On the Laplacian coefficients of unicyclic graphs with prescribed matching number

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Let \( \phi(G, \lambda) = \sum_{k=0}^{n} (-1)^k c_k(G) \lambda^{n-k} \) be the characteristic polynomial of the Laplacian matrix of a graph \( G \) of order \( n \). We give some transformations of connected graphs that decrease all Laplacian coefficients \( c_k(G) \), we then derive the unicyclic graphs with the minimum Laplacian coefficients in the set of all connected unicyclic graphs with prescribed order and matching number. Furthermore, we determine the unique connected unicyclic graph with the minimal Laplacian coefficients among all connected unicyclic graphs of order \( n \) except \( S_5' \), where \( S_5' \) is the unicyclic graph obtained from the \( n \)-vertex star \( S_n \) by joining two of its pendent vertices with an edge.

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1. Introduction

Throughout the paper, we only consider graphs without loops and multiedges. Let \( G = (V(G), E(G)) \) be a simple graph with \( n = \sigma(G) = |V(G)| \) vertices and \( e(G) = |E(G)| \) edges. Let \( A(G) \) and \( D(G) \) denote its adjacency matrix and diagonal matrix of vertex degrees, respectively. Then the Laplacian matrix of \( G \) is defined to be the matrix \( L(G) = D(G) - A(G) \), while the Laplacian polynomial of \( G \) is the characteristic polynomial of \( L(G) \), \( \phi(G, \lambda) = \det(\lambda I_n - L(G)) \). Let \( c_k(G) \) \((0 \leq k \leq n)\) be the absolute values of the coefficients of \( \phi(G, \lambda) \), so that

\[
\phi(G, \lambda) = \sum_{k=0}^{n} (-1)^k c_k(G) \lambda^{n-k}.
\]

(1.1)

It is well-known that

\[
c_0(G) = 1, \quad c_n(G) = 0, \quad c_1(G) = 2e(G), \quad c_{n-1}(G) = n\tau(G),
\]

(1.2)

where \( \tau(G) \) is the number of spanning trees of \( G \) (see, e.g. [14]). If \( G \) is a tree the Laplacian coefficient \( c_{n-2}(G) \) is equal to the Wiener index of \( G \), which is the sum of all distances between unordered pairs of vertices of \( G \) and is considered as one of the most used topological indices with high correlation with many physical and chemical properties of molecular compounds. The Wiener index was introduced in 1947 and investigated by many chemists and mathematicians (see, e.g. [2] for more detail). For recent results and applications of the Wiener index see [2,6,19,3].

Let \( G \) and \( H \) be two graphs of order \( n \). We write \( G \leq H \) if \( c_k(G) \leq c_k(H) \) for all \( 0 \leq k \leq n \), and we write \( G < H \) if \( G \leq H \) and \( c_k(G) < c_k(H) \) for some \( 0 \leq k \leq n \). Recently, the Laplacian coefficients of graphs have attracted many scholars' attention. Gutman and Pavlović [5] proved the following result.
Theorem 1.1. Let $P_n$ and $S_n$ be the path and star of order $n$, respectively. Let $T$ be a tree of order $n$ ($n \geq 5$), different from $P_n$ and $S_n$. Then for $k = n - 2$, $n - 3$,
\[ c_k(S_n) < c_k(T) < c_k(P_n). \]

Let $S(G)$ be subdivision of a graph $G$ obtained by inserting a new vertex of degree two on each edge of $G$, and let $m_k(G)$ be the number of matchings of $G$ containing exactly $k$ edges. Zhou and Gutman [21] proved that for every tree $T$ with $n$ vertices,
\[ c_k(T) = m_k(S(T)), \quad 0 \leq k \leq n. \]

Using this correspondence, Zhou and Gutman [21] demonstrated a conjecture proposed by Gutman and Pavlović [5], namely they obtained the following result.

Theorem 1.2. Let $T$ be a tree of order $n$, different from $P_n$ and $S_n$. Then for all $k = 2, 3, \ldots, n - 2$,
\[ c_k(S_n) < c_k(T) < c_k(P_n). \]

According to (1.3), Mohar [15] gave two transformations of trees making Laplacian coefficients monotone and provided a new proof and a strengthening of Theorem 1.2 by means of the two transformations. Zhang et al. [20] answered some problems on ordering trees with the Laplacian coefficients proposed by Mohar [16] and determined the several new minimal trees among all $n$-vertex trees under the partial order $\preceq$. Ilić [8] determined the minimal trees in the set of all $n$-vertex trees with fixed diameter under the partial order $\preceq$. Ilić [9] characterized the minimal trees in the set of all $n$-vertex trees with fixed matching number under the partial order $\preceq$. Ilić and Ilić [10] characterized the minimal trees in the sets of all $n$-vertex trees with fixed pendant vertex number or 2-degree vertex number under the partial order $\preceq$.

The Laplacian coefficients $c_k(G)$ of a graph $G$ can be expressed in terms of subtree structures of $G$ by the following result of Kelmans and Chelnokov [12]. Let $F$ be a spanning forest of $G$ with components $T_i$, $i = 1, 2, \ldots, k$, having $o(T_i)$ vertices, and set
\[ \gamma_i(F) = \prod_{t=1}^{k} o(T_i). \]

Theorem 1.3. Let $\Theta_k(G)$ be the set of all spanning forests of a graph $G$ on order $n$ with exactly $k$ components. Then the Laplacian coefficient $c_{n-k}(G)$ is expressed by
\[ c_{n-k}(G) = \sum_{F \in \Theta_k(G)} \gamma(F). \]

Using Theorem 1.3, Stevanović and Ilić [18] generalized the two transformation of [15] to general graphs and obtained the following results.

Theorem 1.4. Let $C_n$ be the cycle of order $n$ and let $S'_n$ be the graph obtained from $S_n$ by joining two of its pendant vertices with an edge. Let $G$ be a connected unicyclic graph of order $n$, different from $C_n$ and $S'_n$. Then for $k = 2, 3, \ldots, n - 1$,
\[ c_k(S'_n) < c_k(G) < c_k(C_n); \]
and for $k = 2, 3, \ldots, n - 2$,
\[ c_k(G) > c_k(S'_n). \]

Let $\mu_1(G) \geq \mu_{n-1}(G) \geq \cdots \geq \mu_2(G) \geq \mu_1(G) = 0$ be all eigenvalues of $L(G)$ of a graph $G$. Then the Laplacian-like energy of $G$, LEL for short, is defined as follows:
\[ \text{LEL}(G) = \sum_{k=2}^{n} \sqrt{\mu_k}. \]

This concept was introduced in [13], where it was demonstrated that it has similar features as molecular graph energy defined by Gutman [4]. Stevanovic [17] presented a connection between the Laplacian-like energy invariant and the Laplacian coefficients, where its corrected proof was recently presented by Ilić et al. in [11].

Theorem 1.5. Let $G$ and $H$ be two graphs with $n$ vertices. Then $\text{LEL}(G) \leq \text{LEL}(H)$ if $G \preceq H$, and $\text{LEL}(G) < \text{LEL}(H)$ if $G < H$.

Motivated by the results in [17] concerning the minimal Laplacian coefficients and the minimal Laplacian-like energy of $n$-vertex unicyclic graphs, this paper will characterize the $n$-vertex unicyclic graphs with fixed matching number which simultaneously minimize all Laplacian coefficients, and consecutively Laplacian-like energy.

