Cyclic actions and divisible polynomials

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Abstract

Let \( g : M^{2n} \to M^{2n} \) be an orientation preserving \( P L \) map of period \( m > 2 \). Suppose that the cyclic action defined by \( g \) is locally linear \( P L \), fixing a locally flat submanifold \( F \) with components only of dimension 0 or \( 2n - 2 \), and regular. Let \( \phi(m) \) be Euler’s number and \( \rho(m) = \phi(m) - 1 \) if \( m \) is a power of 2 and \( \rho(m) = \phi(m) \) otherwise. If \( \text{Sign}(g, M) \) is a rational integer, then \( \text{Sign}(g, M) \equiv \text{Sign}(F) \mod 2^\rho(m) \). This congruence is used to show that a codimension-2 locally flat submanifold of cohomology complex projective \( n \)-space fixed by \( g \) must have degree one if \( m \neq 4 \) or \( 10 \) and \( n < \phi(m) + 4 \).

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1. Introduction

Let \( M^{2n} \) be a piecewise linear (\( PL \)), closed, oriented \( 2n \)-manifold. Let \( G_m \) denote the cyclic group of order \( m > 2 \) and let \( g : M^{2n} \to M^{2n} \) be a piecewise linear map of period \( m \) which preserves the preferred orientation of \( M^{2n} \). Suppose that the \( G_m \) action defined by \( g \) has as fixed point set a locally flat submanifold \( F \subseteq M^{2n} \) with normal block bundle \( \nu \). We will assume throughout this paper that the \( G_m \) action is locally linear \( PL \) and tame ([15], p. 189) and so \( \nu \) admits a complex bundle structure compatible with the \( g \)-action. The tame hypothesis is automatically fulfilled if the dimensions of the components of \( F \) are either 0 or \( 2n - 2 \) ([5], Theorem 1.1, [9], and [10], p. 254). We will also assume that the orientation of \( \nu \) is the one determined by its complex structure. This orientation and the preferred orientation of \( M^{2n} \) determine an orientation of \( F \).

Let \( \text{Sign}(g, M) \) be the \( g \)-signature of the action [2]. The \( g \)-signature is an algebraic integer, \( \text{Sign}(g, M) \in \mathbb{Z}[^2]\), where \( \lambda = \exp(2\pi i/m) \). If \( \text{Sign}(g, M) \) is a rational integer, \( \text{Sign}(g, M) \in \mathbb{Z} \), then it is related to the signatures of \( F \) and the transverse self-intersection of \( F, F \cap F \), if the action is regular. The action is regular if there is a fixed irreducible representation of \( G_m \) which determines every normal slice type (Definition 2.1). Let \( F_{\text{even}}(F_{\text{odd}}) \) be the union of all components of \( F \) where the restriction of \( \nu \) has even (odd) complex dimension. Let \( \phi \) be Euler’s totient function and put \( \rho(m) = \phi(m) - 1 \) if \( m = 2^e \) and \( \rho(m) = \phi(m) \) if \( m \neq 2^e \).

**Theorem A.** Suppose that \( g : M^{2n} \to M^{2n} \) is an orientation preserving piecewise linear map of period \( m > 2 \). If the \( G_m \) action defined by \( g \) is locally linear \( PL \), the dimensions of the components of \( F \) are either 0 or \( 2n - 2 \), the
action is regular and \( \text{Sign}(g, M) \in \mathbb{Z} \), then \( \text{Sign}(g, M) \equiv \text{Sign} F_{\text{even}} \pmod{2^{\rho(m)}} \) and \( \text{Sign} F_{\text{odd}} \equiv 0 \pmod{2^{\rho(m)}} \) and so

\[
\text{Sign}(g, M) \equiv \text{Sign} F \pmod{2^{\rho(m)}}.
\]

If \( p \) is an odd prime and \( m = 2p^e \), then \( \text{Sign}(g, M) \equiv \text{Sign}(F \cap F) \pmod{p} \).

**Theorem A** is an extension to the PL category of a result for smooth actions with arbitrary fixed point set ([14], Theorem 4) and it is related to results in the literature. For any smooth action with \( \text{Sign}(g, M) \in \mathbb{Z} \), odd implies that \( \text{Sign}(g, M) \equiv \text{Sign} F \pmod{4} \) and if the action is regular, then \( \text{Sign}(g, M) \equiv \text{Sign} F \pmod{2^{\phi(m)}} \) ([1], Theorems 1 and 4). **Theorem A** extends the latter congruence to certain \( PL G_m \) actions, \( m > 2 \). If \( p \) is an odd prime and \( n < p - 1 \), then for smooth regular \( G_p \) actions \( \text{Sign} M \equiv \text{Sign} F \pmod{p} \) ([8], Theorem 2.2). **Theorem A** offers a congruence relating an integral \( g \)-signature to \( \text{Sign}(F \cap F) \) for certain \( PL G_m \) actions with \( m = 2p^e \). If \( n < \phi(m) \) and the action is smooth in **Theorem A** and the fixed point set is arbitrary, then \( \text{Sign}(g, M) = \text{Sign} F_{\text{even}} \) and \( \text{Sign} F_{\text{odd}} = 0 \) and so \( \text{Sign}(g, M) = \text{Sign} F \) ([14], Theorem 1).

Let \( \alpha_{j/m} = (\lambda^j + 1)(\lambda^j - 1)^{-1}, 1 \leq j \leq m - 1 \), and let \( M_m(x) \) be the minimal polynomial of \( \alpha_{j/m} \) over \( \mathbb{Q} \) if \( (j, m) = 1 \). If \( m > 2 \), then \( M_m(x) \in \mathbb{Z}[x^2] \) (Proposition 3.6). It follows that if \( a(x) \in \mathbb{Q}[x] \) and \( a(x) \) is divisible by \( M_m(x) \), then \( a(x)\text{even} \) and \( x^{-1}a(x)\text{odd} \) are both divisible by \( M_m(x) \) where \( a(x)\text{even} \) is the part of \( a(x) \) with even powers of \( x \) and \( a(x)\text{odd} \) is the part of \( a(x) \) with odd powers of \( x \). This observation is at the heart of the first two congruences in **Theorem A** and it suggests that the study of \( a(x) \in \mathbb{Q}[x^2] \) with \( a(x) \) divisible by \( M_m(x) \) might be useful. The degree of \( M_m(x) \) is \( \phi(m) \) (Proposition 3.3) and so we put \( M_m(x) = \sum_{k=0}^{\mu} m_{2k}x^{2k} \) where \( \mu = \phi(m)/2 \), \( m > 2 \).

**Theorem B.** If \( m > 2 \) and \( r \geq \mu \), then there is a set of integers

\[
A^{(r)}(m) = \left\{ A_{i,j}^{(r)}(m) : 1 \leq i \leq \mu, i - 1 \leq j \leq r \right\}
\]

with the property that if \( a(x) = \sum_{k=0}^{r} a_{2k}x^{2k} \in \mathbb{Q}[x^2] \) is divisible by \( M_m(x) \), then

\[
\sum_{j=i-1}^{r} A_{i,j}^{(r)}(m)a_{2j} = 0, \quad 1 \leq i \leq \mu.
\]

These sets have the property that if \( r \geq \mu + 1 \), then the equations

\[
A_{i,j}^{(r)}(m) = \begin{cases} 
-m_{\phi(m)}A_{i,j}^{(r-1)}(m), & i - 1 \leq j < r, \\
\sum_{k=1}^{\mu} m_{\phi(m)-2k}A_{i,r-k}^{(r-1)}(m), & j = r,
\end{cases}
\]

express \( A^{(r)}(m) \) in terms of \( A^{(r-1)}(m) \).

**Theorem B** extends to arbitrary \( m > 2 \) a construction of a system of linear equations for polynomials divisible by \( M_{2s}(x) \) ([13], Lemma 4.14). We will apply **Theorems A and B** to \( G_m \) actions on \( PL \) cohomology complex projective \( n \)-space. A \( PL \) cohomology complex projective \( n \)-space is a piecewise linear, closed, oriented \( 2n \)-manifold \( M^{2n} \) such that there is a class \( x \in H^2(M; \mathbb{Z}) \) with the property that \( H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}) \). If \( M^{2n} \) admits a locally linear \( PL \) \( G_m \) action with fixed point set \( F \), then the action is of Type \( \Pi_0 \) if \( F \) is the union of an isolated point and a codimension-2 locally flat submanifold. Actions of Type \( \Pi_0 \) are tame ([15], Theorem 1.1, [9], and [10], p. 254) and the Atiyah–Singer \( g \)-Signature Formula (ASgSF) expresses \( \text{Sign}(g, M) \) in terms of \( v \) and the eigenvalues of the action ([15], p. 189). If \( M^{2n} \) is a \( PL \) cohomology projective \( n \)-space then \( \text{Sign}(g, M) = \pm \text{Sign} M = \pm 1 \) in \( n \) is even and \( \text{Sign}(g, M) = 0 \) if \( n \) is odd ([7], Formula (11), p. 28), and so Type \( \Pi_0 \) regular \( G_m \) actions on \( PL \) cohomology complex projective \( n \)-space are good candidates for application of **Theorems A and B**.

