Note

There is no variational characterization of the cycles in the method of periodic projections

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Abstract

The method of periodic projections consists in iterating projections onto m closed convex subsets of a Hilbert space according to a periodic sweeping strategy. In the presence of $m \geqslant 3$ sets, a long-standing question going back to the 1960s is whether the limit cycles obtained by such a process can be characterized as the minimizers of a certain functional. In this paper we answer this question in the negative. Projection algorithms for minimizing smooth convex functions over a product of convex sets are also discussed.
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1. Introduction

Throughout this paper $(\mathcal{H}, \| \cdot \|)$ is a real Hilbert space. Let C_1 and C_2 be closed vector subspaces of $\mathcal{H}$, and let P_1 and P_2 be their respective projection operators. The method of alternating projections for finding the projection of a point $x_0 \in \mathcal{H}$ onto $C_1 \cap C_2$ is governed by the iterations

$$(\forall n \in \mathbb{N}) \quad \left| \begin{array}{l} x_{2n+1} = P_2 x_{2n} \\ x_{2n+2} = P_1 x_{2n+1}. \end{array} \right. \tag{1.1}$$

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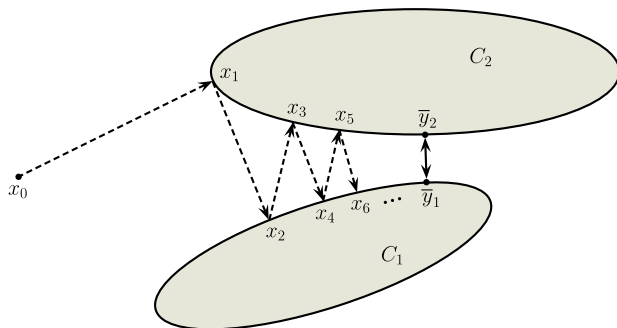
Fig. 1. In the case of $m = 2$ sets, the method of alternating projections produces a cycle $(\bar{y}_1, \bar{y}_2)$ that achieves the minimal distance between the two sets.

This basic process, which can be traced back to Schwarz' alternating method in partial differential equations [26], has found many applications in mathematics and in the applied sciences; see [13,14] and the references therein. The strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ produced by (1.1) to the projection of x_0 onto $C_1 \cap C_2$ was established by von Neumann in 1935 [24, Lemma 22] (see also [7,20] for alternative proofs). The extension of (1.1) to the case when C_1 and C_2 are general nonempty closed convex sets was considered in [10,22]. Thus, it was shown in [10] that, if C_1 is compact, the sequences $(x_{2n})_{n\in\mathbb{N}}$ and $(x_{2n+1})_{n\in\mathbb{N}}$ produced by (1.1) converge strongly to points $\bar{y}_1$ and $\bar{y}_2$, respectively, that constitute a cycle, i.e.,

$$\bar{y}_1 = P_1\bar{y}_2 \quad \text{and} \quad \bar{y}_2 = P_2\bar{y}_1, \tag{1.2}$$

or, equivalently, that solve the variational problem (see Fig. 1)

$$\underset{y_1\in C_1,\, y_2\in C_2}{\text{minimize}} \|y_1 - y_2\|. \tag{1.3}$$

Furthermore, it was shown in [22] that, if C_1 is merely bounded, the same conclusion holds provided strong convergence is replaced by weak convergence. As was proved only recently [19], strong convergence can fail (see also [23]).