Throughout this paper, we use the following notations. Let $M(G)$ denote a maximum matching of a simple graph $G$. For a nonpendent edge $uv$ of $G$, let $E^u_{uv} = E^u_{uv}$ denote the set of all edges incident to $u$ except $uv$ in $G$. Let $U(n, i)$ denote the set of all connected unicyclic graphs with fixed order $n$ and matching number $i$, and let $U(n)$ denote the set of all connected unicyclic graphs with fixed order $n$. 
The rest of the article is organized as follows. In Section 2 we present some new transformations of graphs that decrease all Laplacian coefficients. In Section 3 we determine the minimal connected unicyclic graphs in \( U(n, t) \) under the partial order \( \preceq \). In Section 4 we determine the minimal connected unicyclic graphs in \( U(n) - \{ S'_n \} \) under the partial order \( \preceq \).

2. Some transformations of graphs

In this section we apply the idea from [18] to present some new transformations of graphs that decrease all Laplacian coefficients.

Let \( N_G(v) \) denote the adjacent vertex set of a vertex \( v \) of a graph \( G \) and let \( d_G(v) \) denote the degree of \( v \) in \( G \). An edge of \( G \) is called a pendent edge if it is incident to a vertex of degree 1. A path \( v_0v_1 \cdots v_k \) of \( G \) is called a pendent path of length \( k \) at \( v_0 \) if \( d_G(v_0) \geq 3 \), \( d_G(v_1) = \cdots = d_G(v_{k-1}) = 2 \) and \( d_G(v_k) = 1 \).

**Definition 2.1.** Let \( G \) be a simple connected graph with \( n \) vertices, and let \( uv \) be a nonpendent edge of \( G \) not contained in cycles of length 3. Let \( G_{uv} \) denote the graph obtained from \( G \) in the following way:

1. Delete the edge \( uv \);
2. Identify \( u \) and \( v \), and denote the new vertex by \( w \);
3. Add a pendent edge \( w'w \) to \( w \).

We say that \( G_{uv} \) is a 1-edge-growing transform of \( G \) at \( uv \), where \( G \) and \( G_{uv} \) are shown in Fig. 1 when \( uv \) is a cut edge of \( G \).

**Remark 2.1.** Let \( G \) and \( G_{uv} \) be the two graphs presented in Definition 2.1. Then we have that \( |M(G_{uv})| = |M(G)| \) if \( M(G) \cap E^u_w = \emptyset \) or \( M(G) \cap E^w_v = \emptyset \).

Indeed we may assume, without loss of generality, that \( M(G) \cap E^u_w = \emptyset \). Then there is an edge \( e \in E_{uv} \cup \{ uv \} \) such that \( e \in M(G) \). So \( M(G_{uv}) = (M(G) - \{ e \}) \cup \{ w'w \} \). It follows that \( |M(G_{uv})| = |M(G)| \). \( \square \)

**Theorem 2.2.** Let \( G \) and \( G_{uv} \) be the two graphs presented in Definition 2.1. Then

\[
c_k(G) \geq c_k(G_{uv}), \quad k = 0, 1, \ldots, n.
\]

with equality if and only if \( k \in \{ 0, 1, n - 1, n \} \) when \( uv \) is a cut edge or \( k \in \{ 0, 1, n \} \) otherwise.

**Proof.** From Eq. (1.2), we have that

\[
c_0(G) = 1 = c_0(G_{uv}), \quad c_n(G) = 0 = c_n(G_{uv}), \quad c_1(G) = e(G) = e(G_{uv}) = c_1(G_{uv}).
\]

If \( uv \) is a cut edge of \( G \), then each spanning tree of \( G \) contains the edge \( u \) and each spanning tree of \( G_{uv} \) contains the edge \( w'w \). Thus \( \tau(G) = 1(G_{uv}) \). It follows that

\[
c_{n-1}(G) = n \tau(G) = n \tau(G_{uv}) = c_{n-1}(G_{uv}).
\]

If \( uv \) is not a cut edge of \( G \), then \( uv \) is contained in some cycles of lengths at least 4. Let \( J \) be the set of all spanning trees containing the edge \( uv \) in \( G \). Then \( \Theta_1(G) - J \) is the set of all spanning trees containing no \( uv \) in \( G \). Since \( uv \) is not a cut edge of \( G \), we have \( \Theta_1(G) - J \neq \emptyset \). Note that each spanning tree of \( G_{uv} \) can be obtained from some \( F \in J \) by a 1-edge-growing of \( F \) at \( uv \). So \( |\Theta_1(G_{uv})| = |J| \) and

\[
\tau(G) = |\Theta_1(G)| = |J| + |\Theta_1(G) - J| > |J| = |\Theta_1(G_{uv})| = \tau(G_{uv}).
\]

It follows that

\[
c_{n-1}(G) = n \tau(G) > n \tau(G_{uv}) = c_{n-1}(G_{uv}).
\]

Now assume that \( 2 \leq k \leq n - 2 \) and consider the coefficient \( c_{n-k}(G) \). Let \( F' \) be an arbitrary spanning forest of \( G_{uv} \) with exactly \( k \) components. Let \( T' \) be the tree from \( F' \) containing \( w \). Write

\[
N_G(u) = \{ u, u_1, u_2, \ldots, u_s \}, \quad N_G(v) = \{ u, v_1, v_2, \ldots, v_t \},
\]

\[
E_1 = \{ uu_i : uu_i \in E(T'), \ 1 \leq i \leq s \}, \quad E_2 = \{ vv_i : vv_i \in E(T'), \ 1 \leq i \leq t \},
\]

\[
V'_1 = \{ x : x \in V(T'), \ there \ is \ an \ u_i \ such \ that \ uu_i \in E(T') \ and \ x \ is \ connected \ with \ u_i \ in T' - w \},
\]

\[
V'_2 = \{ x : x \in V(T'), \ there \ is \ an \ u_i \ such \ that \ uv_i \in E(T') \ and \ x \ is \ connected \ with \ u_i \ in T' - w \}.
\]

Then \( V'_1 \cap V'_2 = \emptyset \).
Theorem 1.1

Let \( u \) and \( v \) be two neighboring vertices on a cycle in an unicyclic graph \( G \) such that \( u \) has degree \( t + 1 \) and \( t \) pendent edges incident with \( u \). Let \( \sigma(G, u) \) be the graph obtained from \( G \) by a \( 1 \)-edge-growing transform of \( G \) at \( u \).

Fig. 2. Diagrams of \( G \) and \( G_{uv} \) for the cut edge \( uv \).

If \( T' \) contains the edge \( w'w \), let
\[
T = (T' - \{w, w'\}) \bigcup E_1 \bigcup E_2 \bigcup \{uv\},
\]
\[
F = (F' - \{T'\}) \bigcup \{T\},
\]
then \( F \) is a spanning forest of \( G \) corresponding to \( F' \) with \( k \) components and it is obvious that \( \gamma(F) = \gamma(F') \).

