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On Characterizing Finite Chevalley Groups of Type E_6 and Their Twisted Analogs

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Let $E_6(q)$ be the adjoint Chevalley group of type E_6 over the finite field of order $q = p^n$, p odd. We shall denote the simple twisted analogue of $E_6(q^2)$ by ${}^2E_6(q)$. Let $\epsilon = \pm 1$, $G_\epsilon{}^* = E_6(q)$ when $\epsilon = 1$ and $G_\epsilon{}^* = {}^2E_6(q)$ when $\epsilon = -1$. Suppose z^* is an involution in the center of a Sylow 2-subgroup of $G_\epsilon{}^*$. The purpose of this paper is to prove the following.

THEOREM. *Let G be a finite group with an involution z such that the centralizer $H_\epsilon = C_G(z)$ of z in G is isomorphic to $C_{G_\epsilon{}^*}(z^*)$. Then either $G = O(G)H_\epsilon$ or G is isomorphic to $G_\epsilon{}^*$.*

We begin the proof with a study of the structure of $C_{G_\epsilon{}^*}(z^*)$. For this, we use the method of Iwahori-Ree [6], with which we assume the reader is familiar. To give a decription of the structure, it is necessary to introduce a large number of notations for elements and subsets of $G_\epsilon{}^*$. Our notation generally follows that of [9]. We also refer the reader to [9] for the construction of and standard facts on Chevalley groups. We then analyze the fusion of classes of involutions in H_ϵ. The information is used to construct a subgroup G_0 isomorphic to $G_\epsilon{}^*$ in the interesting case. We show finaly that $G_0 = G$.

Our group-theoretic notation is standard except that A^x will denote xAx^{-1} for some subset A and element x of a group X .The reason for this deviation will be obvious.

1. NOTATION AND STRUCTURE OF H_ϵ

Let Φ be the set of roots of the complex semi-simple Lie algebra **G** of type E_6 relative to a Cartan subalgebra of **G**. For some fixed ordering of Φ, let

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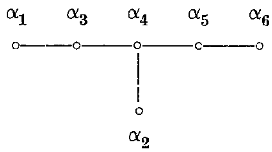
Φ^+ be the set of positive roots. Φ^+ contains a simple system $\{\alpha_i \mid 1 \leqslant i \leqslant 6\}$ with the following Dynkin diagram.



Let F_ϵ be the finite field of order q when $\epsilon = 1$ and of order q^2 when $\epsilon = -1$. The universal Chevalley group E_ϵ of $\mathbf{G}$ over F_ϵ contains the one-parameter unipotent subgroups $X_\alpha{}^* = \langle x_\alpha{}^*(t) \mid t \in F_\epsilon \rangle$, $\alpha \in \Phi$. The following elements play important roles in the study of Chevalley groups.

$$\begin{aligned} w_\alpha{}^*(t) &= x_\alpha{}^*(t)\, x_{-\alpha}^*(-t^{-1})\, x_\alpha{}^*(t), \\ h_\alpha{}^*(t) &= w_\alpha{}^*(t)\, w_\alpha{}^*(1)^{-1}, \\ \omega_\alpha{}^* &= w_\alpha{}^*(1). \end{aligned}$$

For convenience, we record below some relations that are especially important to us.

$$\begin{aligned} \omega_\alpha{}^* h_{\alpha'}^*(t)\, \omega_\alpha^{*-1} &= h_{w_\alpha(\alpha')}^*(t), \\ \omega_\alpha{}^* x_{\alpha'}^*(t)\, \omega_\alpha{}^{*-1} &= x_{w_\alpha(\alpha')}^*(ct), \qquad c = \pm 1, \qquad (*) \\ h_\alpha{}^*(t)\, x_{\alpha'}^*(t')\, h_\alpha{}^*(t)^{-1} &= x_{\alpha'}^*(t^{\langle \alpha', \alpha \rangle} t'). \end{aligned}$$

Let σ be the identity automorphism of E_ϵ when $\epsilon = 1$ and the product of the graph automorphism of E_ϵ and the field automorphism of E_ϵ induced by: $t \to \bar{t} = t^q$, $t \in F_\epsilon$ when $\epsilon = -1$. Recall that in the later case, if ρ denotes the permutation of the roots induced by the graph automorphism, we have