Extending the above results to $m \geqslant 3$ nonempty closed convex subsets $(C_i)_{1\leqslant i\leqslant m}$ of $\mathcal{H}$ poses interesting challenges. For instance, there are many strategies for scheduling the order in which the sets are projected onto. The simplest one corresponds to a periodic activation of the sets, say

$$(\forall n \in \mathbb{N}) \quad \left| \begin{array}{l} x_{mn+1} = P_m x_{mn} \\ x_{mn+2} = P_{m-1} x_{mn+1} \\ \quad\vdots \\ x_{mn+m} = P_1 x_{mn+m-1}, \end{array} \right. \tag{1.4}$$

where $(P_i)_{1\leqslant i\leqslant m}$ denote the respective projection operators onto the sets $(C_i)_{1\leqslant i\leqslant m}$. In the case of closed vector subspaces, it was shown in 1962 that the sequence $(x_n)_{n\in\mathbb{N}}$ thus generated converges strongly to the projection of x_0 onto $\bigcap_{i=1}^{m} C_i$ [18]. This provides a precise extension of the von Neumann result, which corresponds to $m = 2$. Interestingly, however, for nonperiodic sweeping strategies with closed vector subspaces, only weak convergence has been established in general [1] and, since 1965, it has remained an open problem whether strong convergence holds
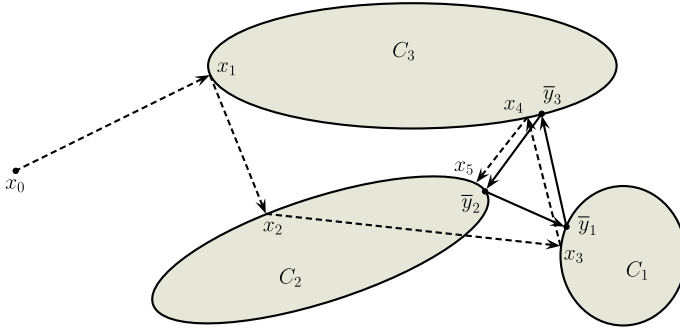
Fig. 2. Example with $m = 3$ sets: the method of periodic projections initialized at x_0 produces the cycle $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$.

(see [2,9,3] for special cases). Another long-standing open problem is the one that we address in this paper and which concerns the asymptotic behavior of the periodic projection algorithm (1.4) for general closed convex sets. It was shown in 1967 [17] (see also [12, Section 7] and [15]) that, if one of the sets is bounded, the sequences $(x_{mn})_{n \in \mathbb{N}}$, $(x_{mn+1})_{n \in \mathbb{N}}$, ..., $(x_{mn+m-1})_{n \in \mathbb{N}}$ converge weakly to points $\bar{y}_1, \bar{y}_m, \ldots, \bar{y}_2$, respectively, that constitute a cycle, i.e. (see Fig. 2),

$$\bar{y}_1 = P_1 \bar{y}_2, \quad \ldots, \quad \bar{y}_{m-1} = P_{m-1} \bar{y}_m, \quad \bar{y}_m = P_m \bar{y}_1. \tag{1.5}$$

However, it remains an open question whether, as in the case of $m = 2$ sets, the cycles can be characterized as the solutions to a variational problem. We formally formulate this problem as follows.

Definition 1.1. Let m be an integer at least equal to 2 and let $(C_1, \ldots, C_m)$ be an ordered family of nonempty closed convex subsets of $\mathcal{H}$ with associated projection operators $(P_1, \ldots, P_m)$. The set of cycles associated with $(C_1, \ldots, C_m)$ is

$$\mathrm{cyc}(C_1, \ldots, C_m)$$
$$= \left\{ (\bar{y}_1, \ldots, \bar{y}_m) \in \mathcal{H}^m \mid \bar{y}_1 = P_1 \bar{y}_2, \ldots, \bar{y}_{m-1} = P_{m-1} \bar{y}_m, \bar{y}_m = P_m \bar{y}_1 \right\}. \tag{1.6}$$

Question 1.2. Let m be an integer at least equal to 3. Does there exist a function $\Phi : \mathcal{H}^m \to \mathbb{R}$ such that, for every ordered family of nonempty closed convex subsets $(C_1, \ldots, C_m)$ of $\mathcal{H}$, $\mathrm{cyc}(C_1, \ldots, C_m)$ is the set of solutions to the variational problem

$$\underset{y_1 \in C_1, \ldots, y_m \in C_m}{\text{minimize}} \Phi(y_1, \ldots, y_m)? \tag{1.7}$$