If \( T' \) does not contain the edge \( w'w \), let
\[
T_1 = (T' - (V'_1 \bigcup \{w\})) \bigcup E_1, \quad T_2 = (T' - (V'_2 \bigcup \{w\})) \bigcup E_2,
\]
\[
F = (F' - \{T', w'\}) \bigcup \{T_1, T_2\},
\]
then \( F \) is a spanning forest of \( G \) corresponding to \( F' \) with \( k \) components. Write
\[
a = o(T' - V'_1), \quad b = o(T' - V'_2).
\]
Then \( o(T') = a + b - 1 \), \( o(T_1) = a \) and \( o(T_2) = b \). Therefore, we get that
\[
\gamma(F') = o(T'c = (a + b - 1)c,
\]
\[
\gamma(F) = o(T_1)o(T_2)c = abc,
\]
where \( c \) is the product of orders of all components except \( T_1 \) and \( T_2 \) in \( F \). So
\[
\gamma(F) - \gamma(F') = c(a - 1)(b - 1) \geq 0.
\]
Since \( uv \) is a nonpendent edge of \( G \) which is not contained in cycles of length 3, we have \( s \geq 1 \) and \( t \geq 1 \). These indicate that there must be a spanning forest \( F' \) of \( G_{uv} \) such that \( a \geq 2 \) and \( b \geq 2 \), i.e. \( \gamma(F') > \gamma(F) \). It is easy to see that the correspondence from \( F' \) to \( F \) above is an injection. So by Theorem 1.1 we get that
\[
c_{n-k}(G) = \sum_{F' \in \mathcal{F}_k(G_{uv})} \gamma(F') - \sum_{F' \in \mathcal{F}_k(G_{uv})} \gamma(F') = c_{n-k}(G_{uv}).
\]
By setting \( k \mapsto n - k \), we complete the proof. \( \square \)

Remark 2.3. Stevanović and Ilić [18] call \( \sigma(G, v) \) and \( \tau(G, u, v) \) a \( \sigma \)-transform and a \( \tau \)-transform of \( G \), respectively. They proved that both the \( \sigma \)-transform and the \( \tau \)-transform of \( G \) decrease all Laplacian coefficients \( c_k(G) \) for \( 2 \leq k \leq n - 2 \). It is easy to see that these results are special cases of Theorem 2.2.

Definition 2.2. Let \( G \) be a simply connected graph of order \( n \) with at most a cycle, and let \( uv \) be an edge of \( G \) such that it is not contained in cycles of length 3, \( d_{c}(u) \geq 3 \), \( d_{c}(v) \geq 3 \) and \( uu' \) is apendent edge. Let \( G_{uv} \) be the graph obtained from \( G \) in the following way:

1. Delete the edge \( uv \) and vertex \( u' \);
2. Identify \( u \) and \( v \), and denote the new vertex by \( w \);
3. Add a pendent path \( w'w' \) to the vertex \( w \).

We say that \( G_{uv} \) is a 1I-edge-growing transform of \( G \) at \( uv \), where \( G \) and \( G_{uv} \) are shown in Fig. 2 when \( uv \) is a cut edge of \( G \).
Remark 2.4. Let $G$ and $G_{uv}$ be the two graphs presented in Definition 2.2. Then we have that $|M(G)| \leq |M(G_{uv})| \leq |M(G)| + 1$.

Indeed we may assume that $N_G(v) = \{u, v_1, v_2, \ldots, v_t\}$, $N_G(u) = \{v, u', u_1, u_2, \ldots, u_t\}$. Since $uu'$ is a pendant edge of $G$, we also may assume that $uu' \in M(G)$.

If $E_{uv} \cap M(G) \neq \emptyset$, assume that $E_{uv} \cap M(G) = \{v_j\}$, then

$$M(G_{uv}) = (M(G) - \{v_j\}) \cup \{u_j\}.$$

Next assume that $E_{uv} \cap M(G) = \emptyset$. If for each $j = 1, 2, \ldots, t$, there is an edge $e_j$ incident to $u_j$ such that $e_j \in M(G)$, then $M(G_{uv}) = (M(G) - \{uu'\}) \cup \{w_j, w'j\}$. If there is an $j (1 \leq j \leq t)$ such that $uu_j$ is a pendant edge or all edges incident to $u_j$ are not in $M(G)$, then $M(G_{uv}) = (M(G) - \{uu'\}) \cup \{w_j, w'j\}$.

By the discussion above we get that $|M(G)| \leq |M(G_{uv})| \leq |M(G)| + 1$. □

Theorem 2.5. Let $G$ and $G_{uv}$ be the two graphs presented in Definition 2.2. Then

$$c_k(G) \geq c_k(G_{uv}), \quad k = 0, 1, \ldots, n,$$

with equality if and only if for either $k \in \{0, 1, n - 1, n\}$ when $uv$ is a cut edge or $k \in \{0, 1, n\}$ otherwise.

Proof. For $k \in \{0, 1, n - 1, n\}$, the proof is similar to Theorem 2.2. Thus now suppose that $2 \leq k \leq n - 2$ and consider the coefficient $c_{n-k}(G)$. Let $F'$ be any spanning forest of $G_{uv}$ with exactly $k$ components and let $T'$ be the tree from $F'$ containing $w$. Write

$$N_G(u) = \{u, u', u_1, u_2, \ldots, u_t\}, \quad N_G(v) = \{v, u_1, v_2, \ldots, v_t\},$$

$$E_1 = \{uu_1 : uu_1 \in E(T'), \ 1 \leq i \leq s\} \cup \{uu_1 : uu_1 \in E(T'), \ 1 \leq i \leq t\},$$

$$E_2 = \{vv_1 : vv_1 \in E(T'), \ 1 \leq i \leq t\},$$

$$V'_1 = \{x : x \in V(T'), \text{ there is an } v_i \text{ such that } vv_1 \in E(T') \text{ and } x \text{ is connected with } v_i \text{ in } T' - w\},$$

$$V'_2 = \{x : x \in V(T'), \text{ there is an } u_i \text{ such that } uu_1 \in E(T') \text{ and } x \text{ is connected with } u_i \text{ in } T' - w\}.$$

Then $\bigcap V'_i = \emptyset$, and from $d_C(u) \geq 3$ and $d_C(v) \geq 3$, we have $s \geq 1$ and $t \geq 2$.

If $F'$ contains the edge $ww'$, let

$$T = (T' - \{w, w', u\}) \cup E_1 \cup E_2 \cup \{uv\},$$

$$F = (F' - \{T'\}) \cup \{T\},$$

then $F$ is a spanning forest of $G$ corresponding to $F'$ with exactly $k$ components and it is easy to see that $\gamma(F) = \gamma(F')$.

Next assume that $F'$ does not include the edge $ww'$. Let $c(F')$ be the product of orders of all components not containing vertices $w$, $w'$ or $u'$ in $F'$. $a = o(T' - V'_1)$ and $b = o(T' - V'_2)$. Then $a \geq 1, b \geq 1$ and $o(T') = a + b - 1$.