$$\sigma(x_\alpha{}^*(t)) = x_{\rho(\alpha)}^*(\epsilon_\alpha \bar{t}),$$

where $\epsilon_\alpha = \pm 1$ and $\epsilon_\alpha = 1$ if $\pm\alpha$ is a simple root. For any subset $A \subseteq E_\sigma$, let A_σ denote the set of fixed points of σ in A. Let X_α, $x_\alpha(t)$, $w_\alpha(t)$, and ω_α, be the images of $X_\alpha{}^*$, $x_\alpha{}^*(t)$, $w_\alpha{}^*(t)$, and $\omega_\alpha{}^*$, respectively, in $\bar{E}_\epsilon = E_\epsilon / Z(E_{\epsilon,\sigma})$. We shall use the so-called 'bar' convention for homomorphic images of subsets of E_ϵ in $\bar{E}_\epsilon$. Recall that $\mid Z(E_{\epsilon,\sigma}) \mid = d = (3, q - \epsilon)$. (See [9].) If X is σ-invariant, σ acts canonically on $\bar{X}$. Because σ is involutory *and* $\mid Z(E_{\epsilon,\sigma}) \mid$ is odd, it follows that $(\bar{X})\sigma = (\overline{X_\sigma})$. This fact will be used later without comment. We note that $\overline{E_{\epsilon,\sigma}} = G_\epsilon{}^*$. The resulting relations in $\bar{E}_\epsilon$ arising from (*) will be denoted by $\overline{(*)}$.

Next we introduce notations for elements and subsets of $\bar{E}_\epsilon$.

$$L_\alpha = \langle X_\alpha , X_{-\alpha} \rangle, \qquad \alpha \in \bar{\Phi},$$
$$\langle t_\alpha \rangle = Z(L_\alpha).$$

Recall that since Φ has only one root length, if $\alpha = \sum c_i \alpha_i$, then $t_\alpha = \prod_i t_{\alpha_i}^{c_i}$. Therefore $t_\alpha = \prod t_{\alpha_j}$ over all j such that c_j is odd. For brevity, we also denote t_α by $t_{j_1 j_2 \ldots}$ where c_{j_k} is odd.

$$\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \text{, highest root of } \Phi,$$
$$\beta = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 ,$$
$$\gamma = \alpha_3 + \alpha_4 + \alpha_5 ,$$
$$L_0 = L_{\alpha_0,\sigma} , \; Z(L_0) = \langle t_{146} \rangle = \langle t_0 \rangle,$$
$$L_b = L_{\beta,\sigma} , \; Z(L_b) = \langle t_{13456} \rangle = \langle t_\beta \rangle,$$
$$L_c = L_{\gamma,\sigma} , \; Z(L_c) = \langle t_{345} \rangle = \langle t_\gamma \rangle.$$

Let ρ be the identity permutation on Φ when $\epsilon = 1$ and the permutation on Φ defined earlier when $\epsilon = -1$.

For all $i \in \{1, 2,\ldots, 6\}$, let

$$L_i = L_{\alpha_i,\sigma} \cong SL(2, q), \qquad \text{when} \quad i = \rho(i),$$
$$L_{i\rho(i)} = \langle x_{\alpha_i}(t)\, x_{\rho(\alpha_i)}(\bar{t}) \mid t \in F_\epsilon \rangle,$$
$$\cong SL(2, q^2), \qquad \text{when} \quad i = \rho(i),$$

κ is a generator of the multiplicative group of F_1, λ is a generator of the multiplicative group of F_{-1} such that $\lambda^{q+1} = \kappa$.

$$h = \begin{cases} h_{\alpha_1}(\kappa), & \text{when} \quad \epsilon = 1, \\ y x h_{\alpha_1}(\lambda^{q-1})\, x^{-1} y^{-1}, & \text{when} \quad \epsilon = -1, \end{cases}$$

where

$$x^{-1} = x_\beta(t)\, w_\beta(t)\, x_\beta(\tfrac{1}{2}t),$$
$$y^{-1} = x_\gamma(t)\, w_\gamma(t)\, x_\gamma(\tfrac{1}{2}t),$$
$$t = \lambda^{1/2(q+1)}.$$

and

$$h' = \begin{cases} h_{\alpha_3}(\kappa), & \text{when} \quad \epsilon = 1, \\ y' y h_{\alpha_3}(\lambda^{q-1})\, y^{-1} (y')^{-1}, \end{cases}$$

where y is as above and $(y')^{-1} = x_{\alpha_4}(t) w_{\alpha_4}(t) x_{\alpha_4}(\frac{1}{2})$; $t = \lambda^{\frac{1}{2}(q-1)}$.

Using relations $(\overline{*})$, we compute that $h, h' \in \overline{E_{\epsilon,\sigma}}$ and obviously $|h| = |h'| = q - \epsilon$. We refer the reader to [1, p. 260] for pertinent facts necessary to above computations.

Finally if X is a Chevalley group, we shall call the automorphism σ' of X such that the twisted analogue of X is the set of fixed points of σ', the twisting automorphism of X. Also to shorten some of the later proofs we introduce the following notations

$$GL_\epsilon(m, q) = \begin{cases} GL(m, q), & \text{when } \epsilon = 1, \\ GU(m, q), & \text{when } \epsilon = -1, \end{cases}$$

$$SL_\epsilon(m, q) = \begin{cases} SL(m, q), & \text{when } \epsilon = 1, \\ SU(m, q), & \text{when } \epsilon = -1, \end{cases}$$

$$PSL_\epsilon(m, q) = SL_\epsilon(m, q)/Z(SL_\epsilon(m, q),$$

and $\mathrm{Spin}_\epsilon(2m, q) = \mathrm{Spin}(2m, q)$, the universal Chevalley group of type D_m over the field of q elements when $\epsilon = 1$ and $\mathrm{Spin}_\epsilon(2m, q)$ is the set of fixed points in $\mathrm{Spin}(2m, q^2)$ of its twisting automorphism when $\epsilon = -1$.