Let us note that the motivations behind Question 1.2 are not purely theoretical but also quite practical. Indeed, the variational properties of the cycles when $m = 2$ have led to fruitful applications in applied physics and signal processing; see [8] and the references therein. Since the method of periodic projections (1.4) is used in scenarios involving $m \geqslant 3$ possibly nonintersecting sets [11], it is therefore important to understand the properties of its limit cycles and, in particular, whether they are optimal in some sense. Since the seminal work [17] in 1967 that first established the existence of cycles, little progress has been made towards this goal beyond the observation that simple candidates such as

$\Phi : (y_1, \dots, y_m) \mapsto \|y_1 - y_2\| + \cdots + \|y_{m-1} - y_m\| + \|y_m - y_1\|$ fail [5,6,11,21]. The main result of this paper is that the answer to Question 1.2 is actually negative. This result will be established in Section 2. Finally, in Section 3, projection algorithms that are pertinent to extensions of (1.3) to $m \geqslant 3$ sets will be discussed.

2. A negative answer to Question 1.2

We denote by $S(x; \rho)$ the sphere of center $x \in \mathcal{H}$ and radius $\rho \in [0, +\infty[$, and by P_C the projection operator onto a nonempty closed convex set $C \subset \mathcal{H}$; in particular, $P_C 0$ is the element of minimal norm in C.

Our main result hinges on the following variational property, which is of interest in its own right.

Theorem 2.1. *Suppose that* $\dim \mathcal{H} \geqslant 2$ *and let* $\varphi : \mathcal{H} \to \mathbb{R}$ *be such that its infimum on every nonempty closed convex set* $C \subset \mathcal{H}$ *is attained at* $P_C 0$. *Then the following assertions hold.*

(i) φ *is radially increasing, i.e.,*

$$(\forall x \in \mathcal{H}) \, (\forall y \in \mathcal{H}) \quad \|x\| < \|y\| \quad \Rightarrow \quad \varphi(x) \leqslant \varphi(y). \tag{2.1}$$

(ii) *Suppose that, for every nonempty closed convex set* $C \subset \mathcal{H}$, $P_C 0$ *is the unique minimizer of* φ *on* C. *Then* φ *is strictly radially increasing, i.e.,*

$$(\forall x \in \mathcal{H}) \, (\forall y \in \mathcal{H}) \quad \|x\| < \|y\| \quad \Rightarrow \quad \varphi(x) < \varphi(y). \tag{2.2}$$

(iii) *Except for at most countably many values of* $\rho \in [0, +\infty[$, φ *is constant on* $S(0; \rho)$.

Proof. (i): Let us fix x and y in $\mathcal{H}$ such that $\|x\| < \|y\|$. If $x = 0$, property (2.1) amounts to the fact that 0 is a global minimizer of φ, which follows from the assumption with $C = \mathcal{H}$. We now suppose that $x \neq 0$. Let V be a 2-dimensional vector subspace of $\mathcal{H}$ containing x and y, and let $\alpha \in [0, \pi]$ be the angle between x and y. For every integer $n \geqslant 3$, consider a polygonal spiral built as follows: set $x_{n,0} = y$ and for $k = 1, \dots, n$ define $x_{n,k} = P_{R_{n,k}} x_{n,k-1}$, where $(R_{n,k})_{1 \leqslant k \leqslant n}$ are n angularly equispaced rays in V between the rays $[0, +\infty[y$ and $[0, +\infty[x = R_{n,n}$ (see Fig. 3). Clearly, for the segment $C = [x_{n,k-1}, x_{n,k}]$, we have $P_C 0 = x_{n,k}$, so that the assumption on φ gives $\varphi(x_{n,k}) \leqslant \varphi(x_{n,k-1})$, and therefore $\varphi(x_{n,n}) \leqslant \varphi(x_{n,0}) = \varphi(y)$. On the other hand, $x_{n,n}$ and x are collinear with $\|x_{n,n}\| = \|y\|(\cos(\alpha/n))^n$ so that for n large enough we have $\|x_{n,n}\| > \|x\|$ and, therefore, the segment $C = [x, x_{n,n}]$ satisfies $P_C 0 = x$, from which we get $\varphi(x) \leqslant \varphi(x_{n,n}) \leqslant \varphi(y)$.