If $F'$ does not include the edge $ww'$, then denote such $F'$ by $F'_1(a, b)$ and put

$$T_1 = \left( T' - \left( V'_1 \cup \{w\} \right) \right) \cup E_1, \quad T_2 = \left( T' - \left( V'_2 \cup \{w\} \right) \right) \cup E_2,$$

$$F_1(a, b) = (F'_1(a, b) - \{T', w'\}) \cup \{T_1, T_2\}.$$ 

So $F_1(a, b)$ is a spanning forest of $G$ corresponding to $F'_1(a, b)$ with exactly $k$ components, $o(T_1) = a$ and $o(T_2) = b$. It is easy to see that

$$\gamma(F_1(a, b)) - \gamma(F'_1(a, b)) = o(T_1)o(T_2)c(F'_1(a, b)) - o(T')c(F'_1(a, b))$$

$$= (a - 1)(b - 1)c(F'_1(a, b)) \geq 0. \quad (2.1)$$

According to the assumptions of $G$, there must be a spanning forest $F'_1(a, b)$ of $G_{uv}$ with $a \geq 2$ and $b \geq 2$ such that Eq. (2.1) is positive.

Now suppose that $F'$ includes the edge $ww'$ and denote such $F'$ by $F'_2(a, b)$.

If $b \geq 2$, then let

$$T_1 = \left( T' - \left( V'_1 \cup \{w\} \right) \right) \cup E_1, \quad T_2 = \left( T' - \left( V'_2 \cup \{w\} \right) \right) \cup E_2,$$

$$F_2(a, b) = (F'_2(a, b) - \{T', w'\}) \cup \{T_1, T_2\}.$$ 

So $F_2(a, b)$ is a spanning forest of $G$ corresponding to $F'_2(a, b)$ with exactly $k$ components, $o(T_1) = a + 1$ and $o(T_2) = b$. It is easy to see that

$$\gamma(F_2(a, b)) - \gamma(F'_2(a, b)) = o(T_1)o(T_2)c(F'_2(a, b)) - 2o(T')c(F'_2(a, b))$$

$$= (a - 1)(b - 2)c(F'_2(a, b)) \geq 0. \quad (2.2)$$
According to the assumptions of $G$, there must be a spanning forest $F'_1(a, b)$ of $G_{uv}$ with $a \geq 2$ and $b \geq 3$ such that Eq. (2.2) is positive.

Next assume that $b = 1$. Since $t \geq 2$ and $G$ is a tree or connected unicyclic graph, there is a component $H$ of $F'_2(a, 1)$ except $T'$ such that it contains some vertex $v_1$ and has the minimal order $p$. Let $T''$ be the tree obtained from $T'$ and $H$ by joining $v_1$ and $w$ with an edge. Set

$$F'_1(a, p + 1) = (F'_2(a, 1) - \{T', H, w'u\}) \cup \{T'', w', u'\}.$$ 

Then $F'_1(a, p + 1)$ is a spanning forest of $G_{uv}$ with exactly $k$ components corresponding to $F'_2(a, 1)$ and the correspondence from $F'_2(a, 1)$ to $F'_1(a, p + 1)$ is an injection. It is obvious that $F'_2(a, 1) - \{T', H, w'u\} = F'_1(a, p + 1) - \{T'', w', u'\}$. Let $c$ denote the products of orders of all components in $F'_1(a, p + 1) - \{T'', w', u'\}$. Then

$$\gamma(F'_1(a, p + 1)) = o(T'')c = (a + p)c,$$
$$\gamma(F'_2(a, 1)) = 2o(T)o(H)c = 2apc.$$

Let

$$T_1 = (T' - w) \cup E_1, \quad F_2(a, 1) = (F'_2(a, 1) - \{T', w'u\}) \cup \{T_1, v\}$$

$$\tilde{T}_1 = (T' - w) \cup (E_1 - uu'), \quad \tilde{T}_2 = H + v_lv,$$

$$F_1(a, p + 1) = (F'(a, p + 1) - \{T'', w'\}) \cup \{\tilde{T}_1, \tilde{T}_2\}.$$ 

Then $F_2(a, 1)$ and $F_1(a, p + 1)$ are two spanning forests of $G$ with exactly $k$ components corresponding to $F'_2(a, 1)$ and $F'_1(a, p + 1)$, respectively. It is easy to see that

$$\gamma(F_1(a, p + 1)) = o(T_1)o(T_2)c = a(p + 1)c,$$
$$\gamma(F_2(a, 1)) = o(T_1)o(H)c = (a + 1)pc.$$

So we have that

$$[\gamma(F_1(a, p + 1)) + \gamma(F_2(a, 1))] - [\gamma(F'_1(a, p + 1)) + \gamma(F'_2(a, 1))] = 0.$$ 

It is easy to see that the correspondence from $F'$ to $F$ defined above is an injection. By summing over possible subsets of $k$-components spanning forests of $G_{uv}$, from Theorem 1.1 we get that

$$c_{n-k}(G) = \sum_{F \in \mathcal{F}_{k}(G)} \gamma(F) > \sum_{F \in \mathcal{F}_{k}(G_{uv})} \gamma(F) = c_{n-k}(G_{uv}).$$ 

By setting $k \mapsto n - k$, we complete the proof. \(\square\)

Let $G$ be a connected graph and let $u$ be a vertex of $G$. Let $G_{u,v}^{t}$ be the graph obtained from $G$ in the following way: Add $t$ pendent edges at $u$, then join $u$ and $v$ of another star with $s$ pendent vertices with an edge.

**Theorem 2.6.** Let $st \neq 0$ and let $G_{u,v}^{t}$ be the graph of order $n$ defined above. Then for all $i$ such that $\max\{1, s - t + 1\} \leq i \leq s$, we have that

$$c_k(G_{u,v}^{t}) \geq c_k(G_{u,v}^{t+1}), \quad k = 0, 1, 2, \ldots, n,$$

with equality if and only if $k$ is in $\{0, 1, n - 1, n\}$.

**Proof.** For $k \in \{0, 1, n - 1, n\}$, it is easy to see that $c_k(G_{u,v}^{t}) = c_k(G_{u,v}^{t+1})$.

Now assume $2 \leq k \leq n - 2$ and consider the coefficient $c_{n-k}(G_{u,v}^{t+1})$. Write

$$U = V(G_{u,v}^{t+1}) - V(G_{u,v}^{t}).$$ 

Let $F'$ be any spanning forest of $G_{u,v}^{t+1}$ with exactly $k$ components. Let $T'$ be the tree from $F'$ containing the vertex $u$. Let $a$ be the order of $T' - U$ and let $b + 1$ be the number of components in the vertex-induced subgraph $F'[V(T') \cup U]$. Then $a \geq 1$ and $0 \leq b \leq s + t + 1$.

If $b = 0$, then let $T$ be the tree obtained from $T'$ by deleting $i$ pendent edges at $u$ and adding $i$ new pendent edges at $v$. So $F = (F' - \{T'\}) \cup \{T\}$ is a spanning forest of $G_{u,v}^{t}$ with exactly $k$ components and, obviously, $\gamma(F) = \gamma(F')$.

If $b = s + t + 1$, then each vertex of $U$ is a component of $F'$. Take $F = F'$. So $F$ is a spanning forest of $G$ with exactly $k$ components and $\gamma(F) = \gamma(F')$.

Now assume that $1 \leq b \leq s + t$. Consider the subset $I^*$ of those spanning forests of $G_{u,v}^{t+1}$ with exactly $k$ components which coincide on $G_{u,v}^{t}$ without vertices from $\{u\} \cup U$ and for which numbers $a$ and $b$ are fixed. For each $F' \in I^*$, let $p$ be the number of pendent vertices adjacent to $u$ as a component of $F'$. 