Suppose that \( M^{2n} \) is a \( PL \) cohomology complex projective \( n \)-space with a locally linear \( PL \) \( G_m \) action of Type \( \Pi_0 \) with \( F \) equal to the union of \( F^{2n-2} \) and an isolated point. We say that the degree of \( F^{2n-2} \) is \( d \in \mathbb{Z} \) if \( i_n[F^{2n-2}] \) is dual to \( dx \) where \( i : F^{2n-2} \subset M^{2n} \) is the inclusion. If \( p \) is a prime and \( p \) divides \( m \), then \( d \neq 0 \pmod{p} \) ([4], pp. 378–383) and so \( d \neq 0 \). A \( G_m \) action of Type \( \Pi_0 \) is standard if it is regular and \( d = \pm 1 \). The Pontrjagin class of \( M^{2n} \), \( p_*(M^{2n}) \subset H^*(M; \mathbb{Q}) \), is standard if \( p_*(M^{2n}) = (1 + x^2)^{n+1} \). The Pontrjagin class of complex projective
Theorems A and B to study these rigidity conditions for regular \( G_m \) actions of Type \( \text{II}_0 \). We will produce divisibility conditions for \( d \) involving the row sums of the integers \( A_{i,j}^{(r)}(m) \) in Theorem B
\[
R_i^{(r)}(m) = \sum_{j=-1}^{r} A_{i,j}^{(r)}(m)
\]
where \( 1 \leq i \leq \mu \). Let \( f(n) \) be \( n! \) divided by a maximal power of 2.

**Theorem C.** Suppose that \( M^{2n} \) is a PL cohomology complex projective \( n \)-space which admits a regular locally linear \( PL \) \( G_m \) action of Type \( \text{II}_0 \) where \( m > 2 \) and \( m \neq 4 \). If \( n < \phi(m) + 2 \), then the action is standard and the Pontrjagin class of \( M^{2n} \) is standard. If \( n \geq \phi(m) + 2 \) and \( d \) is the degree of the fixed codimension-2 locally flat submanifold, then \( d^{2i} \) divides \( f(n)R_i^{[(n/2)−1]}(m) \) if \( n \) is even and \( d^{2i+1} \) divides \( f(n)R_i^{[(n/2)−1]}(m) \) if \( n \) is odd.

Theorem C strengthens an earlier result that if \( p \) is an odd prime and \( n < p + 1 \), then any regular, locally linear \( PL \) \( G_p \) action of Type \( \text{II}_0 \) is standard and the Pontrjagin class of \( M^{2n} \) is standard ([13], Theorem B). If \( p = 3 \) or 5, the range in which any \( G_p \) action of Type \( \text{II}_0 \) must be standard and the Pontrjagin class of \( M^{2n} \) can be improved to \( n < p + 3 \) ([13], Theorem A). We use Theorems A and B to show that this improved range is valid for some regular \( G_m \) actions of Type \( \text{II}_0 \). Note that if \( n = \phi(m) + 2 \) or \( \phi(m) + 3 \), then \( [n/2] − 1 = \mu \) and
\[
R_i^{(n)}(m) = m^{\phi(m)−2} − m^{\phi(m)} \text{ where } M_m(x) = \sum_{k=0}^{\mu} m_{2k}x^{2k} \text{ (Formula (4.11)) and so our next theorem about the prime factorization of } m^{\phi(m)−2} − m^{\phi(m)} \text{ will be useful in view of the divisibility conditions in Theorem C at level } i = \mu. \text{ If } t \text{ is an integer and } p \text{ is a prime, let } \text{ord}_p t \text{ denote the exponent of } p \text{ in the prime factorization of } t.

**Theorem D.** If \( m > 2 \) and \( m \neq 4, 10 \), then \( \text{ord}_2(\phi(m)−2 − m^{\phi(m)}) < \phi(m) \) and if \( p \) is an odd prime, then \( \text{ord}_p(\phi(m)−2 − m^{\phi(m)}) < \phi(m)/2 \).

We will see that the inequalities in Theorem D are sharp and that they imply \( d = \pm 1 \) if \( n = \phi(m) + 2 \) or \( \phi(m) + 3 \) and \( m > 2, m \neq 4, 10 \). This fact plus some further analysis for the case \( m > 2, m \text{ odd, are contained in our final principal result.}

**Theorem E.** Suppose that \( M^{2n} \) is a PL cohomology complex projective \( n \)-space which admits a regular locally linear \( PL \) \( G_m \) action of Type \( \text{II}_0 \). If \( m > 2, m \neq 4, 10 \), and \( n < \phi(m) + 4 \), then the action is standard. If \( m > 2 \) and \( m \text{ odd and } n < \phi(m) + 4 \), then the action is standard and the Pontrjagin class of \( M^{2n} \) is standard.

Theorem E is sharp in the sense that if \( n \geq \phi(m) + 4 \), then the Pontrjagin class of \( M^{2n} \) need not be standard. If \( p \) is an odd prime and \( n \geq p + 3 \), then there are infinitely many \( PL \) homotopy complex projective \( n \)-spaces with nonstandard Pontrjagin classes which admit standard locally linear \( PL \) \( G_p \) actions of Type \( \text{II}_0 \) ([5], Proposition 0.3 (\( \ell, \ell \text{.PL} \))). If \( n \geq 2p + 9 \), then there are infinitely many smooth homotopy complex projective \( n \)-spaces with nonstandard Pontrjagin classes which admit standard smooth \( G_p \) actions of Type \( \text{II}_0 \) ([5], Theorem 0.3 (diff)) All the known examples of actions of Type \( \text{II}_0 \) are standard. A standard action is the only possibility in some cases. If \( n \leq 4 \) and \( M^{2n} \) is a smooth cohomology complex projective \( n \)-space with a smooth \( G_p \) action of Type \( \text{II}_0 \), then the action and the Pontrjagin class of \( M^{2n} \) are standard ([5], Theorem A (i)(ii) \( n \leq 3, p \geq 3, n = 4, p \geq 5 \), [6], Theorem E \( n = 4, p = 3 \)). Every smooth involution of Type \( \text{II}_0 \) on \( CP^n \) is standard ([6], Theorem A) and if \( n \text{ is odd and } p \text{ is an odd prime, then every smooth } G_p \text{ action of Type } \text{II}_0 \text{ on } CP^n \text{ is standard ([12], Theorem B).}

This paper is organized as follows. In Section 2, we review the Atiyah–Singer \( g \)-Signature Formula for smooth \( G_m \) actions as formulated by Berend and Katz ([3], Theorem 2.2) and extend this formula to tame regular actions. Section 3 contains a discussion of the polynomials \( M_m(x) \), the proof of Theorem A (Theorem 3.14), and the proof of the first assertion in Theorem C (Theorem 3.21). In Section 4, we prove Theorem B (Theorem 4.1) and the second assertion on Theorem C (Theorem 4.12). Cotangent sums and Rademacher Reciprocity are used in Section 5 to obtain formulas for \( m^{\phi(m)−2} \) (Formulas (5.9) and (5.12)) and to prove Theorem D (Theorem 5.18). Section 6 contains the proof of Theorem E (Theorem 6.1).

2. The Atiyah–Singer \( g \)-Signature Formula

Suppose that \( M^{2n} \) is a piecewise linear, closed, oriented \( 2n \)-manifold. Let \( g : M^{2n} \to M^{2n} \) be a piecewise linear map of period \( m \). We assume that the \( G_m \) action defined by \( g \) is locally linear \( PL \). Let \( F \) be the locally flat
submanifold fixed by this $G_m$ action and let $v$ be its normal block bundle. We assume that the action is tame ([15], p. 189), and so over each connected component of $F$, $v$ splits into a direct sum of eigebundles $v_j$. Each $v_j$ is a complex bundle of complex dimension $\mu_j$ and $g$ acts on $v_j$ as multiplication by $\lambda^j$ where $\lambda = \exp(2\pi i/m)$ and $1 \leq j \leq m - 1$. This associates with every connected component of $F$ a normal slice type, an $m - 1$-tuple of nonnegative integers $\mu = (\mu_1, \mu_2, \ldots, \mu_{m-1})$. Let $F_\mu$ be the union of all components with slice type $\mu$ and let $v_\mu$ be the normal bundle of $F_\mu$ in $M^{2n}$. Note that $\dim_{\mathbb{C}} v_\mu = \sum_{j=1}^{m} \mu_j$. For smooth actions, Berend and Katz define quasi-signatures, $S_0(v_\mu)$, for each $\omega \in \prod_{j=1}^{m-1} \mathbb{Z}_{+}^{\mu_j} / \prod_{j=1}^{m-1} S(\mu_j)$, where $\mathbb{Z}_{+}$ is the set of nonnegative integers and if $t$ is a positive integer, $S(t)$ is the symmetric group on $t$ letters ([3], Section 3). These quasi-signatures are integers in the smooth category ([3], Proposition 2.1) and if $g$ is a diffeomorphism of order $m$, $\text{Sign}(g, M)$ can be expressed in terms of these integers and the algebraic numbers $\alpha_{j/m} = (\lambda^j + 1)(\lambda^j - 1)^{-1}, 1 \leq j \leq m - 1$ ([3], Theorem 2.2).

