial boundary value problem (1.2), (1.3) of IBq equation (1.4) (i.e., $a=1$ and $p=2$ in (1.8)), if we take initial data

$$\varphi(x)=\varphi_0/2+\varphi_1\cos\pi x+\varphi_2\cos 2\pi x,$$
$$\psi(x)=\psi_1\cos\pi x+\psi_2\cos 2\pi x, \tag{4.35}$$

where $\varphi_0\leqslant-1$, $\varphi_1\leqslant-[(1+\pi^2)(1+4\pi^2)]^{1/2}/\sqrt{8}\pi^2$, $\varphi_2\leqslant-[2(1+\pi^2)^3\varphi_1^2/(1+4\pi^2)\pi^4]^{1/4}$, $\psi_2\leqslant[(1+\pi^2)/(1+4\pi^2)]^{1/2}\psi_1<0$, then a simple verification shows that the assumptions of Theorem 2.1 and assumptions (i) and (ii)(H_2) of Theorem 2.2 hold. So, by Theorems 2.1 and 2.2, the corresponding problem (1.4), (1.2), (1.3) admits a unique generalized solution $u\in W^{2,\infty}([0,T];A)$, $0<T<T^0$, and there exists a finite constant $\widetilde{T}$ such that

$$u(0,t)\to-\infty,\quad \big\|u(t)\big\|\to+\infty\quad \text{as } t\to\widetilde{T}^-. \tag{4.36}$$

Now we give a numerical experiment to demonstrate the correctness of Example 1. Let $\varphi_0=-2000$, $\varphi_1=\varphi_2=\psi_1=\psi_2=-1000$ in (4.35), and rewrite Eq. (1.4) as

$$v_t-u_{xx}-v_{xxt}=(u^2)_{xx},\qquad u_t=v. \tag{4.37}$$

Let $t=j\tau$, where j is a nonnegative integer, $\tau=0.002$ is the time step length and $h=0.05$ is the space step length. By the ordinary difference method

$$v_t[i,j] = \frac{v[i,j] - v[i,j-1]}{\tau} + o(\tau), \tag{4.38}$$

$$u_{xx}[i,j] = \frac{u[i-1,j] - 2u[i,j] + u[i+1,j]}{h^2} + o(h^2), \tag{4.39}$$

$$v_{xxt}[i,j] = \frac{v[i-1,j] - v[i-1,j-1] - 2v[i,j] + 2v[i,j-1] + v[i+1,j]}{h^2\tau} - \frac{v[i+1,j-1]}{h^2\tau} + o(h^2\tau), \tag{4.40}$$

$$(u^2)_{xx}[i,j] = \frac{u^2[i-1,j] - 2u^2[i,j] + u^2[i+1,j]}{h^2} + o(h^2), \tag{4.41}$$

$$u_t[i,j] = \frac{u[i,j+1] - u[i,j]}{\tau} + o(\tau), \tag{4.42}$$

we get the following difference scheme:

$$\begin{aligned} &-v[i-1,j] + (2+h^2)v[i,j] - v[i+1,j] \\ &\quad = h^2 v[i,j-1] + \tau\big(u[i-1,j] - 2u[i,j] + u[i-1,j]\big) \\ &\qquad + \tau\big(u^2[i+1,j] - 2u^2[i,j] + u^2[i-1,j]\big) \\ &\qquad + \big(-v[i-1,j-1] + 2v[i,j-1] - v[i+1,j-1]\big), \\ &u[i,j+1] = u[i,j] + \tau v[i,j]. \end{aligned} \tag{4.43}$$

And by the scheme we get the graphs of the numerical solutions of the corresponding problem (1.4), (1.2), (1.3) at $j = 0, 6, 10, 13, 15, 20$ and 30, respectively, which show that the solutions $u(x,t)$ develop a pronounced negative spike gradually at the point $x = 0$ as $t \to \widetilde{T}^-$, see Figs. 1, 2 and 3. And this fact corresponds with (4.36).