If $F'$ contains the edge $vu$, then the sum of all such $\gamma(F')$ is equal to

$$\Delta_1^* = \left(\frac{s+t}{b}\right)(a + s + t + 1 - b)c,$$

where $c$ is the product of orders of all components in $F' - (V(T') \cup U)$. If $F'$ does not contain the edge $vu$, then the sum of all such $\gamma(F')$ is equal to

$$\Delta_2^*(i) = \sum_{p=0}^{b-1} \left(\frac{t+i}{p}\right) \left(\frac{s-i}{b-1-p}\right)(a + t + i)(s - i + 2 - b + p)c.$$

So we have $\Delta^*(b) = \sum_{F \in \mathcal{F}} \gamma(F') = \Delta_1^* + \Delta_2^*(i)$.

For each $F' \in \mathcal{F}^*$, let

$$V'(F') = \{w : wu \in E(F'), wv \in E(G_{v,u}^{n,i})\},$$

$$F = (T' - \{wu : w \in V'(F')\}) \bigcup \{wv : w \in V'(F')\}.$$

Then $F$ is a spanning forest of $G_{v,u}^{n,i}$ with exactly $k$ components corresponding to $F'$ with $F - U = F' - U$. Let $I'$ be the subset of such spanning forests $F$ of $G$. Then the correspondence from $\mathcal{I}^*$ to $I'$ above is an injection. If $F$ contains the edge $vu$, then the sum of all such $\gamma(F)$ equals $\Delta_1^*$. If $F$ does not contain the edge $vu$, then the sum of all such $\gamma(F)$ equals $\Delta_2^*(0)$. So we have $\Delta(b) = \sum_{F \in \mathcal{F}} \gamma(F) = \Delta_1^* + \Delta_2^*(0)$.

Write $\delta_x = a + t + x, \delta_t = s - x + 2 - b, h(p, q) = \left(\frac{t+q}{p}\right) \left(\frac{s-q}{b-1-p}\right).$ By the well-known formula

$$\sum_{p=0}^{t} \left(\frac{t}{p}\right) \left(\frac{s}{r-p}\right) = \left(\frac{s+t}{r}\right).$$

we get that

$$\frac{1}{c} \Delta_1^*(t) = \delta_t \sum_{p=0}^{b-1} h(p, i) + (\delta_t - \delta_t) \sum_{p=0}^{b-1} ph(p, i) - \sum_{p=0}^{b-1} p^2 h(p, i)
\quad = \delta_t \left(\frac{s+t}{b-1}\right) + (\delta_t - \delta_t)(t + i) \left(\frac{s+t-1}{b-2}\right) - \left[(t+i)(t+i+1) \left(\frac{s+t-2}{b-3}\right) + (t+i) \left(\frac{s+t-1}{b-2}\right)\right].$$

So

$$\Delta(b) - \Delta^*(b) = \Delta^*_2(0) - \Delta_2^*(i) = ic\Phi(b),$$

where

$$\Phi(b) = \left[\left(\frac{s+t}{b-1}\right) - \left(\frac{s+t-1}{b-2}\right)\right](a + t - s + b - 2 + i) - \left[\left(\frac{s+t-1}{b-2}\right) - \left(\frac{s+t-2}{b-3}\right)\right](2t+i-1)
\quad = \left(\frac{s+t-1}{b-1}\right)(a + t - s + b - 2 + i) - \left(\frac{s+t-2}{b-2}\right)(2t+i-1).$$

It is easy to see, from the assumptions of $i$, that

$$\Delta(1) - \Delta^*(1) = ic\Phi(1) = [a + i - (s - t + 1)]ic > 0.$$

Next assume that $b \geq 2$. It is easy to see, from the assumptions of $i$, that

$$\Phi(b) = \left(\frac{s+t-2}{b-2}\right) \frac{1}{b-1}[(t+i-s)(s+t-b) + (a-1)(s+t-1)] \geq 0.$$
Proof. It is obvious that
\[ c_0(G) = 1 = c_0(G - uv), \quad c_0(G) = 0 = c_p(G - uv). \]

Now suppose that \( 1 \leq k \leq n - 1 \) and consider the coefficient \( c_{n-k}(G) \). Let \( J \) denote the set of all spanning forests containing no \( uv \) with exactly \( k \) components in \( G \). Then \( \Theta_k(G - uv) = J \) and \( \Theta_k(G) - J \) is the set of all spanning forests containing \( uv \) with exactly \( k \) components in \( G \). It is well-known that \( G \) has a spanning tree \( T \) containing the edge \( uv \). Let \( \bar{E} \) be arbitrary set of \( k - 1 \) edges of \( T \) not containing the edge \( uv \). Then \( T - \bar{E} \) is a spanning forest of \( G \) such that it contains the edge \( uv \) and has exactly \( k \) components. Therefore, \( \Theta_k(G) - J \neq \emptyset \). So by Theorem 1.1, we have
\[ c_{n-k}(G) = \sum_{F \in \Theta_k(G)} \gamma(F) > \sum_{F \in \Theta_k(G - uv)} \gamma(F) = c_{n-k}(G - uv). \]

By setting \( k \leftrightarrow n - k \), we complete the proof. \( \square \)

Remark 2.8. Let \( K_n \) denote the complete graph of order \( n \) and let \( K'_n \) denote the graph obtained from \( K_n \) by deleting an arbitrary edge. Let \( G \notin \{ K_n, K'_n, S_n \} \) be a connected graph of order \( n \). From Theorems 2.2 and 2.7, we have
\[ S_n < G < K'_n < K_n. \]

Remark 2.9. From Theorem 1.5 it is easy to see that the four transformations of graphs defined in Theorems 2.2 and 2.5–2.7, respectively, decrease the LEL of graphs. In particular, if \( G \notin \{ K_n, K'_n, S_n \} \) is a connected graph of order \( n \), then
\[ \text{LEL}(S_n) < \text{LEL}(G) < \text{LEL}(K'_n) < \text{LEL}(K_n). \]

3. The Laplacian coefficients of unicyclic graphs in \( U(n, i) \)

In this section we use Theorems 2.2 and 2.5 to characterize the connected unicyclic graphs in \( U(n, i) \) which simultaneously minimize all Laplacian coefficients. It is easy to show that the following result holds.

Lemma 3.1. Let \( f(\lambda) \) and \( g(\lambda) \) be two real polynomials arranged according to decreasing exponents. If their coefficients are alternate about positive and negative, then the coefficients of \( f(\lambda)g(\lambda) \) also are alternate about positive and negative.