We will develop an expression like this for $\text{Sign}(g, M)$ in the case of locally linear $PL$, tame, regular actions. The quasi-signatures $S_0(v_\mu)$ are rational numbers in general in this case, but we will show that if the action satisfies the additional hypothesis that all of the components of $F$ are either of dimension $0$ or $2n - 2$, then the quasi-signatures $S_0(v_\mu)$ are integers.

**Definition 2.1.** A tame, locally linear $PL$ $G_m$ action is regular if there exists $j_0, 1 \leq j_0 \leq m - 1$, such that $(j_0, m) = 1$ and for every slice type $\mu = (\mu_1, \mu_2, \ldots, \mu_{m-1})$, $\mu_j = 0$ if $j \neq j_0$.

If $\mu = (\mu_1, \mu_2, \ldots, \mu_{m-1})$ and $\mu_j = 0$ if $j \neq j_0$, then $\mu_j = \dim_{\mathbb{C}} v_\mu$. We assume that $\mu_{j_0} > 0$. If a regular action has $s$ slice types, each can be identified with a complex codimension $c_j$ and $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ where $F^{2n-2c_i}$ is the union of all components of $F$ with complex codimension $c_i$. Let $v_c$ be the normal block bundle of $F^{2n-2c_i}$ in $M^{2n}$. The quasi-signatures $S_0(v_c)$ can be described as follows. If $c \neq 0$, let $x_k, 1 \leq k \leq c$, be integral Wu cohomology classes of degree $2$ with the property that the Chern classes of $v_c$ are the elementary symmetric polynomials in the variables $x_k$. If $\omega \in \mathbb{Z}_{+}^{c}/S(c)$ is covered by $\tilde{\omega} = (\omega_k) \in \mathbb{Z}_{+}^{c}$, let

$$T_\omega(v_c) = |S_{\tilde{\omega}}|^{-1} \sum_{\omega \in \mathcal{S}(c)} \prod_{k=1}^{c} \tanh \omega_k x_k$$

where $|S_{\tilde{\omega}}|$ is the order of the stabilizer of $\tilde{\omega}$ ([3], Formula (2.5)). If $L(F^{2n-2c})$ is the Hirzebruch $L$-class of the $PL$ manifold $F^{2n-2c}$ and $[F^{2n-2c}]$ is the fundamental class of $F^{2n-2c}$, let

$$S_0(v_c) = T_\omega(v_c) L(F^{2n-2c}) [F^{2n-2c}].$$

Note that $T_\omega(v_c)$ is a polynomial over $\mathbb{Q}$ in the Chern classes of $v_c$ and since $L(F^{2n-2c})$ is defined using the rational combinatorial Pontrjagin classes of $F^{2n-2c}$, it is clear that $S_0(v_c)$ is a rational number. These numbers are a special case of numbers defined for arbitrary slice types of arbitrary smooth actions ([3], Formula (2.6)).

It is useful to define certain sums of the rational numbers $S_0(v_c)$ using two norms on $\mathbb{Z}_{+}^{c}/S(c)$, $\| \cdot \|$ and $| \cdot |$. If $\omega \in \mathbb{Z}_{+}^{c}/S(c)$, then $\| \omega \|$ is the sum of the entries in $\omega$ and $| \omega \|$ is the number of nonzero entries in $\omega$. If $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ and $c \in \{c_1, c_2, \ldots, c_s\}$ and $1 \leq j \leq c$ and $j \leq k \leq n - c$, then we define

$$S_c(j, k)(v_c) = \sum_{\omega \in \mathcal{S}(c, j, k)} S_0(v_c)$$

where $s(c, j, k) = \{\omega \in \mathbb{Z}_{+}^{c}/S(c) : |\omega| = j, \|\omega\| = k\}$ ([14], Definition 2.5).

**Definition 2.5.** If $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ is the fixed point set of a regular, tame, locally linear $PL$ $G_m$ action and $c \in \{c_1, c_2, \ldots, c_s\}$ then the polynomial $p_c(x) \in \mathbb{Q}[x]$ is defined by the conditions that $p_0(x) = p_n(x) = 0$ and if $0 < c < n$, then

$$p_c(x) = \sum_{j=1}^{c} \sum_{k=1}^{n-c} (-1)^{k} x^{c+k-2j} (x^2 - 1)^{j-1} S_c(j, k)(v_c).$$

The polynomial $p_c(x)$ plays a role in the ASgSF for regular $G_m$ actions. The next step is to define two more polynomials which will be used in the signature formula.
Definition 2.7. If $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ is the fixed point set of a regular, tame, locally linear $PL G_m$ action, then the polynomial $p(x) \in \mathbb{Q}[x]$ is defined by

$$p(x) = \sum_{i=1}^{s} p_{c_i}(x).$$

(2.8)

Definition 2.9. If $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$ is the fixed point set of a locally linear $PL G_m$ action, then the polynomial $s(x) \in \mathbb{Z}[x]$ is defined by

$$s(x) = \sum_{i=1}^{s} \operatorname{Sign} F^{2n-2c_i} x^{c_i}.$$  

(2.10)

Theorem 2.11. Suppose that $g : M^{2n} \to M^{2n}$ is a piecewise linear map of period $m \geq 2$. If the $G_m$ action defined by $g$ is locally linear $PL$, tame, regular and $F = \bigcup_{i=1}^{s} F^{2n-2c_i}$, then there exists a polynomial with rational coefficients $p(x) \in \mathbb{Q}[x]$ such that the degree of $p(x)$ is at most $n - 2$ and there is an algebraic number $\alpha \in \{\alpha_{j/m} : 1 \leq j \leq m - 1, (j, m) = 1\}$ such that

$$\operatorname{Sign}(g, M) = s(\alpha) + (\alpha^2 - 1) p(\alpha).$$

(2.12)

Proof. There exists $j_0$ with $(j_0, m) = 1$ and $\mu_j = 0, j \neq j_0$, for every slice type $\mu = (\mu_1, \mu_2, \ldots, \mu_{m-1})$. The ASgSF holds for locally linear $PL$ tame $G_m$ actions ([15], p. 189) and so it follows from (2.1) in [3] that for the regular action above

$$\operatorname{Sign}(g, M) = \sum_{i=1}^{s} \prod_{k=1}^{c_i} \lambda_{j_k}\lambda_{j_k}^{2x_{j_k}k} + 1 L(F^{2n-2c_i})[F^{2n-2c_i}]$$

(2.13)

where $x_{j_k}k, 1 \leq k \leq c_i$, are the Wu classes of $\nu_{c_i}, 1 \leq i \leq s$. It follows from a hyperbolic tangent expansion ([3], p. 944) and the combinatorial conventions described above that

$$\operatorname{Sign}(g, M) = \sum_{c} \sum_{\omega \in \mathbb{Z}_{c}^{s}/S(c)} (-1)^{\|\omega\| \alpha^c + \|\omega\| - 2|\omega|} (\alpha^2 - 1)^{|\omega|} S_{\omega}(\nu_c)$$

(2.14)

where $\alpha = \alpha_{j_0/m}$. Formula (2.12) follows from (2.14) by putting $|\omega| = j, \|\omega\| = k$ and using (2.4), (2.6), (2.8) and (2.10) and the fact that $w_{(0,0,\ldots,0)}(\nu_c) = \operatorname{Sign} F^{2n-2c}$. We have observed that $S_{\omega}(\nu_c) \in \mathbb{Q}$ and so it follows from (2.4) and (2.6) and (2.8) that $p(x) \in \mathbb{Q}[x]$. It follows from (2.6) that degree $p_{\nu_c}(x) \leq$ degree $p_1(x) \leq n - 2$ and so the degree of $p(x)$ is at most $n - 2$ by (2.8). \qed

We now turn to the task of showing that $p(x)$ is a polynomial with integer coefficients if it is assumed that the components of $F$ have dimension either 0 or $2n - 2$. We begin with a lemma that concerns the following data. Suppose that $M^{2n}$ is a piecewise linear ($PL$), closed, orientable $2n$-manifold and that $K^{2n-2} \subset M^{2n}$ is a closed, orientable locally flat submanifold with normal block bundle $v$, a real orientable 2-plane bundle ([9], [10], p. 254) with Euler class $u \in H^2(K; \mathbb{Z})$. If $s = 0, 1, 2, \ldots$, then the $s$-fold self-intersection of $K$ in $M$ is defined inductively using transversality in the $PL$ category: $K^{(0)} = M, K^{(1)} = K$ and if $K^{(s-1)} \subset M, s > 1$, and $j : K^{(s-1)} \to M$ is transverse to $K$, then $K^{(s)} = j^{-1}(K)$. The dimension of $K^{(s)}$ is $2n - 2s$, there is a chain of submanifolds $K^{(n)} \subset K^{(n-1)} \subset \cdots \subset K \subset M$ and $K^{(n)}$ is a set of points.