(ii): If the minimizer of φ on every nonempty closed convex set $C \subset \mathcal{H}$ is unique, then all the inequalities above are strict.

(iii): Define $g : [0, +\infty[\to \mathbb{R}$ and $h : [0, +\infty[\to \mathbb{R}$ by $(\forall \rho \in [0, +\infty[)$ $g(\rho) = \inf \varphi(S(0; \rho))$ and $h(\rho) = \sup \varphi(S(0; \rho))$. It follows from (2.1) that

$$\left(\forall \rho \in [0, +\infty[\right) \left(\forall \rho' \in [0, +\infty[\right) \quad \rho < \rho' \quad \Rightarrow \quad g(\rho) \leqslant h(\rho) \leqslant g(\rho') \leqslant h(\rho'). \tag{2.3}$$

Hence, g and h are increasing and therefore, by Froda's theorem [16], [25, Theorem 4.30], the set of points at which they are discontinuous is at most countable. Furthermore, it follows from (2.3)
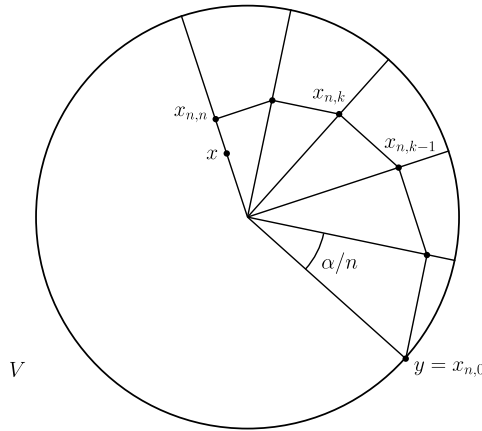
Fig. 3. A polygonal spiral from $y = x_{n,0}$ to $x_{n,n}$ in V.

that g and h have the same right and left limits at every point. Therefore, they have the same points of continuity and their values coincide at these points. We conclude that, except for at most countably many $\rho \in [0, +\infty[$, we have $g(\rho) = h(\rho)$ so that φ is constant on $S(0; \rho)$. $\quad\square$

As a consequence, we get the following result.

Corollary 2.2. *Suppose that* $\dim \mathcal{H} \geqslant 2$ *and let* $\varphi : \mathcal{H} \to \mathbb{R}$ *be such that its infimum on every nonempty closed convex set* $C \subset \mathcal{H}$ *is attained at* $P_C 0$. *If* φ *is lower or upper semicontinuous, then it is a radial function, namely* $\varphi = \theta \circ \|\cdot\|$, *where* $\theta : [0, +\infty[\to \mathbb{R}$ *is increasing. Furthermore, if* $P_C 0$ *is the unique minimizer of* φ *on every nonempty closed convex set* $C \subset \mathcal{H}$, *then* θ *is strictly increasing.*

Proof. Define $g : [0, +\infty[\to \mathbb{R}$ and $h : [0, +\infty[\to \mathbb{R}$ by $(\forall \rho \in [0, +\infty[)$ $g(\rho) = \inf \varphi(S(0; \rho))$ and $h(\rho) = \sup \varphi(S(0; \rho))$. To establish that φ is radial we must show that $g(\rho) = h(\rho)$ for every $\rho \in [0, +\infty[$. This is obvious for $\rho = 0$ since $g(0) = h(0) = \varphi(0)$. Suppose now that $\rho > 0$. If φ is lower semicontinuous, we derive from Theorem 2.1(i) that

$$\big(\forall x \in S(0; \rho)\big) \quad \varphi(x) \leqslant \varliminf_{n \to +\infty} \varphi\big((1 - 1/n)x\big) \leqslant \varliminf_{n \to +\infty} h\big((1 - 1/n)\rho\big) \leqslant g(\rho). \quad (2.4)$$