Let \( G_p(s_1, t_1, s_2, t_2, \ldots, s_p, t_p) \) be the connected unicyclic graph of order \( n \) obtained from a cycle \( u_1u_2 \cdots u_pu_1 \) by adding \( s_j \) pendant paths of length 2 and \( t_j \) pendant edges at the vertex \( u_j \) \((j = 1, 2, \ldots, p)\), where \( n = 2 \sum_{j=1}^{p} s_j + \sum_{j=1}^{p} t_j + p \). In particular, write
\[ \tilde{G}_p(t_1, t_2, \ldots, t_p) = G_p(0, t_1, 0, t_2, \ldots, 0, t_p) \]
\[ U^1_{n,i} = G_3(i - 2, n - 2i + 1, 0, 0, 0), \quad U^2_{n,i} = G_3(i - 3, n - 2i + 1, 0, 1, 0, 1). \]

Set
\[ \eta = \lambda^2 - 3\lambda + 1, \]
\[ f(s, t) = \eta[\lambda^2 - (s + t + 3)\lambda + (s + 2)] - s(\lambda - 1)^2. \]

By an elementary calculation, we have
\[ \phi(G_3(s_1, t_1, s_2, t_2, s_3, t_3, \lambda)) = (\lambda - 1)^{s_1+t_2+t_3-3}\eta^{s_1+t_2+t_3-3}g(s_1, t_1, s_2, t_2, s_3, t_3), \]
where
\[ g(s_1, t_1, s_2, t_2, s_3, t_3) = \prod_{j=1}^{3} f(s_j, t_j) - \eta^3(\lambda - 1)^2 \sum_{j=1}^{3} f(s_j, t_j) + 2\eta^3(\lambda - 1)^3. \]

Lemma 3.2. Let \( G_3(s_1, t_1, s_2, t_2, s_3, t_3) \) be the graph defined above.

1. If \( \max\{0, t_2 - s_1 - t_1\} < r \leq t_2 \), then for \( k = 2, 3, \ldots, n - 2 \),
   \[ c_k(G_3(s_1 + s_2, t_1 + r, 0, t_2 - r, s_3, t_3)) > c_k(G_3(s_1 + s_2, t_1 + r, 0, t_2 - r, s_3, t_3)). \]

2. If \( i \geq 3 \) and \( n \geq 9 \), then for \( k = 0, 1, 2, \ldots, n - 1, n \),
   \[ c_k(U^2_{n,i}) \geq c_k(U^1_{n,i}). \]

and for \( k = 2, 3, \ldots, n - 3 \), the inequalities are strict.

Proof. (1) Set \( \theta = s_1 + t_1 + r - t_2 \). From \( \phi(G_3(s_1, t_1, s_2, t_2, s_3, t_3, \lambda)) \), by an elementary calculation we have
\[ \phi(G_3(s_1, t_1, s_2, t_2, s_3, t_3, \lambda)) - \phi(G_3(s_1 + s_2, t_1 + r, 0, t_2 - r, s_3, t_3, \lambda)) \]
\[ = \frac{\lambda^2(\lambda - 1)^{s_1+t_2+t_3-3}\eta^{s_1+t_2+t_3-3}f(s_3, t_3)((s_2 + r)\eta + s_2)[\theta\eta + s_1]}{(s_2 + r)\eta + s_2}. \]
On one hand, Eq. (3.1) is a polynomial on $\lambda$ with order $n - 2$. On the other hand, each factor in Eq. (3.1) is a real polynomial with alternate coefficients on positive and negative. So by Lemma 3.1, Eq. (3.1) also is a real polynomial with alternate coefficients on positive and negative. Assume that

$$\nabla = \sum_{k=2}^{n-2} (-1)^k b_k \lambda^{n-k},$$

where $b_k > 0$ for $k = 2, 3, \ldots, n - 2$. Then by Eq. (1.1) we have

$$c_i(G_3(s_1, t_1, s_2, t_2, s_3, t_3)) = c_0(G_3(s_1 + s_2, t_1 + r, 0, t_2 - r, s_3, t_3)) = b_k > 0.$$

This completes the proof of (1).

(2) For $j = 1, 2$, from $\phi(G_3(s_1, t_1, s_2, t_2, s_3, t_3), \lambda)$ it is easy to get that

$$\phi(U_{n,i}^j, \lambda) = \lambda (\lambda - 1)^{n-2i} \eta^{-3} f_j(\lambda),$$

where

$$f_1(\lambda) = (\lambda^2 - 4\lambda + 3)[\lambda^2 - (n - i + 5)\lambda^2 + (3n - 3i + 7)\lambda - n],$$

$$f_2(\lambda) = (\lambda^2 - 5\lambda + 3)[\lambda^2 - (n - i + 4)\lambda^2 + (3n - 3i + 4)\lambda - n].$$

So we get that

$$\phi(U_{n,i}^2, \lambda) = \phi(U_{n,i}^1, \lambda) = \lambda^2 (\lambda - 1)^{n-2i} \eta^{-3} [(n - i - 3)\lambda^2 + (3n - 3i - 11)\lambda + (n - 9)].$$

So in a similar way to (1) by Lemma 3.1 it is easy to see that the result holds. \hfill \Box

Let $G$ be a connected unicyclic graph and let $u$ be a vertex of $G$ not on its unique cycle $C$. Let $v$ be the vertex on $C$ such that the distance between it and $u$ is smaller than those among the other vertices of $C$ and $u$. We call $v$ the root of $u$ on $C$ and call the distance of $u$ and $v$ the height of $u$ on $C$.

**Theorem 3.3.** If $i \geq 3$ and $n \geq 9$, then $U_{n,i}^i$ is the unique minimal element of $U(n, i)$ under the partial order $\leq$.

**Proof.** Let $U$ be a minimal element of $U(n, i)$ under the partial order $\leq$. Let $C$ denote the unique cycle of $U$ and let $M(U)$ denote a maximum matching of $U$ containing the most pendent edges. Now we only need prove $U \cong U_{n,i}^i$.

**Claim 1.** Each pendant path of $U$ has length at most 2.

Suppose, for a contradiction, that there is a pendant path $v_0v_1 \cdots v_k$ of $U$ at $v_0$ with length $k \geq 3$. Note that $e = v_{k-2}v_{k-1}$ is not on $C$. So by a l-edge-growing transform of $U$ at $e$, we can get a connected unicyclic graph $U_{n,i}^e$ of order $n$. By the assumption of $M(U)$, we have that $v_{k-1}v_k \notin M(U)$. It follows that $v_{k-2}v_{k-1} \notin M(U)$. So from $E_{e}^{v_{k-2}} = \{v_{k-2}v_{k-1}\}$, we have that $E_{e}^{v_{k-2}} \cap M(U) = \emptyset$. Thus by Remark 2.1 we have that $|M(U_e)| = |M(U)|$. So $U_e \in U(n, i)$, and by Theorem 2.2 we have

$$c_k(U) > c_k(U_e), \quad k = 2, 3, \ldots, n - 2,$$

a contradiction to the choice of $U$.

**Claim 2.** Each vertex of $U$ not on $C$ has degree at most 2.

Suppose, for a contradiction, that there are vertices of $U$ not on $C$ with degree at least 3. Let $u$ be such a vertex with the largest height. Let $uvu' \cdots$ be the unique path from $u$ to its root on $C$. By Claim 1 the other vertices adjacent to $u$ except $v$ lie on pendant paths of lengths at most 2.

(a) Suppose that there exists a pendant edge $uv'$ at $u$.

Assume $d_{u}(u) \geq 3$. By a l-edge-growing transform of $U$ at $uv$, we get a connected unicyclic graph $U_{uv}'$ of order $n$. By Remark 2.4 we have that $|M(U_{uv}')| = i, i + 1$.

If $|M(U_{uv}')| = i$, then $U_{uv}' \in U(n, i)$, and by Theorem 2.5 we have that

$$c_k(U) > c_k(U_{uv}'), \quad k = 2, 3, \ldots, n - 2,$$

a contradiction to the choice of $U$.