Lemma 2.15. If $M^{2n}$ is a piecewise linear ($PL$), closed, orientable $2n$-manifold and $K^{2n-2} \subset M^{2n}$ is a closed, orientable locally flat submanifold with normal block bundle $v$ with Euler class $u \in H^2(K; \mathbb{Z})$, then if $s \geq 1$,

$$\operatorname{Sign} K^{(s)} = \tanh^{s-1} u L(K)[K].$$

(2.16)

If $K$ is dual to $w \in H^{2n}(M; \mathbb{Z})$, that is $w_K = w \cap [M]$ if $i : K \subset M$ is the inclusion, then

$$\operatorname{Sign} K^{(s)} = \tanh^{s} w L(M)[M].$$

(2.17)
Proposition 2.18. If \( n \geq 2 \), then
\[
(-1)^{n-1} p_1(x) = \begin{cases} 
\sum_{k=1}^{\lfloor n/2 \rfloor} \text{Sign} F_1^{(2k)} x^{2k-2}, & n \text{ even}, \\
\sum_{k=1}^{\lfloor n/2 \rfloor} \text{Sign} F_1^{(2k+1)} x^{2k-1}, & n \text{ odd}.
\end{cases}
\]  
\tag{2.19}

In particular \( p_1(x) \) is a polynomial with integer coefficients.

Proof. Note that \( \mathbb{Z}_+^n / S(1) = \mathbb{Z}_+ \) and so \( \omega \) is a nonnegative integer in the expression \( S_{\omega}(v_1) \). It follows from (2.2) and (2.3) and (2.16) with \( K = F_1 \) and \( u = x_1 \) that \( S_{\omega}(v_1) = \text{Sign} F_1^{(\omega+1)} \). Formula (2.19) now follows from (2.4) and (2.6). \( \square \)

Theorem 2.20. Suppose that \( g : M^{2n} \to M^{2n} \) is a piecewise linear map of period \( m \geq 2 \). If the \( G_m \) action defined by \( g \) is locally linear PL, \( F = F^0 \cup F^{2n-2} \) and the action is regular, then there exists a polynomial with integer coefficients \( p(x) \in \mathbb{Z}[x] \) such that the degree of \( p(x) \) is at most \( n - 2 \) and there is an algebraic number \( \alpha \in \{ \alpha_{j/m} : 1 \leq j \leq m - 1, (j, m) = 1 \} \) such that
\[
\text{Sign}(g, M) = s(\alpha) + (\alpha^2 - 1) p(\alpha). \tag{2.21}
\]

Proof. The assertions above follow immediately from Theorem 2.11, since actions such that \( F = F^0 \cup F^{2n-2} \) are tame ([5], Theorem 1.1, [9], [10], p. 254), together with (2.19) because \( p(x) = p_1(x) \) if \( F = F^0 \cup F^{2n-2} \). \( \square \)

Our next result records a formula for \( p(x) \) in the special case of Type II\(0 \) actions on cohomology complex projective \( n \)-space. If \( M^{2n} \) is a PL cohomology complex projective \( n \)-space, \( d \in \mathbb{Z} \), and \( x \in H^2(M; \mathbb{Z}) \) is the generator of the cohomology algebra, \( K^{2n-2} \subset M^{2n} \) a locally flat submanifold dual to \( dx \), then we define integers \( s_k(d), 0 \leq k \leq \lfloor n/2 \rfloor \), by the formulas
\[
s_k(d) = \begin{cases} 
\text{Sign} K^{(2k)}, & n \text{ even}, \\
\text{Sign} K^{(2k+1)}, & n \text{ odd}.
\end{cases} \tag{2.22}
\]

Note that it follows from (2.17) with \( w = dx \) that \( \text{Sign} K^{(s)} = \tanh^s dxL(M)[M] \). This expression leads to certain numerical congruences for the integers \( s_k(d) \) ([13], Corollary 2.6) which will be useful (Theorem 4.12).

Proposition 2.23. If \( n \geq 2 \) and \( M^{2n} \) is a PL cohomology complex projective \( n \)-space which admits a locally linear PL, regular \( G_m \) action of Type II\(0 \) fixing a codimension-2 locally flat submanifold of degree \( d \), then
\[
(-1)^{n-1} p(x) = \begin{cases} 
\sum_{k=1}^{\lfloor n/2 \rfloor} s_k(d) x^{2k-2}, & n \text{ even}, \\
\sum_{k=1}^{\lfloor n/2 \rfloor} s_k(d) x^{2k-1}, & n \text{ odd}.
\end{cases} \tag{2.24}
\]

Proof. Formula (2.24) follows from (2.19) with \( p_1(x) = p(x) \) and (2.22). \( \square \)
3. The polynomial $M_m(x)$

We begin this section with a discussion of the minimal polynomial over $\mathbb{Q}$ of the numbers $\alpha_{j/m} = (\lambda^j + 1)(\lambda^j - 1)^{-1}$, $1 \leq j \leq m - 1$, $(j, m) = 1$ and $\lambda = \exp(2\pi i / m)$.

**Definition 3.1.** If $m \geq 2$ and $\phi(m) = |\{ j : 1 \leq j \leq m - 1, (j, m) = 1 \}|$ and $\Phi_m(x)$ is the $m$th cyclotomic polynomial, then the polynomial $M_m(x) \in \mathbb{Z}[x]$ is defined by

$$M_m(x) = (x - 1)^{\phi(m)} \Phi_m((x + 1)(x - 1)^{-1}).$$

(3.2)

**Proposition 3.3.** If $m \geq 2$ and $(j, m) = 1$, then $M_m(x)$ is the minimal polynomial of $\alpha_{j/m}$ over $\mathbb{Q}$ and it satisfies the equations below where $p$ is an odd prime.

$$M_m(0) = \begin{cases} 
0, & m = 2, \\
2, & m = 2^e, e > 1, \\
p, & m = 2^e p, e \geq 1, \\
1, & \text{otherwise}.
\end{cases}$$

(3.4)

$M_m(\pm 1) = (\pm 1)^{\phi(m)} 2^{\phi(m)}$.

(3.5)

**Proof.** The identity $\lambda^j = (\alpha_{j/m} + 1)(\alpha_{j/m} - 1)^{-1}$ implies that $M_m(\alpha_{j/m}) = 0$ and $M_m(x)$ is irreducible over $\mathbb{Q}$ and has degree $\phi(m)$ since $\Phi_m(x)$ has these two properties. Formula (3.4) follows from the values of $\Phi_m(-1)$ ([11], pp. 203–208) and (3.5) from the fact that $\Phi_m(x)$ is monic with degree $\phi(m)$ and $\Phi_m(0) = 1$ ([11], pp. 203–208).

**Proposition 3.6.** If $m > 2$, then

$$M_m(x) = \Phi_m(1) \prod_{j=1}^{[m/2]} (x^2 - \alpha_{j/m}^2).$$

(3.7)

In particular, $M_m(x) \in \mathbb{Z}[x^2]$ and if $M_m(x) = \sum_{k=0}^{m} m_{2k} x^{2k}$, then

$$m_{\phi(m) - 2} = -\Phi_m(1) \sum_{j=1}^{[m/2]} \alpha_{j/m}^2.$$

(3.8)

$$m_{\phi(m)} = \Phi_m(1).$$

(3.9)

**Proof.** If $m > 2$ and $1 \leq j \leq [m/2], (j, m) = 1$, then $(j, m - j) = 1, j \neq m - j$ and $\alpha_{j/m} = -\alpha_{(m-j)/m}$ and so the $\phi(m)$ roots of $M_m(x)$ are divided into $\phi(m)/2$ pairs of a root and its negative. Formulas (3.7) and (3.9) follow from this observation since (3.2) implies that $m_{\phi(m)} = \Phi_m(1)$. Formula (3.8) follows from (3.7).

**Proposition 3.10** ([14, Proposition 3.13]). If $m$ is not a power of 2, then $M_m(x)$ is primitive. If $e \geq 1$, then $2^{-1} M_{2e}(x)$ is primitive.

We will apply (3.4), (3.5) and Proposition 3.10 in the proof of Theorem A. We begin with congruences for values of integral polynomials divisible by $M_m(x)$. Recall that if $a(x) \in \mathbb{Q}[x]$, then $a(x)_{\text{even}}$ and $a(x)_{\text{odd}}$ are the parts of $a(x)$ with even and odd powers of $x$, respectively and that $\rho(m) = \phi(m) - 1$ if $m = 2^e$ and $\rho(m) = \phi(m), m \neq 2^e$.