Therefore, taking the supremum for $x \in S(0; \rho)$ we get $h(\rho) \leqslant g(\rho)$, hence $h(\rho) = g(\rho)$. Similarly, if φ is upper semicontinuous, we have

$$\big(\forall x \in S(0; \rho)\big) \quad \varphi(x) \geqslant \varlimsup_{n \to +\infty} \varphi\big((1 + 1/n)x\big) \geqslant \varlimsup_{n \to +\infty} g\big((1 + 1/n)\rho\big) \geqslant h(\rho), \quad (2.5)$$

and taking the infimum for $x \in S(0; \rho)$ gives $g(\rho) \geqslant h(\rho)$, hence $h(\rho) = g(\rho)$. Altogether, using Theorem 2.1(i), we conclude that $\varphi = \theta \circ \|\cdot\|$ for some increasing function $\theta : [0, +\infty[\to \mathbb{R}$. The case of strict monotonicity of θ follows from Theorem 2.1(ii). $\quad\square$

Using Theorem 2.1 we can provide the following answer to Question 1.2.

Theorem 2.3. *Suppose that* $\dim \mathcal{H} \geqslant 2$ *and let m be an integer at least equal to 3. There exists no function* $\Phi : \mathcal{H}^m \to \mathbb{R}$ *such that, for every ordered family of nonempty closed convex subsets* $(C_1, \ldots, C_m)$ *of* $\mathcal{H}$, $\mathrm{cyc}(C_1, \ldots, C_m)$ *is the set of solutions to the variational problem*

$$\underset{y_1 \in C_1, \ldots, y_m \in C_m}{\text{minimize}} \Phi(y_1, \ldots, y_m). \tag{2.6}$$

Proof. Suppose that Φ exists and set $(\forall i \in \{1, \ldots, m-2\})$ $C_i = \{0\}$. Moreover, take $z \in \mathcal{H}$ and set $C_{m-1} = \{z\}$. Then, for every nonempty closed convex set $C_m \subset \mathcal{H}$ we have

$$\underset{y_1 \in C_1, \ldots, y_m \in C_m}{\text{Argmin}} \Phi(y_1, \ldots, y_m) = \mathrm{cyc}(C_1, \ldots, C_m) = \{(0, \ldots, 0, z, P_{C_m} 0)\}. \tag{2.7}$$

Hence, Theorem 2.1 implies that, except for at most countably many values of $\rho \in [0, +\infty[$, the function $\Phi(0, \ldots, 0, z, \cdot)$ is constant on $S(0; \rho)$.

Now suppose that $z \in S(0; 1)$ and take $\rho \in]1, +\infty[$ so that $\Phi(0, \ldots, 0, z, \cdot)$ and $\Phi(0, \ldots, 0, -z, \cdot)$ are constant on $S(0; \rho)$. Clearly,

$$\mathrm{cyc}(\{0\}, \ldots, \{0\}, [-z, z], \{\rho z\}) = \{(0, \ldots, 0, z, \rho z)\} \tag{2.8}$$

and

$$\mathrm{cyc}(\{0\}, \ldots, \{0\}, [-z, z], \{-\rho z\}) = \{(0, \ldots, 0, -z, -\rho z)\}, \tag{2.9}$$

so that

$$\begin{aligned}
\Phi(0, \ldots, 0, z, \rho z) &< \Phi(0, \ldots, 0, -z, \rho z) \\
&= \Phi(0, \ldots, 0, -z, -\rho z) \\
&< \Phi(0, \ldots, 0, z, -\rho z) \\
&= \Phi(0, \ldots, 0, z, \rho z),
\end{aligned} \tag{2.10}$$

where the inequalities come from the fact that the minima of Φ characterize the cycles, while the equalities follow from the constancy of the functions on $S(0; \rho)$. Since these strict inequalities are impossible it follows that Φ cannot exist. $\quad \square$

3. Related projection algorithms

We have shown that the cycles produced by the method of cyclic projections (1.4) are not characterized as the solutions to a problem of the type (1.7), irrespective of the choice of the function $\Phi : \mathcal{H}^m \to \mathbb{R}$. Nonetheless, alternative projection methods can be devised to solve variational problems over a product of closed convex sets. Here is an example.