If $|M(U_{uv}')| = i + 1$, then by using a l-edge-growing transform at $uu'$, we can get a connected unicyclic graph $W$ of order $n$ with $|M(W)| = i$. So $W \in U(n, i)$, and by Theorems 2.2 and 2.5 we have that

$$c_k(U) > c_k(U_{uv}'), c_k(W), \quad k = 2, 3, \ldots, n - 2,$$

a contradiction to the choice of $U$.

Assume $d_{u}(v) = 2$. By the assumption of $M(U)$ we may assume that $uu' \in M(U)$, so $uu' \notin M(U)$. Since $uu'$ is not on $C$, by a l-edge-growing transform of $U$ at $vv'$, we get a connected unicyclic graph $U_{vv}'$ of order $n$. From $E_{e}^{v_{k-2}} \cap M(U) = \emptyset$ and
Remark 2.1 we get that $|M(U_{vw})| = |M(U)|$. So $U_{vw} \in U(n, i)$, and by Theorem 2.2 we have

$$c_k(U) > c_k(U_{vw}), \quad k = 2, 3, \ldots, n - 2,$$

a contradiction to the choice of $U$.

(b) Suppose that there do not exist pendant edges at $u$.

By a 1-edge-growing transform of $U$ at $u$, we get a connected unicyclic graph $U_{uw}$ of order $n$. From the assumption of $M(U)$ we know that all pendant edges of pendant paths of length 2 at $u$ belong to $M(U)$. It follows that $E_{uw}^2 \cap M(U) = \emptyset$. So by Remark 2.1 we get $|M(U_{uw})| = |M(U)|$. Thus $U_{uw} \in U(n, i)$, and by Theorem 2.2 we have that

$$c_k(U) > c_k(U_{uw}), \quad k = 2, 3, \ldots, n - 2,$$

a contradiction to the choice of $U$.

Claim 3. Each vertex of $U$ not on $C$ has height at most 2.

Let $u$ be any vertex of $U$ not on $C$. By Claim 2 $u$ is in some pendant path $P$, and by Claim 1 the length of $P$ is at most 2. It follows that the height of $u$ is at most 2.

Claim 4. The length of $C$ is equal to 3.

Suppose, for contradiction, that the length $p$ of $C$ is at least 4. By Claims 2 and 3, there are nonnegative integers $s_j, t_j$ ($j = 1, 2, \ldots, p$) such that

$$U \cong G_p(s_1, t_1, s_2, t_2, \ldots, s_p, t_p) \in U(n, i).$$

If there exists an edge $e = u_iu_{i+1}$ of $C$ with $e \in M(U)$ (here $i + 1$ is equal to 1 when $i = p$), then by a 1-edge-growing transform of $U$ at $e$ we can get a connected unicyclic graph $U_e$ of order $n$. From $e \in M(U)$ we know that $(E_{u_i}^2 \cup E_{u_{i+1}}^2) \cap M(U) = \emptyset$. So by Remark 2.1 we get $|M(U_e)| = |M(U)|$. It follows that $U_e \in U(n, i)$, and from Theorem 2.2 we have that

$$c_k(U) > c_k(U_e), \quad k = 2, 3, \ldots, n - 2,$$

a contraction to the choice of $U$.

Next assume that each edge of $C$ is not in $M(U)$. For $1 \leq i \leq p$, if $t_i = t_{i+1} = 0$, then $M(U) \cup \{u_iu_{i+1}\}$ is a matching of $U$, a contradiction to the assumption of $M(U)$. Therefore, without loss of generality, assume that $t_i \neq 0$. Let $u_{i+1}u'_{i+1}$ be a pendant edge at $u_{i+1}$ and $u_{i+1}u'_{i+1} \in M(U)$. We again distinguish the three following cases.

Assume that $t_i = 0$. Let $u_iu'_i$ be a pendant edge at $u_i$ and $u_iu'_i \in M(U)$. By a 1-edge-growing transform of $U$ at $e = u_iu_{i+1}$, we get a connected unicyclic graph $U'_e$ of order $n$. It is easy to see that

$$M(U'_e) = (M(U) - \{u_iu'_i, u_{i+1}u'_{i+1}\}) \cup \{wu'_i, w'u'_{i+1}\}.$$  

These indicate that $U'_e \in U(n, i)$, and from Theorem 2.5 we have that

$$c_k(U) > c_k(U'_e), \quad k = 2, 3, \ldots, n - 2,$$

a contradiction to the choice of $U$.

Assume that $t_i = 0$ and $s_i \neq 0$. Let $u_iu_{i+1}$ be a pendant path of length 2 at $u_i$. By a 1-edge-growing transform of $U$ at $e = u_iu_{i+1}$, we get a connected unicyclic graph $U_{e}$ of order $n$. From the assumption of $M(U)$ we know that all pendant edges of pendant paths of length 2 at $u_i$ belong to $M(U)$. It follows that $E_{u_i}^2 \cap M(U) = \emptyset$. So by Remark 2.1 we get $|M(U_e)| = |M(U)|$. Thus $U_e \in U(n, i)$, and from Theorem 2.2 we have that

$$c_k(U) > c_k(U_e), \quad k = 2, 3, \ldots, n - 2,$$

a contradiction to the choice of $U$.

Assume that $t_i = s_i = 0$. By a 1-edge-growing transform of $U$ at $e = u_iu_{i+1}$, we get a connected unicyclic graph $U_e$ of order $n$. Since $E_{u_i}^2 = \{u_iu_i\}$ and $u_i - u_{i+1} \notin M(U)$ (here $i + 1$ is equal to 1 when $i = 1$), by Remark 2.1 we have that $|M(U_e)| = |M(U)|$. Thus $U_e \in U(n, i)$, and from Theorem 2.2 we have that

$$c_k(U) > c_k(U_e), \quad k = 2, 3, \ldots, n - 2,$$

a contradiction to the choice of $U$.

Claim 5. $U \cong U_{n,i}^1$.

By Claims 2–4, there exist nonnegative integers $s_j$ and $t_j$ ($j = 1, 2, 3$) such that

$$U \cong G_3(s_1, t_1, s_2, t_2, s_3, t_3) \in U(n, i).$$

If $t_1t_2t_3 \neq 0$, then $U_{n,i}^2 \cong G_3(s_1 + s_2 + s_3, t_1 + t_2 + t_3 - 2, 0, 0, 0, 1)$, and by Lemma 3.2, we have, for $k = 2, 3, \ldots, n - 2$, that

$$c_k(U) \geq c_k(U_{n,i}^1) > c_k(U_{n,i}^1),$$

a contradiction to the choice of $U$. 

a contradiction to the choice of \(U\). Thus \(t_1t_2t_3 = 0\). Without loss of generality, assume that \(s_1 + t_1 \geq s_2 + t_2 \geq s_3 + t_3\). Since \(i \geq 3\), we have \(s_1 + t_1 \neq 0\). If \(s_2 + t_2 \neq 0\), then
\[
U' = G_2(s_1 + t_2 + s_3, t_1 + t_2 + t_3, 0, 0, 0, 0) \in U(n, i),
\]
and by Lemma 3.2(1), we have \(c_k(U) > c_k(U')\) for \(k = 2, 3, \ldots, n - 2\), a contradiction to the choice of \(U\). Thus \(s_2 + t_2 = s_3 + t_3 = 0\). So from \(i \geq 3\) we have \(s_1 \neq 0\). If \(t_1 = 0\), then by a 1-edge-growing of \(U\) at a nonpendent edge not on \(C\), we can get
\[
U' = G_2(s_1 - 1, t_1 + 2, 0, 0, 0, 0) \in U(n, i),
\]
and by Theorem 2.2, we have \(c_k(U) > c_k(U')\) for \(k = 2, 3, \ldots, n - 2\), a contradiction to the choice of \(U\). Hence \(t_1 \neq 0\). So
\[
U \cong G_3(s_1, t_1, s_2, t_2, s_3, t_3) = G(s_1, t_1, 0, 0, 0, 0) \cong U_{n,i}^1.
\]
This completes the proof. \(\Box\)

Assume \(i \geq 4\). By a 1-edge-growing transform of \(U_{n,i}^1\) at a nonpendent edge not on triangle, \(U_{n,i}^1\) is transformed into \(U_{n,i-1}^1\).