**Proposition 3.11.** Suppose that $a(x) \in \mathbb{Q}[x]$ and that $m > 2$. If $a(x)$ is divisible by $M_m(x)$, then $a(x)_{\text{even}}$ and $x^{-1} a(x)_{\text{odd}}$ are divisible by $M_m(x)$. If $a(x)$ is divisible by $M_m(x)$ and the degree of $a(x)$ is less than $\phi(m)$, then $a(x)$ is identically zero. If $a(x) \in \mathbb{Z}[x]$ and $a(x)$ is divisible by $M_m(x)$, then

$$a(1)_{\text{even}} \equiv 0 \pmod{2^{\rho(m)}}, \quad a(1)_{\text{odd}} \equiv 0 \pmod{2^{\rho(m)}}.$$ 

(3.12)
If \( a(x) \in \mathbb{Z}[x] \) and \( a(x) \) is divisible by \( M_{2^p}(x) \) where \( p \) is an odd prime and \( e \geq 1 \), then
\[
a(0) \equiv 0 \pmod{p}.
\] (3.13)

**Proof.** The first assertion follows from the fact that \( M_m(x) \in \mathbb{Z}[x^2] \) if \( m > 2 \) (Proposition 3.6) and the second assertion follows from the facts that the degree of \( M_m(x) \) is \( \phi(m) \) and \( M_m(x) \) is the minimal polynomial of \( \alpha_{j/m} \) over \( \mathbb{Q} \) (Proposition 3.3). Formulas (3.12) and (3.13) now follow from (3.4), (3.5) and Proposition 3.10. \(\square\)

We will use Proposition 3.11 to prove Theorem A and to offer a refined version of (2.21) for Type II \( G_m \) actions. Our next theorem is the same as Theorem A.

**Theorem 3.14.** Suppose that \( g : M^{2n} \to M^{2n} \) is an orientation preserving piecewise linear map of period \( m > 2 \). If the \( G_m \) action defined by \( g \) is locally linear PL, \( F = F^0 \cup F^{2n-2} \) and the action is regular and \( \text{Sign} \,(g, M) \in \mathbb{Z} \), then \( \text{Sign}(g, M) \equiv \text{Sign} F_{\text{even}} \,(\text{mod } 2^{\rho(m)}) \) and \( \text{Sign} F_{\text{odd}} \equiv 0 \,(\text{mod } 2^{\rho(m)}) \) and so
\[
\text{Sign}(g, M) \equiv \text{Sign} F \,(\text{mod } 2^{\rho(m)}).
\] (3.15)

If \( p \) is an odd prime and \( m = 2p^e \), then \( \text{Sign}(g, M) \equiv \text{Sign}(F \cap F) \,(\text{mod } p) \).

**Proof.** Let \( a(x) \) be defined by the equation
\[
a(x) = s(x) + (x^2 - 1)p(x) - \text{Sign}(g, M)
\] (3.16)
where the polynomials \( s(x) \) and \( p(x) \) are defined by (2.10) and (2.8) respectively. If \( \text{Sign}(g, M) \in \mathbb{Z} \), then Theorem 2.20 implies that \( a(x) \in \mathbb{Z}[x] \) and (2.21) and (3.12) yield the first two congruences and (3.15). If \( p \) is an odd prime and \( m = 2p^e \), then (2.19), (2.21) and (3.13) give the last congruence. \(\square\)

We now turn to the application of (3.15) to Type II \( G_m \) actions. We begin by defining two polynomials associated with a cohomology complex projective \( n \)-space and its self-intersection signatures \( s_k(d), 0 \leq k \leq \lfloor n/2 \rfloor \) (2.22).

**Definition 3.17.** If \( M^{2n} \) is a cohomology complex projective \( n \)-space and \( d \) is an integer, then the two polynomials \( b_{\pm}(x) \in \mathbb{Z}[x] \) are defined by the equations
\[
b_{\pm}(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} (s_k(d) \pm 1)x^{2k-2}.
\] (3.18)

**Theorem 3.19.** Suppose that \( M^{2n} \) is a PL cohomology complex projective \( n \)-space which admits a regular locally linear PL \( G_m \) action of Type II \( G_m \) where \( m > 2, m \neq 4 \). Suppose that the action fixes a codimension-2 locally flat submanifold of degree \( d \). If \( n \) is even, then there exists \( \alpha \in \{ \alpha_{j/m} : 1 \leq j \leq m - 1, (j, m) = 1 \} \) such that \( b_{\pm}(\alpha) = 0 \) and, if \( g^* \) is the identity on \( H^0(M; \mathbb{Q}) \), then \( b_{-}(\alpha) = 0 \). If \( n \) is odd, then there exists \( \alpha \in \{ \alpha_{j/m} : 1 \leq j \leq m - 1, (j, m) = 1 \} \) such that \( b_{-}(\alpha) = 0 \) if \( d > 0 \) and \( b_{+}(\alpha) = 0 \) if \( d < 0 \).

**Proof.** Let \( g : M^{2n} \to M^{2n} \) be a generator of the action and let \( F = F^{2n-2} \cup F^0 \) where \( F^{2n-2} \) is a connected locally flat submanifold of degree \( d \) and \( F^0 \) is an isolated point. We begin by using Theorem 3.14 to study the relationship between \( \text{Sign}(g, M) \) and \( \text{Sign} F \). If \( n \) is even, choose the orientation for \( M^{2n} \) such that \( \text{Sign} M = +1 \). It follows that \( \text{Sign}(g, M) = \pm \text{Sign} M \) and, if \( g^* \) is the identity in \( H^0(M; \mathbb{Q}) \), that \( \text{Sign}(g, M) = +1 \) ([7], Formula (11), p. 28). It follows from the first congruence in Theorem 3.14 that \( \text{Sign}(g, M) = \text{Sign} F^0 \) since \( \rho(m) \geq 2 \) if \( m > 2, m \neq 4 \).

If \( n \) is arbitrary and \( p \) is a prime dividing \( m \), then \( M^{2n} \) admits a \( G_p \) action of Type II \( G_m \) fixing \( F^{2n-2} \) and so \( d \neq 0 \,(\text{mod } p) \) ([4], pp. 378–383) and so \( d \neq 0 \). If \( n \) is odd, \( \text{Sign} F^{2n-2} = \pm 1 \) and \( \text{Sign} F^{2n-2} \) and \( d \) have the same algebraic sign ([6], Lemma 4.1). It follows from the second congruence in Theorem 3.14 that \( \text{Sign} F^{2n-2} = \text{Sign} F^0 \) since \( \rho(m) \geq 2 \) if \( m > 2, m \neq 4 \).

We now use (2.24) to write the polynomial \( a(x) \) (3.16) in the case of Type II \( G_m \) actions as
\[
a(x) = \begin{cases} 
\text{Sign} F^0 x^n - (x^2 - 1) \sum_{k=1}^{\lfloor n/2 \rfloor} s_k(d)x^{2k-2} - \text{Sign}(g, M), & \text{if } n \text{ even}, \\
\text{Sign} F^0 x^n + \text{Sign} F^{2n-2} x + (x^2 - 1) \sum_{k=1}^{\lfloor n/2 \rfloor} s_k(d)x^{2k-1} & \text{if } n \text{ odd}.
\end{cases}
\] (3.20)
The fact that $\text{Sign}(g, M) = 0$ if $n$ is odd is used in (3.20). The assertions above follow from (3.20) and the observations made in the preceding two paragraphs. If $n$ is even, $\text{Sign}(g, M)$ and $\text{Sign} F^0$ are either both $+1$ or both $-1$ and so $a(\alpha) = 0$ for some $\alpha \in \{ \alpha_{j/m} : 1 \leq j \leq m - 1, (j, m) = 1 \}$ (2.21) implies that $b_{\pm}(\alpha) = 0$. If $g^*$ is the identity on $H^*(M; \mathbb{Q})$, $\text{Sign}(g, M) = \text{Sign} F^0 = +1$ and so $b_{-}(\alpha) = 0$. If $n$ is odd, then $\text{Sign} F^{2n-2}$ and $\text{Sign} F^0$ are both $\pm 1$ and are of opposite signs with $\text{Sign} F^{2n-2}$ and $d$ having the same algebraic sign. Therefore $a(\alpha) = 0$ for some $\alpha \in \{ \alpha_{j/m} : 1 \leq j \leq m - 1, (j, m) = 1 \}$ (2.21) implies that $b_{-}(\alpha) = 0$ if $d > 0$ and $b_{+}(\alpha) = 0$ if $d < 0$. □

We use Theorem 3.19 and the fact that the degree of $b_{\pm}(x)$ is $2([n/2] - 1)$ to prove the first assertion in Theorem C.

**Theorem 3.21.** Suppose that $M^{2n}$ is a cohomology complex projective $n$-space which admits a regular locally linear PL $G_m$ action of Type $\Pi_0$ where $m > 2$ and $m \neq 4$. If $n < \phi(m) + 2$, then the action is standard and the Pontrjagin class of $M^{2n}$ is standard.

**Proof.** Since $n < \phi(m) + 2$ implies that the degree of $b_{\pm}(x)$ is less than $\phi(m)$, it follows from the second assertion in Proposition 3.11 and Theorem 3.19 that $b_{\pm}(x) \equiv 0$ if $n$ is even and if $n$ is odd, $b_{-}(x) \equiv 0$ if $d > 0$ and $b_{+}(x) \equiv 0$ if $d < 0$.