Example 2. For initial boundary value problem (1.2), (1.3) of the IMBq equation (1.6), if we take $a = -1$ and initial data

$$\varphi(x) = \varphi_0/2, \qquad \psi(x) = \psi_1 \cos \pi x, \tag{4.44}$$

where $\varphi_0 \geqslant 2/\sqrt{3}$, $\psi_1 > 0$. Obviously $\varphi, \psi \in A$, $\tilde{\varphi} \geqslant 0$ and $\tilde{\psi} \geqslant 0$, i.e., the assumptions of Theorem 2.1 and assumptions (i) and (ii)(H_6) of Theorem 2.3 hold. Therefore,

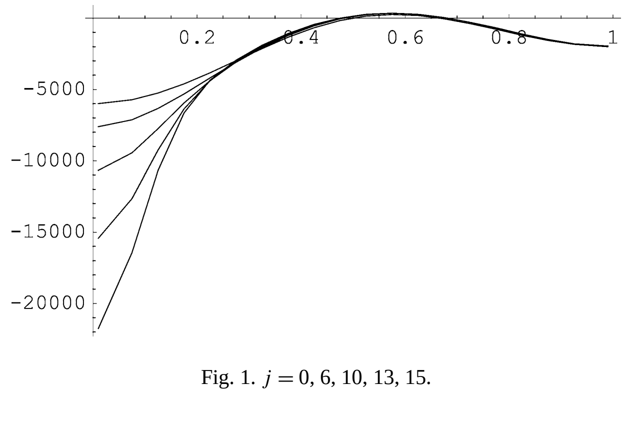

Fig. 1. $j = 0, 6, 10, 13, 15$.

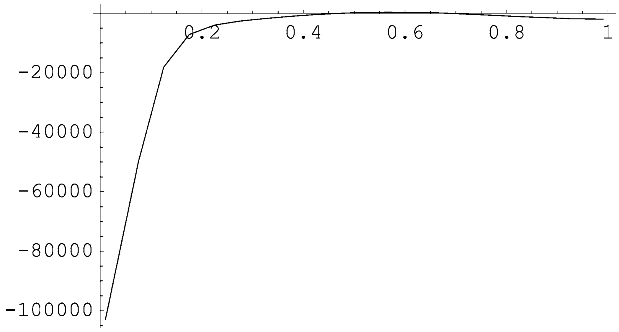

Fig. 2. $j = 20$.

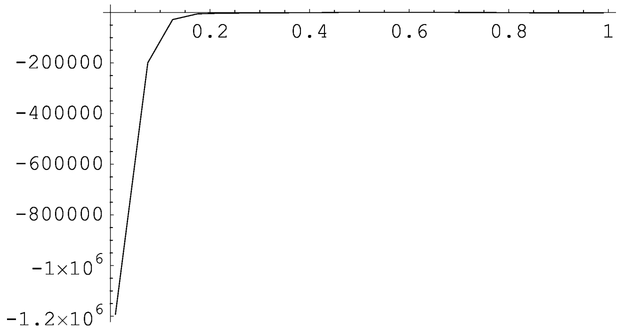

Fig. 3. $j = 30$.

the corresponding problem (1.6), (1.2), (1.3) admits a unique generalized solution $u \in W^{2,\infty}([0,T]; A)$, $0 < T < T^0$, and u blows up in finite time $\widetilde{T}$, i.e.,

$$u(0,t) \to +\infty, \quad \|u(t)\| \to +\infty \quad \text{as } t \to \widetilde{T}^-. \tag{4.45}$$

Similarly, take $\varphi_0 = 2400$, $\psi_1 = 1000$, and rewrite Eq. (1.6) as

$$v_t - u_{xx} - v_{xxt} = -(u^3)_{xx}, \qquad u_t = v. \tag{4.46}$$

Let $t = j\tau$, where $\tau = 0.0005$ and $h = 0.05$ are respectively the time and space step length. By the same difference scheme as shown in (4.38)–(4.40) and

$$(u^3)_{xx}[i,j] = \frac{u^3[i-1,j] - 2u^3[i,j] + u^3[i+1,j]}{h^2} + o(h^2), \tag{4.47}$$

$$u_t[i,j] = \frac{u[i,j+1] - u[i,j-1]}{2\tau} + o(\tau^2), \tag{4.48}$$

$$\begin{aligned} &-v[i-1,j] + (2+h^2)v[i,j] - v[i+1,j] \\ &\quad = h^2 v[i,j-1] + \tau\big(u[i-1,j] - 2u[i,j] + u[i-1,j]\big) \\ &\qquad - \tau\big(u^3[i+1,j] - 2u^3[i,j] + u^3[i-1,j]\big) \\ &\qquad + \big(-v[i-1,j-1] + 2v[i,j-1] - v[i+1,j-1]\big), \\ &u[i,j+1] = u[i,j-1] + 2\tau v[i,j], \end{aligned} \tag{4.49}$$

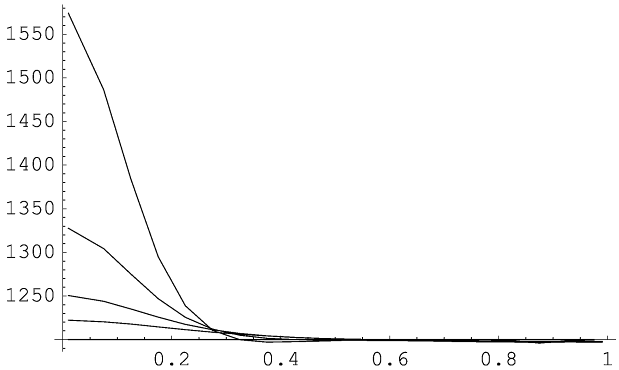

Fig. 4. $j = 0, 5, 6, 7, 8$.