So by Theorem 2.2, for \(k = 2, 3, \ldots, n - 2\),
\[
c_k(U_{n,i}^1) > c_k(U_{n,i-1}^1).
\]
Therefore, by Theorem 3.3 we immediately obtain the following corollary.

**Corollary 3.4.** Let \(i \geq 3\) and \(n \geq 9\). Then in the set of all connected unicyclic graph of order \(n\) and matching number at least \(i\), \(U_{n,i}^1\) is the unique minimal element under the partial order \(\preceq\).

By Theorem 1.5 and Corollary 3.4 we immediately obtain the following corollary.

**Corollary 3.5.** Let \(i \geq 3\) and \(n \geq 9\). Then in the set of all connected unicyclic graph of order \(n\) and matching number at least \(i\), \(U_{n,i}^1\) is the unique graph with the minimal LEL. In particular, \(U_{n,1}^1\) is the unique graph in \(U(n, i)\) with the minimal LEL.
4. The Laplacian coefficients of unicyclic graphs in $U(n)$

Theorem 1.4 indicates that $S_n'$ is the unique minimal connected unicyclic graph in $U(n)$ on the partial order $\preceq$. In this section we further determine the minimal connected unicyclic graphs in $U(n) - \{S_n'\}$ on the partial order $\preceq$.

Let $n \geq 5$ and $\delta_n = \lambda(\lambda - 1)^{n-5}$. By an elementary calculation, we have that

$$\phi(H_1(n), \lambda) = \delta_n[\lambda^4 - (n + 5)\lambda^3 + (7n - 1)\lambda^2 - (13n - 21)\lambda + 3n].$$

From $\phi(G_3(s_1, t_1, s_2, t_2, s_3, t_3), \lambda)$, we have that

$$\phi(\tilde{G}_3(n - 4, 1, 0), \lambda) = \delta_n[\lambda^4 - (n + 5)\lambda^3 + (6n + 3)\lambda^2 - (9n - 5)\lambda + 3n].$$

Thus by $\phi(U_{n,i}, \lambda)$, for $n \geq 5$ we easily get that

$$\tilde{G}_3(n - 4, 1, 0) \prec H_1(n).$$

(4.1)

(4.2)

It is easy to see that $U(5) = \{S_5', \tilde{G}_3(1, 1, 0), H_1(5), \tilde{G}_4(1, 0, 0, 0), C_5\}$. By using a 1-edge-growing transform of $C_5$ at $u_1u_2$, $C_5$ can be transformed into $\tilde{G}_4(1, 0, 0, 0)$. Again by using a 1-edge-growing transform of $\tilde{G}_4(1, 0, 0, 0)$ at $u_2u_3$, $\tilde{G}_4(1, 0, 0, 0)$ can be transformed into $\tilde{G}_3(1, 1, 0)$. Therefore, we have that

$$\tilde{G}_3(1, 1, 0) \prec \tilde{G}_4(1, 0, 0, 0) \prec C_5.$$

(4.3)

**Theorem 4.1.** Assume $n \geq 5$. Then $\tilde{G}_3(n - 4, 1, 0)$ is the unique minimal connected unicyclic graph in $U(n) - \{S_n'\}$ on the partial order $\preceq$.

**Proof.** Let $U \in U(n) - \{S_n', \tilde{G}_3(n - 4, 1, 0)\}$ and let $i$ be the matching number of $U$. From $n \geq 5$, we have $i \geq 2$. Now we only need prove $\tilde{G}_3(n - 4, 1, 0) \prec U$.

If $i \geq 3$ and $n \geq 9$, then by Eq. (4.2) and Corollary 3.4, we have

$$\tilde{G}_3(n - 4, 1, 0) \prec U_{n,3} \preceq U.$$

**Fig. 3.** Diagrams of unicyclic graphs in $U(7, 3)$. 
If \((n, i) \in \{ (8, 4), (n, 3) : n = 6, 7, 8 \}\), i.e. \(U \in (6, 3) \cup U(7, 3) \cup U(8, 3) \cup U(8, 4)\), then by Eq. (4.2) and Remark 3.7, we have \(\tilde{G}_3(n - 4, 1, 0) \prec U_{n,i}^I \subseteq U\).

Now assume that \(i = 2\). Then \(U \in U(n, 2) - \{ S'_2, \tilde{G}_3(n - 4, 1, 0)\}\), namely \(U\) is one of the following \(n\)-vertex graphs:

\[ H_1(n), \quad \tilde{G}_3(s, t, 0) \quad (s \geq t \geq 2), \quad \tilde{G}_4(t_1, 0, t_3, 0) \quad (t_1 \geq t_3), \quad C_5.\]

If \(U \cong H_1(n)\), then by Eq. (4.1) we have \(\tilde{G}_3(n - 4, 1, 0) \prec H_1(n) \cong U\).

If \(U \cong \tilde{G}_3(s, t, 0)\), then by Lemma 3.2(1) we have

\[ \tilde{G}_3(n - 4, 1, 0) = \tilde{G}_3(s + t - 1, 1, 0) \prec \tilde{G}_3(s, t, 0) \cong U.\]

If \(U \cong \tilde{G}_4(t_1, 0, t_3, 0)\), then by using a 1-edge-growing transform of \(\tilde{G}_4(t_1, 0, t_3, 0)\) at \(u_2u_3\), we get \(\tilde{G}_3(t_1, t_3 + 1, 0)\). So by Lemma 3.2(1) and Theorem 2.2 we have

\[ \tilde{G}_3(n - 4, 1, 0) = \tilde{G}_3(t_1 + t_3, 1, 0) \preceq \tilde{G}_3(t_1, t_3 + 1, 0) \prec \tilde{G}_4(t_1, 0, t_3, 0) \cong U.\]

If \(U \cong C_5\), then by Eq. (4.3) we have \(\tilde{G}_3(n - 4, 1, 0) = \tilde{G}_3(1, 1, 0) \prec C_5 \cong U\).

This proof is completed. \(\square\)

By Theorems 1.4, 1.5 and 4.1, we have the following result.

**Corollary 4.2.** Assume \(n \geq 5\). Then \(\tilde{G}_3(n - 4, 1, 0)\) is the unique graph of \(U(n)\) with the second smallest LEL.

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**References**