If $n$ is even, $b_{\pm}(x) \not\equiv 0$ since $s_{[n/2]}(d) = d^{n}$ ([13], p. 657) and so it must be the case that $b_{-}(x) \equiv 0$. It follows that $b_{\pm}(x) = b_{\pm}(\alpha) = 1 \leq k < [n/2]$. This means $s_k(1) = 1, 1 \leq k < [n/2]$ since $s_k(d)$ is an even function of $d$ if $n$ is even ([13], Formulas (2.2) and (2.5)) and so the Pontrjagin class of $M^{2n}$ is standard ([13], Proposition 2.8).

If $n$ is odd and $d > 0$, then $b_{-}(x) \equiv 0$ and it follows that $d = +1$ since $s_{[n/2]}(d) = d^{n}$ ([13], p. 657) and so the action is standard and $s_k(1) = 1, 1 \leq k < [n/2]$, and so the Pontrjagin class of $M^{2n}$ is standard ([13], Proposition 2.8). If $n$ is odd and $d < 0$, then $b_{-}(x) \equiv 0$ and it follows that $d = -1$ since $s_{[n/2]}(d) = d^{n}$ ([13], p. 657) and so the action is standard and $s_k(-1) = -1, 1 \leq k < [n/2]$. This means $s_k(1) = 1, 1 \leq k < [n/2]$ since $s_k(d)$ is an odd function of $d$ if $n$ is odd ([13], Formulas (2.2) and (2.5)) and so the Pontrjagin class of $M^{2n}$ is standard ([13], Proposition 2.8). □

4. Linear systems

The purpose of this section is to prove Theorem B and the second assertion in Theorem C. Our starting point is Proposition 3.11 where we observed that to study the divisibility of polynomials in $\mathbb{Q}[x]$ by $M_m(x)$, it is enough to settle this question for $\mathbb{Q}[x^2]$. Recall that $m > 2$, $\mu = \phi(m)/2$ and $M_m(x) = \sum_{k=0}^{\mu} m_{2k} x^{2k}$. Our next result is the same as Theorem B.

**Theorem 4.1.** If $m > 2$ and $r \geq \mu$, then there is a set of integers $A^{(r)}(m) = \{ A^{(r)}_{i,j}(m) : 1 \leq i \leq \mu, i - 1 \leq j \leq r \}$ with the property that if $a(x) = \sum_{k=0}^{r} a_{2k} x^{2k} \in \mathbb{Q}[x^2]$ is divisible by $M_m(x)$, then

$$\sum_{j=i-1}^{r} A^{(r)}_{i,j}(m) a_{2j} = 0, \quad 1 \leq i \leq \mu. \quad (4.2)$$

These sets have the property that if $r \geq \mu + 1$, then the equations

$$A^{(r)}_{i,j}(m) = \begin{cases} -m_{\phi(m)} A^{(r-1)}_{i,j}(m), & i - 1 \leq j < r, \\ \sum_{k=1}^{\mu} m_{\phi(m)-2k} A^{(r-1)}_{i,r-k}(m), & j = r, \end{cases} \quad (4.3)$$

express $A^{(r)}(m)$ in terms of $A^{(r-1)}(m)$.

**Lemma 4.4.** If $a(x) = \sum_{k=0}^{\mu} a_{2k} x^{2k} \in \mathbb{Q}[x^2]$ is divisible by $M_m(x)$, then the coefficients of $a(x)$ satisfy (4.2) at level $r = \mu$ with

$$A^{(\mu)}_{i,j}(m) = \begin{cases} (-1)^{\mu-j} (m_{2(j+1)} + m_{2(j-1)}), & i - 1 < j \leq \mu, \\ (-1)^{\mu-i+1} m_{2i}, & j = i - 1. \end{cases} \quad (4.5)$$
Proof. There is a rational \( q_0 \in \mathbb{Q} \) such that \( a_{2k} = m_{2k} q_0, 0 \leq k \leq \mu \). The unknown \( q_0 \) can be eliminated to produce the equations, with alternating signs added,
\[
(-1)^{\mu-k} (m_{2k-2} a_{2k} - m_{2k} a_{2k-2}) = 0,
\]
where \( 1 \leq k \leq \mu \). The alternating signs have been added to (4.6) so that the systems (4.2) and (4.5) will produce a system in the case \( m = 5 \) which was obtained by other means ([13], Lemma 4.14). Eq. (4.2) at level \( i = \mu \) is (4.6) with \( k = \mu \), (4.2) at level \( i = \mu - 1 \) is the sum of (4.6) at \( k = \mu \) and \( k = \mu - 1 \), and in general, (4.2) at level \( i \) is the sum of Eq. (4.6) for \( \mu \leq k \leq i \). Formula (4.5) gives an expression for \( A_{i,j}^{(\mu)}(m) \) modulo the convention that \( m_{2(\mu+1)} = 0 \). □

Proof of Theorem 4.1. The proof is by induction on \( r \). The initial step, \( r = \mu \), is just Lemma 4.4. For the inductive step, assume that \( r > \mu \) and that the set \( A^{(r-1)}(m) \subset \mathbb{Z} \) exists and satisfies (4.2) for divisible polynomials of degree \( 2r - 2 \). Suppose that \( a(x) \in \mathbb{Q}[x^2] \) and that the degree of \( a(x) \) is \( 2r \) and that \( a(x) \) is divisible by \( M_m(x) \). This means there exists \( q(x) \in \mathbb{Q}[x^2] \) with \( a(x) = M_m(x) q(x) \). If \( q(x) = \sum_{k=0}^{r-\mu-1} q_{2k} x^{2k} \), then
\[
a(x) = M_m(x) q_{2r-\phi(m)} x^{2r-\phi(m)} + M_m(x) \sum_{k=0}^{r-\mu-1} q_{2k} x^{2k}.
\]
The next step is to define \( A^{(r)}(m) \subset \mathbb{Z} \) in terms of \( A^{(r-1)}(m) \) via (4.3), that is, we put
\[
A_{i,j}^{(r)}(m) := \begin{cases} 
-m_{\phi(m)} A_{i,j}^{(r-1)}(m), & i - 1 \leq j < r, \\
\sum_{k=1}^{\mu} m_{\phi(m)-2k} A_{i,r-k}^{(r-1)}(m), & j = r.
\end{cases}
\]
Clearly \( A^{(r)}(m) \subset \mathbb{Z} \). The second term on the right of (4.7) is a polynomial of degree \( 2r - 2 \) which is divisible by \( M_m(x) \). If the inductive hypothesis is applied to this polynomial, (4.8) implies that the coefficients of \( a(x) \) satisfy (4.2) at level \( r = \mu \). □

Definition 4.9. The \( i \)th row sum of the system of linear (4.2) is defined for \( 1 \leq i \leq \mu \) by
\[
R_{i}^{(r)}(m) = \sum_{j=-1}^{r} A_{i,j}^{(r)}(m).
\]
The most important row sum for our purposes is the row sum with \( r = i = \mu \). This row is (4.6) with \( k = \mu \) and so
\[
R_{\mu}^{(\mu)}(m) = m_{\phi(m)-2} - m_{\phi(m)}.
\]
We will return to this row sum after we prove the second assertion in Theorem C. Recall that \( f(n) \) is \( n! \) divided by a maximal power of 2.

Theorem 4.12. Suppose that \( M^{2n} \) is a PL cohomology projective \( n \)-space which admits a regular locally linear PL \( G_m \) action of Type II\( _0 \) fixing a codimension-2 locally flat submanifold of degree \( d \). If \( m > 2 \), \( m \neq 4 \), and \( n \geq \phi(m) + 2 \), then for every \( i \) such that \( 1 \leq i \leq \mu \), \( d^{2i} \) divides \( f(n) R_{i}^{([n/2]-1)}(m) \) if \( n \) is even and \( d^{2i+1} \) divides \( f(n) R_{i}^{([n/2]-1)}(m) \) if \( n \) is odd.

Proof. Since \( n \geq \phi(m) + 2 \) is the same as \([n/2] - 1 \geq \mu \) and the degree of \( b_\pm(x) = 2([n/2]) - 1 \) (3.18), it follows from Theorems 3.19 and 4.1 that for \( 1 \leq i \leq \mu \),
\[
\sum_{j=-1}^{[n/2]-1} A_{i,j}^{([n/2]-1)}(m) s_{j+1}(d) = \pm R_{i}^{([n/2]-1)}(m).
\]
The divisibility conditions follow by multiplying both sides of (4.13) by \( f(n) \) and using the facts that \( f(n)s_{k}(d) \equiv 0 \) (mod \( d^{2k} \)) if \( n \) is even and \( f(n)s_{k}(d) \equiv 0 \) (mod \( d^{2k+1} \)) if \( n \) is odd ([13], Corollary 2.6). □
Corollary 4.14. Suppose that $M^{2n}$ satisfies the hypotheses of Theorem 4.12. If $n = \phi(m) + 2$, then $d^{\phi(m)}$ divides $f(n)(m_{\phi(m)}-2 - m_{\phi(m)})$ and if $n = \phi(m) + 3$, then $d^{\phi(m)+1}$ divides $f(n)(m_{\phi(m)}-2 - m_{\phi(m)})$.

