JOURNAL OF ALGEBRA 25, 519–521 (1973)

Semiprime Rings with Krull Dimension are Goldie

ROBERT GORDON AND J. C. ROBSON

University of Utah and University of Leeds

Received March 17, 1972

The main result of this note is that a semiprime ring with Krull dimension is a right order in a semisimple artinian ring.

In [3] this theorem is cast in the more general context of rings with enough monoform right ideals. In fact, semiprime right Goldie rings are precisely those semiprime rings with finite uniform dimension and enough monoform right ideals.

The *Krull dimension*, $K \dim M$, of a module M is an ordinal and was defined originally by Rentschler and Gabriel [8] for finite ordinals, then extended to include transfinite ordinals by Krause [7], (or see [3]). Examples of Jategaonkar [6] show that there are rings of arbitrary Krull dimension (see [3, Example 10.3]). Also, any noetherian module has Krull dimension [1].

Modules with Krull dimension are, by definition, analogous to artinian modules; in fact, the modules of Krull dimension zero are precisely nonzero artinian modules. The modules corresponding in this analogy with simple modules are critical, or restricted, modules as defined in [5] and [2]. This suggests the following lemmas, the proofs of which are straightforward.

LEMMA 1. *Every nonzero module with Krull dimension has a critical submodule.*

LEMMA 2. *Every nonzero endomorphism of a critical module is a monomorphism.*

THEOREM. *A semiprime ring with Krull dimension is a right order in a semisimple artinian ring.*

Proof. Let R be a semiprime ring with Krull dimension. Since it is known (see, for example, [7, Proposition 4]) that R has finite uniform dimension, it is enough to show that the singular ideal $Z(R)$ is zero. But if $Z(R) \neq 0$ then $Z(R)$ contains a critical submodule C and, since $C^2 \neq 0$, there is an element $c \in C$ with $cC \neq 0$. Thus the left multiplication map $c : C \to C$ is, by Lemma 2, a monomorphism. This means that the right annihilator of c intersects C trivially, against the definition of $Z(R)$.

The effect of the presence of critical right ideals is further brought out by the next result which is a special case of [3, Theorem 3.7].

COROLLARY 1. *A ring which is a direct sum of critical right ideals is a split extension of a nilpotent ring by a finite direct product of prime right Goldie rings.*

Proof. Order these critical right ideals by their Krull dimensions. One can then apply the theorem together with [4, Section 1, Corollaries 3 and 4].

COROLLARY 2. *Any ring with Krull dimension has the ascending chain condition on semiprime ideals.*

Proof. The proof for prime ideals is a straightforward consequence of the theorem, using the argument in [8, g]. The extension to semiprime ideals is immediate from the fact [3, Theorem 7.7] that a ring with ascending chain condition for primes also has ascending chain condition for finite intersections of primes.

Corollary 2 enables one to define an ordinal *classical Krull dimension* of any ring with Krull dimension, following Gabriel [1, p. 425, Corollaire 2] and Krause [7, Definition 11]. The method of [8, g] yields the next result.

COROLLARY 3. *If R is a ring with Krull dimension α then the classical Krull dimension of R is at most α.*

Jategaonkar's examples, mentioned above, show that there are rings with arbitrary classical Krull dimension.

Vis-à-vis the theorem we point out that semiprime rings with Krull dimension need not be right noetherian. For example, a rank 2 valuation ring R whose value group is $\mathbf{Z} \oplus \mathbf{Z}$, ordered lexicographically, has Krull dimension 2. Furthermore R has a factor ring of Krull dimension 1 which is Goldie but not noetherian, and another factor ring of Krull dimension 1 which is not Goldie, failing to have ascending chain condition on annihilators; see [3, Examples 10.7 and 10.8] for a more complete description of these rings.

On the other hand, a semiprime right Goldie ring need not have a Krull dimension. The polynomial ring in infinitely many commuting indeterminates over a field is such an example. This is one consequence of

PROPOSITION. *If $R[x]$ has Krull dimension then R is right noetherian.*

Proof. If not, then R has an infinite strictly ascending chain of right ideals, $I_0 \subset I_1 \subset \cdots$. Consider the right ideals

$$A = I_1 + I_2 x + I_3 x^2 + \cdots \qquad \text{and} \qquad B = I_0 + I_1 x + I_2 x^2 + \cdots$$

of $R[x]$. Since $Ax \subsetneq B$,

$$A/B \cong I_1/I_0 \oplus I_2/I_1 \oplus \cdots$$

as $R[x]$-modules, contradicting the assumption that $R[x]$ has Krull dimension.

In contrast to the negative property of Krull dimension afforded by this proposition, there is a positive one: the Krull dimension of a ring is a Morita invariant. This follows from the fact that the endomorphism ring of a finitely generated projective module over a ring R with Krull dimension has Krull dimension at most that of R. The hypothesis of projectivity is not redundant since in [3, Section 10] an example is given of a finitely generated module over an artinian ring whose endomorphism ring does not have Krull dimension.

In a similar vein, there is an example of a critical right ideal in a prime noetherian ring whose endomorphism ring does not have Krull dimension. There is also an example of a critical module over a domain of Krull dimension 1 whose endomorphism ring is a proper subfield of the endomorphism ring of its injective envelope. This provides an example of a critical module which is not compressible; see [2].

References

1. P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
2. A. W. Goldie, Properties of the idealiser, *in* "Ring Theory" (R. Gordon, Ed.), Academic Press, New York, 1972.
3. R. Gordon and J. C. Robson, Krull dimension, *Amer. Math. Soc. Mem.*, to appear.
4. R. Gordon and L. W. Small, Piecewise domains, *J. Algebra* **23** (1972), 553–564.
5. R. Hart, Krull dimension and global dimension of simple Ore-extensions, *Math. Zeitschr.* **121** (1971), 341–345.
6. A. V. Jategaonkar, A counter-example in ring theory and homological algebra, *J. Algebra* **12** (1969), 418–440.
7. G. Krause, On the Krull-dimension of left noetherian left Matlis-rings, *Math. Zeitschr.* **118** (1970), 207–214.
8. R. Rentschler and P. Gabriel, Sur la dimension des anneaux et ensembles ordonnés, *C. R. Acad. Sci. Paris* **265** (1967), 712–715.