Theorem 3.1. *For every* $i \in I = \{1, \ldots, m\}$, *let* $\mathcal{H}_i$ *be a real Hilbert space and let* C_i *be a nonempty closed convex subset of* $\mathcal{H}_i$ *with projection operator* P_i. *Let* $\mathcal{H}$ *be the Hilbert direct sum of the spaces* $(\mathcal{H}_i)_{1 \leqslant i \leqslant m}$, *and let* $\Phi : \mathcal{H} \to \mathbb{R}$ *be a differentiable convex function such*

that $\nabla\Phi : \mathcal{H} \to \mathcal{H} : y \mapsto (G_i y)_{i \in I}$ is $1/\beta$-lipschitzian for some $\beta \in]0, +\infty[$ and such that the problem

$$\underset{y_1 \in C_1, \ldots, y_m \in C_m}{\text{minimize}} \Phi(y_1, \ldots, y_m) \tag{3.1}$$

admits at least one solution. Let $\gamma \in]0, 2\beta[$, set $\delta = \min\{1, \beta/\gamma\} + 1/2$, let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{n \in \mathbb{N}} \lambda_n(\delta - \lambda_n) = +\infty$, and let $\boldsymbol{x}_0 = (x_{i,0})_{i \in I} \in \mathcal{H}$. Set

$$(\forall n \in \mathbb{N}) \ (\forall i \in I) \quad x_{i,n+1} = x_{i,n} + \lambda_n\big(P_i(x_{i,n} - \gamma G_i \boldsymbol{x}_n) - x_{i,n}\big). \tag{3.2}$$

Then, for every $i \in I$, $(x_{i,n})_{n \in \mathbb{N}}$ converges weakly to a point $\bar{y}_i \in C_i$, and $(\bar{y}_i)_{i \in I}$ is a solution to (3.1).

Proof. Set $\boldsymbol{C} = \bigtimes_{i \in I} C_i$. Then $\boldsymbol{C}$ is a nonempty closed convex subset of $\mathcal{H}$ with projection operator $P_{\boldsymbol{C}} : \boldsymbol{x} \mapsto (P_i x_i)_{i \in I}$ [7, Proposition 28.3]. Accordingly, we can rewrite (3.2) as

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \lambda_n\big(P_{\boldsymbol{C}}(\boldsymbol{x}_n - \gamma \nabla\Phi(\boldsymbol{x}_n)) - \boldsymbol{x}_n\big). \tag{3.3}$$

It follows from [7, Corollary 27.10] that $(\boldsymbol{x}_n)_{n \in \mathbb{N}}$ converges weakly to a minimizer $\bar{\boldsymbol{y}}$ of Φ over $\boldsymbol{C}$, which concludes the proof. $\square$

The projection algorithm described in the next result solves an extension of (1.3) to $m \geqslant 3$ sets.

Corollary 3.2. Let m be an integer at least equal to 3. For every $i \in I = \{1, \ldots, m\}$, let C_i be a nonempty closed convex subset of $\mathcal{H}$ with projection operator P_i, and let $x_{i,0} \in \mathcal{H}$. Suppose that one of the sets in $(C_i)_{i \in I}$ is bounded and set

$$(\forall n \in \mathbb{N}) \ (\forall i \in I) \quad x_{i,n+1} = P_i\left(\frac{1}{m-1} \sum_{j \in I \setminus \{i\}} x_{j,n}\right). \tag{3.4}$$

Then for every $i \in I$, $(x_{i,n})_{n \in \mathbb{N}}$ converges weakly to a point $\bar{y}_i \in C_i$, and $(\bar{y}_i)_{i \in I}$ is a solution to the variational problem

$$\underset{y_1 \in C_1, \ldots, y_m \in C_m}{\text{minimize}} \sum_{\substack{(i,j) \in I^2 \\ i < j}} \|y_i - y_j\|^2. \tag{3.5}$$

Moreover, $\bar{y} = (1/m) \sum_{i \in I} \bar{y}_i$ is a minimizer of the function $\varphi : \mathcal{H} \to \mathbb{R} : y \mapsto \sum_{i \in I} \|y - P_i y\|^2$.