Proof. This is Theorem 4.12 with $[n/2] − 1 = \mu$ and $i = \mu$ together with (4.11). □

We have now completed the proofs of Theorem A (Theorem 3.14), Theorem B (Theorem 4.1) and Theorem C (Theorem 3.21 contains the first assertion and Theorem 4.12 the second.). We turn to the proof of Theorem D. Theorem D and Corollary 4.14 will be our main tools in the proof of Theorem E. A complete description of $A^{(r)}(m)$ is included because there is evidence that the full linear system with alternating signs (Formulas (4.2) and (4.5)) is useful ([13], Lemmas 4.14 and 5.9). Recall that $m_{\phi(m)} = \Phi_m(1)$ (3.9) and $\Phi_m(1) = p$ if $m = p^e$, $p$ a prime, and $\Phi_m(1) = 1$ otherwise ([11], pp. 203–208). These observations and Corollary 4.14 lead us to study $m_{\phi(m)-2}$ in Section 5.

5. Cotangent sums and the coefficient $m_{\phi(m)-2}$

The function $(x)$, $x \in \mathbb{R}$, is defined by the conditions $(x) = x − [x] − \frac{1}{2}$, $x \not\in \mathbb{Z}$, and $(x) = 0$, $x \in \mathbb{Z}$, and it appears in the generalized Dedekind sum ([7], (20) p. 96) which is defined if $(q, m) = (r, m) = 1$ by the equation

$$
s(q, r; m) = \sum_{k=1}^{m} \left( \left( \frac{kq}{m} \right) \left( \frac{kr}{m} \right) \right).
$$

These sums occur in certain cotangent sums which we can use since $\alpha_{j/m} = -i \cot(j\pi/m)$.

Theorem 5.2 ([7], Theorem 1, p. 100). If $(q, m) = (r, m) = 1$, then

$$
\sum_{j=1}^{m-1} \alpha_{jq/m} \alpha_{jr/m} = -4ms(q, r; m).
$$

Rademacher reciprocity can be used to compute the right hand side of (5.3) in some cases.

Theorem 5.4 ([7], Theorem 2, p. 96). If $(q, m) = (r, m) = (q, r) = 1$, then

$$
s(q, r; m) + s(m, r; q) + s(m, q; r) = \frac{m^2 + q^2 + r^2 - 3mqr}{12mqr}.
$$

Corollary 5.6. If $m \geq 2$, then

$$
\sum_{j=1}^{m-1} \alpha^2_{j/m} = -\frac{(m-1)(m-2)}{3}.
$$

Proof. Formula (5.7) follows from (5.3) and (5.5) with $q = r = 1$. □

Proposition 5.8. If $m > 2$, then

$$
m_{\phi(m)-2} = \Phi_m(1) \left[ \frac{(m-1)(m-2)}{6} - \sum_{d|m, 2<d<m} \frac{m_{\phi(d)-2}}{\Phi_d(1)} \right].
$$

Proof. It follows from (3.8) and (5.7) that

$$
m_{\phi(m)-2} = \Phi_m(1) \left[ \frac{(m-1)(m-2)}{6} + \sum_{d|m, 2<d<m} \left( \sum_{j=1}^{[d/2]} \alpha_{j/d}^2 \right) \right]
$$

and so (5.9) follows by applying (3.8) to the inner sum in (5.10). □
Table 5.31

<table>
<thead>
<tr>
<th>m</th>
<th>(\Phi_m((m-1)(m-2)/6 - 1))</th>
<th>(\phi(m))</th>
<th>(\phi(m)/2)</th>
<th>(3\phi(m)/2)</th>
<th>(m_{\phi(m)}-2)</th>
<th>(m_{\phi(m)}-2-m_{\phi(m)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>28</td>
<td>6</td>
<td>3</td>
<td>27</td>
<td>35</td>
<td>(2^2 \cdot 7)</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>4</td>
<td>2</td>
<td>9</td>
<td>12</td>
<td>(2^2 \cdot 5)</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>4</td>
<td>2</td>
<td>9</td>
<td>10</td>
<td>(3^2)</td>
</tr>
<tr>
<td>12</td>
<td>(17 \cdot 3)</td>
<td>4</td>
<td>2</td>
<td>9</td>
<td>14</td>
<td>13</td>
</tr>
<tr>
<td>18</td>
<td>(44 \cdot 3)</td>
<td>6</td>
<td>3</td>
<td>27</td>
<td>33</td>
<td>(2^5)</td>
</tr>
<tr>
<td>24</td>
<td>(83 \cdot 3)</td>
<td>8</td>
<td>4</td>
<td>81</td>
<td>60</td>
<td>59</td>
</tr>
<tr>
<td>30</td>
<td>(134 \cdot 3)</td>
<td>8</td>
<td>4</td>
<td>81</td>
<td>92</td>
<td>(7 \cdot 13)</td>
</tr>
</tbody>
</table>

\(m_{\phi(p^e)} = \begin{cases} \left(\frac{p}{3}\right), & e = 1, \\ p \left[ (p^e - 1)(p^e - 2) - \frac{(p^e - 1)(p^e - 2)}{6} \right], & e > 1. \end{cases} \) \quad (5.12)

**Proof.** Formula (5.12) follows from (5.9) and the fact that \(\Phi_{p^e}(1) = p\), \(p\) prime and \(e \geq 1\).

Formulas (5.9) and (5.12) will be useful in computing \(m_{\phi(m)}-2\) (Table 5.31). We now apply these formulas to finding estimates of \(m_{\phi(m)}-2\).

**Proposition 5.13.** If \(m > 4\), then \(m_{\phi(m)}-2-m_{\phi(m)} > 0\).

**Proof.** This assertion follows from (3.8), (3.9) and the facts that \(\alpha_{j/m} = -i \cot(j\pi/m)\) and \(\cot(\pi/m) > 1\) if \(m > 4\).

**Proposition 5.14.** If \(m > 2\), then

\[ m_{\phi(m)}-2-m_{\phi(m)} \leq \Phi_m(1) \left[ \frac{(m-1)(m-2)}{6} - 1 \right]. \] \quad (5.15)

**Proof.** Inequality (5.15) follows from (3.9) and (5.9) since (3.9), (5.12) and Proposition 5.13 imply that \(m_{\phi(d)}-2 > 0\), \(d > 2\).

**Corollary 5.16.** If \(m > 4\), then

\[ 0 < m_{\phi(m)}-2-m_{\phi(m)} \leq \left(\frac{m}{3}\right) - m. \] \quad (5.17)

**Proof.** The inequalities in (5.17) follow from Proposition 5.13 and (5.15) together with the fact that \(\Phi_m(1) \leq m\).

Inequalities (5.15) and (5.17) will be used in the proof of our next theorem which is equivalent to Theorem D. We will state the theorem now and then prove it after proving some lemmas. If \(t\) is an integer and \(p\) is a prime, let \(\text{ord}_p t\) be the exponent of \(p\) in the prime factorization of \(t\).

**Theorem 5.18.** If \(m > 2\), \(m \neq 4, 10\), and \(p\) is an odd prime, then

\[ \text{ord}_2(m_{\phi(m)}-2-m_{\phi(m)}) < \phi(m). \] \quad (5.19)

\[ \text{ord}_p(m_{\phi(m)}-2-m_{\phi(m)}) < \phi(m)/2. \] \quad (5.20)

**Lemma 5.21.** If \(m > 4\) and

\[ \left(\frac{m}{3}\right) - m < \frac{3\phi(m)}{2}\] \quad (5.22)

then inequalities (5.19) and (5.20) hold.
proof. This observation follows from the inequalities in (5.17). □

Our strategy for the proof of Theorem 5.18 will be to show that (5.22) holds for all but finitely many \( m > 4 \) and so (5.19) and (5.20) hold for these values of \( m \). For the integers left, \( m > 4 \), \( m \neq 10 \), either the right side of (5.15) is strictly less than \( 3^{\phi(m)/2} \) and so (5.19) and (5.20) hold or direct computation using (5.9) shows that (5.19) and (5.20) hold.

Lemma 5.23. If \( m \) has \( s \) prime divisors, then

\[
\frac{m}{s+1} \leq \phi(m). \tag{5.24}
\]

In particular, since \( s \leq \log_2 m + 1 \),

\[
\frac{m}{\log_2 m + 2} \leq \phi(m). \tag{5.25}
\]

Proof. Suppose that \( m = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} \), \( p_i \) prime, \( e_i \geq 1 \), \( 1 \leq i \leq s \). Since \( \phi(m) = \prod_{i=1}^{m} (p_i^{e_i} - p_i^{e_i-1}) = m \prod_{i=1}^{m} (1 - p_i^{-1}) \), (5.24) follows from \((1 - p_i^{-1}) \geq (1 - i^{-1})\). Formula (5.25) follows from (5.24) and the inequality \( s \leq \log_2 m + 1 \). □

Lemma 5.26. If \( m > 278 \), then \( m \) satisfies inequality (5.22). In particular, if \( m > 278 \), then \( m \) satisfies inequalities (5.19) and (5.20).