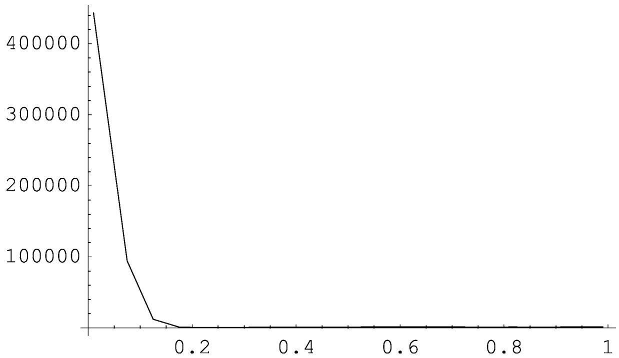

Fig. 5. $j = 11$.

we get the graphs of the numerical solutions of the corresponding problem (1.6), (1.2), (1.3), with $a = -1$, at $j = 0$, 5, 6, 7, 8 and 11, respectively, which show that the solutions $u(x,t)$ develop a pronounced positive spike gradually at $x = 0$ as $t \to \widetilde{T}^-$, see Figs. 4 and 5. And this fact corresponds with (4.45).

Now, we make another experiment to show how the above-mentioned difference scheme works on a non-blowup solution. By the homogeneous balance method, see [14], we easily find a solitary wave solution

$$u(x,t) = \frac{\sqrt{2}}{1+x+t} \tag{4.50}$$

of the IMBq equation (1.6), with $a = -1$, and

$$\begin{aligned} &u_x(0,t) = -\frac{\sqrt{2}}{(1+t)^2}, \qquad u_x(1,t) = -\frac{\sqrt{2}}{(2+t)^2}, \\ &u(x,0) = \frac{\sqrt{2}}{1+x}, \qquad u_t(x,0) = -\frac{\sqrt{2}}{(1+x)^2}. \end{aligned} \tag{4.51}$$

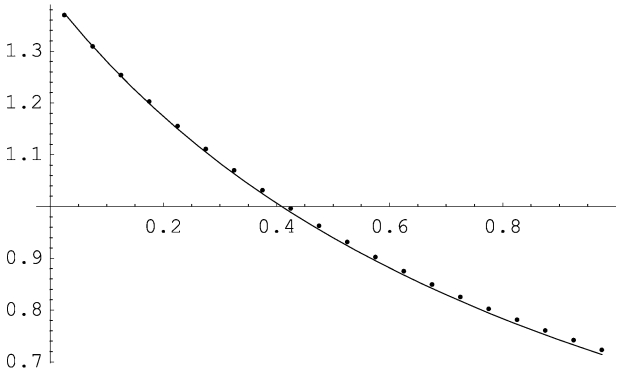

Fig. 6. $j = 10$.

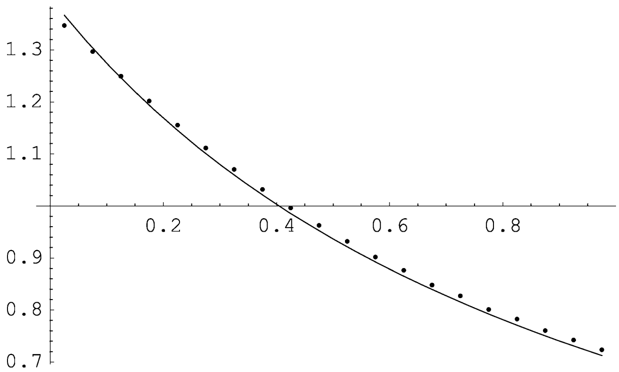

Fig. 7. $j = 20$.

By the same difference scheme as shown in (4.38)–(4.40) and (4.47)–(4.49), we get the graphs of the numerical solutions of problem (1.6), (4.51), with $a = -1$, at $j = 10$ and 20, see Figs. 6 and 7. The comparison of the graphs of the numerical solution with the exact solution of problem (1.6), (4.51), with $a = -1$, shows that the difference scheme is stable at least in time interval $[0, 0.01] \supset [0, 0.0055]$.

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