Proof. We use the notation of Theorem 3.1, with $(\forall i \in I)$ $\mathcal{H}_i = \mathcal{H}$. Set $\beta = 1 - 1/m$, $\gamma = 1$,

$$\Phi : \mathcal{H} \to \mathbb{R} : (y_i)_{i \in I} \mapsto \frac{1}{2(m-1)} \sum_{\substack{(i,j) \in I^2 \\ i < j}} \|y_i - y_j\|^2, \tag{3.6}$$

$C = \bigtimes_{i \in I} C_i$, and $D = \{(y, \ldots, y) \in \mathcal{H} \mid y \in \mathcal{H}\}$. Then [4,11]

$$\text{Fix } P_C P_D = \text{Argmin } \Phi \quad \text{and} \quad \text{Fix } P_D P_C = \big\{(y, \ldots, y) \mid y \in \text{Argmin } \varphi\big\}. \tag{3.7}$$

Since one of the sets in $(C_i)_{i \in I}$ is bounded, $\text{Argmin } \varphi \neq \emptyset$ [11, Proposition 7]. Now let $y \in \text{Argmin } \varphi$, and set $\boldsymbol{y} = (y, \ldots, y)$ and $\boldsymbol{x} = P_C \boldsymbol{y}$. Then (3.7) yields $\boldsymbol{y} = P_D P_C \boldsymbol{y}$ and therefore $\boldsymbol{x} = P_C(P_D P_C \boldsymbol{y}) = P_C P_D \boldsymbol{x}$. Hence $\boldsymbol{x} \in \text{Argmin } \Phi$ and thus $\text{Argmin } \Phi \neq \emptyset$. On the other hand, (3.5) is a special case of (3.1) and the gradient of Φ is the continuous linear operator

$$\nabla \Phi : \boldsymbol{y} \mapsto \left(y_i - \frac{1}{m-1} \sum_{j \in I \setminus \{i\}} y_j \right)_{i \in I} \tag{3.8}$$

with norm $m/(m-1) = 1/\beta$. Note that, since $m > 2$, $2\beta > 1 = \gamma$. Moreover, $\delta = \min\{1, \beta/\gamma\} + 1/2 > 1$. Thus, upon setting, for every $n \in \mathbb{N}$, $\lambda_n = 1 \in]0, \delta[$ in (3.2), we obtain (3.4) and observe that $\sum_{n \in \mathbb{N}} \lambda_n(\delta - \lambda_n) = +\infty$. Altogether, the convergence result follows from Theorem 3.1. Finally, set $\bar{\boldsymbol{y}} = (\bar{y}_1, \ldots, \bar{y}_m)$ and $\bar{z} = P_D \bar{\boldsymbol{y}}$. Then (3.7) yields

$$(\bar{y}, \ldots, \bar{y}) = \bar{z} = P_D \bar{\boldsymbol{y}} = P_D(P_C P_D \bar{\boldsymbol{y}}) = P_D P_C \bar{z} \tag{3.9}$$

and hence $\bar{y} \in \text{Argmin } \varphi$. $\square$

Remark 3.3. Alternative projection schemes can be derived from Theorem 3.1. For instance, Corollary 3.2 remains valid if (3.4) is replaced by

$$(\forall n \in \mathbb{N}) \ (\forall i \in I) \quad x_{i,n+1} = P_i \left(\frac{1}{m} \sum_{j \in I} x_{j,n} \right), \tag{3.10}$$

which amounts to taking $\gamma = \beta$ instead of $\gamma = 1$ in the above proof. We then recover a process investigated in [5,11].

Acknowledgments

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