Proof. Let the function \( f_0(x) \), \( x > 1 \), be defined by

\[
f_0(x) := \exp \left[ \frac{(\ln 2)(\ln 3)x}{2 \ln x + 4 \ln 2} \right] - \frac{x^3 - 3x^2 - 4x}{6}. \tag{5.27}
\]

If \( m_0 \) is the smallest integer such that \( f_0(m_0) > 0 \) and \( x > m_0 \) implies that \( f_0(x) > f_0(m_0) > 0 \), then (5.25) implies that (5.22), and hence (5.19) and (5.20), hold for \( m > m_0 \). A little work with \( f_0'(x) \) and some computations show that \( m_0 = 278 \). □

We now turn to the set \( 4 < m \leq 278 \). We can eliminate some of the integers in this range by noting that \( m \leq 278 \) means that \( m \) has at most 4 prime divisors. We use (5.24) which is stronger than (5.25) to relate (5.22) to the number of prime divisors of \( m \).

Lemma 5.28. Suppose that \( m > 4 \) and that \( m \) has exactly \( s \) prime divisors. Inequality (5.22) and inequalities (5.19) and (5.20) hold if either \( s = 1 \) and \( m > 31 \) or \( s = 2 \) and \( m > 56 \) or \( s = 3 \) and \( m > 84 \) or \( s = 4 \).

Proof. If \( s \geq 1 \), let the function \( f_s(x) \), \( x > 1 \), be defined by

\[
f_s(x) := \exp \left[ \frac{(\ln 3)x}{2x + 2} \right] - \frac{x^3 - 3x^2 - 4x}{6}. \tag{5.29}
\]

If \( m_s \) is the smallest integer such that \( f_s(m_s) > 0 \) and \( x > m_s \) implies that \( f_s(x) > f_s(m_s) > 0 \), then (5.24) implies that (5.22), and hence (5.19) and (5.20), hold if \( m \) has exactly \( s \) prime divisions and \( m > m_s \). A little work with \( f_s'(x) \) and some computations produce the following values of \( m_s \), \( s \leq 4 : m_1 = 31, m_2 = 56, m_3 = 84 \) and \( m_4 = 113 \). The assertions in the lemma now follow since \( s = 4 \) implies \( m > 113 \). □

Lemma 5.30. Suppose that \( m > 4 \) and that \( m \) has exactly \( s \) prime divisors. If either \( s = 1 \) and \( m \leq 31 \) or \( s = 2 \) and \( m \leq 56 \) and \( m \neq 10 \), or \( s = 3 \) and \( m \leq 84 \), then inequalities (5.19) and (5.20) hold.

Proof. If either \( s = 1 \) and \( m \leq 31 \) or \( s = 2 \) and \( m \leq 56 \) or \( s = 3 \) and \( m \leq 84 \), then direct computation shows that either \( \Phi_m(1)(m-1)(m-2)/6 - 11 < 3^{\Phi(m)/2} \), and so (5.19) and (5.20) hold in view of Proposition 5.13 and (5.15), or \( m \) appears in Table 5.31. □
The values of $m_{\phi(m)-2}$ in the table are produced using (5.9) and (5.12) and the prime factorization of $m_{\phi(m)-2} - m_{\phi(m)}$ is shown for convenience. Inspection of Table 5.31 shows that every value of $m$ entered satisfies (5.19) and (5.20) except $m = 10$ and the proof is complete. □

**Proof of Theorem 5.18.** The assertions in the theorem follow for every $m > 2$, $m \neq 3, 4, 10$, from Lemmas 5.26, 5.28 and 5.30. Formula (5.12) implies that $m_{\phi(3)-2} - m_{\phi(3)} = -2$ and so the theorem is proved. □

We remark that $m = 4$ is an exceptional value because by (5.12) $m_{\phi(4)-2} - m_{\phi(4)} = 0$. Note that inequalities (5.19) and (5.20) cannot be improved. From Table 5.31, $\text{ord}_2(m_{\phi(18)-2} - m_{\phi(18)}) = 5 = \phi(18) - 1$ and so (5.19) is sharp. It follows from (5.12) that $m_{\phi(16)-2} - m_{\phi(16)} = 2 \cdot 3^3$ and so $\text{ord}_3(m_{\phi(16)-2} - m_{\phi(16)}) = 3 = \phi(16)/2 - 1$ and so (5.20) is sharp. Note that (5.19) and (5.20) are both sharp at $m = 6$: (5.9) and (5.12) imply that $m_{\phi(6)-2} = 3$.

6. Regular $G_m$ Actions of Type $\mathbb{II}_0$

The purpose of this section is to prove a theorem which is equivalent to Theorem E and in which Theorem 3.21, Corollary 4.14 and Theorem 5.18 are used to show that Type $\mathbb{II}_0$ actions are rigid if $n < \phi(m) + 4$.

**Theorem 6.1.** Suppose that $M^{2n}$ is a PL cohomology complex projective $n$-space which admits a regular locally linear $PL G_m$ action of Type $\mathbb{II}_0$. If $m > 2$, $m \neq 4, 10$, and $n < \phi(m) + 4$, then the action is standard. If $m > 2$ and $m$ is odd and $n < \phi(m) + 4$, then the action is standard and the Pontrjagin class of $M^{2n}$ is standard.

**Proof.** Let $d$ be the degree of the fixed codimension-2 locally flat submanifold of $M^{2n}$. We assert that the action is standard, that is $d = \pm 1$. We may assume that $n = \phi(m) + 2$ or $\phi(m) + 3$ by Theorem 3.21 and so we are in a position to use Corollary 4.14. If $p$ is a prime, then $\text{ord}_p(n!) = (n - \lambda_p(n))(p - 1)^{-1}$, where $\lambda_p(n)$ is the sum of the coefficients in the $p$-adic decomposition of $n$. It follows that if $p$ is an odd prime, then $\text{ord}_p(n!) \leq n/2 - 1$, $n$ even and $\text{ord}_p(n!) \leq [n/2]$, $n$ odd. Therefore if $m > 2$ and $p$ is an odd prime, $\text{ord}_p f(\phi(m) + 2) \leq \phi(m)/2$ and $\text{ord}_p f(\phi(m) + 3) \leq \phi(m)/2 + 1$. If $m > 2, m \neq 4, 10$, these inequalities plus Corollary 4.14, (5.19) and (5.20) imply that $d = \pm 1$.

To see that $m > 2$ and $m$ odd and $n < \phi(m) + 4$ implies that the Pontrjagin class of $M^{2n}$ is standard, we note that we may assume that $n = \phi(m) + 2$ or $\phi(m) + 3$ by Theorem 3.21. We begin with $n = \phi(m) + 2$. Since $m$ is odd, $g^s$ is the identity on $H^{\phi(m)+2}(M, Q)$ so we have $b_-(\alpha) = 0$ for some $\alpha \in \{s_j/m : 1 \leq j \leq m - 1, (j, m) = 1\}$ by Theorem 3.19. Since $d = \pm 1$ and $s_{[n/2]}(d) = d^n$ ([13], p. 657), $b_-(\alpha) = 0$ for $n = \phi(m) + 2$ means

$$
\sum_{k=1}^{\phi(m)/2} (s_k(\pm 1) - 1)\alpha^{2k-2} = 0.
$$

(6.2)

It follows from the second assertion in Proposition 3.11 and (6.2) that $s_k(\pm 1) = 1$, $1 \leq k \leq \phi(m)/2$ and so $s_k(\pm 1) = 1$, $1 \leq k \leq \phi(m)/2$ since $s_k(d)$ is an even function of $d$ if $n$ is even ([13], Formulas (2.2) and (2.5)) and the Pontrjagin class of $M^{2n}$ is standard ([13], Proposition 2.8). If $n = \phi(m) + 3$, we need only assume $m > 2, m \neq 4, 10$, and so $d = \pm 1$. If $d = \pm 1$, then $b_+(\alpha) = 0$, for some $\alpha \in \{s_j/m : 1 \leq j \leq m - 1, (j, m) = 1\}$, by Theorem 3.19, and so $s_{[n/2]}(d) = d^n$ ([13], p. 657) implies that $b_+(\alpha) = 0$ for $n = \phi(m) + 3$ means

$$
\sum_{k=1}^{\phi(m)/2} (s_k(\pm 1) \mp 1)\alpha^{2k-2} = 0.
$$

(6.3)

Therefore the second assertion in Proposition 3.11 and (6.3) imply that $s_k(1) = 1$, $1 \leq k \leq \phi(m)/2$ since $s_k(d)$ is an odd function of $d$ if $n$ is odd ([13], Formulas (2.2) and (2.5)) and so the Pontrjagin class of $M^{2n}$ is standard ([13], Proposition 2.8). □

The exceptional case $m = 4$ is one in which Corollary 4.14 does not detect $d$ because $m_{\phi(4)-2} - m_{\phi(4)} = 0$. If $m = 10$, it follows from Corollary 4.14 and Table 5.31 that $d = \pm 1$ or $d = \pm 3$. 

References