We are now ready to determine the conjugacy classes of involutions in $\overline{E_{\epsilon,\sigma}}$ and their centralizers.

LEMMA 1.1. (i) *There are precisely two conjugacy classes of involutions in* $\overline{E_{\epsilon,\sigma}}$ *with representatives* t_{35} *and* t_0 ;

(ii) $C_{\overline{E_{\epsilon,\sigma}}}(t_{35}) = \langle h\rangle L$ *where* $L = \langle L_0, L_2, L_3, L_4, L_5\rangle$ *when* $\epsilon = 1$; $L = \langle L_0, L_2, L_4, L_{35}\rangle$ *when* $\epsilon = -1$; $L \simeq \mathrm{Spin}_\epsilon(10, q)$ *and* L *is a normal subgroup of index* $(q - \epsilon)/d$;

(iii) $C_{\overline{E_{\epsilon,\sigma}}}(t_0) = \langle h_{\alpha_2}(\kappa)\rangle ML_0$ *where* $M = \langle L_1, L_3, L_4, L_5, L_6\rangle$ *when* $\epsilon = 1$; $M = \langle L_4, L_{35}, L_{16}\rangle$ *when* $\epsilon = -1$; $M \simeq SL_\epsilon(6, q)/Z$ *where* Z *is the subgroup of order* d *in* $Z(SL_\epsilon(6, q))$; $[M, L_0] = 1$ *and* $M \cap L_0 = \langle t_0\rangle$;

(iv) t_{35} *is contained in the center of a Sylow 2-subgroup of* $\overline{E_{\epsilon,\sigma}}$, *whereas* t_0 *is not.*

Proof. The case $\epsilon = 1$ has already been worked out in details by Iwahori [6]; whence we shall assume that $\epsilon = -1$.

E_ϵ may be viewed in a natural manner as a subgroup of a connected linear algebraic group X over the algebraic closure of F_ϵ [6, p. F1]. Both $C_X(t_{35})$ and $C_X(t_{146})$ are connected [6, p. F21]; whence by [10, p. E11] (i) follows as $|Z(E_{\epsilon,\sigma})|$ is odd.

Since $|Z(E_{\epsilon,\sigma})|$ is odd, $C_{\overline{E_{\epsilon,\sigma}}}(t_{35}) = C_{E_{\epsilon,\sigma}}(h^*_{\alpha_3}(-1)\, h^*_{\alpha_5}(-1))\, Z(E_{\epsilon,\sigma})/Z(E_{\epsilon,\sigma})$; whence we may assume without loss of generality that $Z(E_{\epsilon,\sigma}) = 1$. Then, applying the case $\epsilon = 1$ to our situation we get $C_{E_\epsilon}(t_{35}) = \langle h_{\alpha_1}(\lambda)\rangle \tilde{L}$ where $\tilde{L} = \langle L_{\alpha_0}, L_{\alpha_2}, L_{\alpha_3}, L_{\alpha_4}, L_{\alpha_5}\rangle$ and $\tilde{L} \simeq \mathrm{Spin}(10, q^2)$. Now α_0 is in the orbit of a simple root of Φ by the action of $\langle \omega_{\alpha_i} \mid 1 \leqslant i \leqslant 6 \rangle$. [9, p. 268]. Since σ leaves $\langle \omega_{\alpha_i} \mid 1 \leqslant i \leqslant 6 \rangle$ pointwise fixed, it follows $\sigma(x_{\pm\alpha_0}(t)) = x_{\pm\alpha_0}(t)$ as $x_{\pm\alpha_0}(t) = \omega x_{\pm\alpha_i}(t)\omega^{-1}$ for some i and $\omega \in \langle \omega_{\alpha_i} \mid 1 \leqslant i \leqslant 6 \rangle$. Therefore σ restricted to $\tilde{L}$ acts as its twisting automorphism and so $L = (\tilde{L})_\sigma \simeq \mathrm{Spin}_\epsilon(10, q)$.

Because of the uniqueness of Bruhat factorization of elements of E_ϵ, and in view of the action of σ on E_ϵ it follows easily that $C_{\overline{E_{\epsilon,\sigma}}}(t_{35}) = \langle h_{\alpha_1}(\lambda) h_{\alpha_6}(\bar{\lambda}) \rangle \tilde{L}_\sigma$. We note that $(h_{\alpha_1}(\lambda) h_{\alpha_6}(\bar{\lambda}))^{q-\epsilon} \in \tilde{L}_\sigma$ and $h \in C_{\overline{E_{\epsilon,\sigma}}}(t_{35}) - L$, whence $C_{\overline{E_{\epsilon,\sigma}}}(t_{35}) = \langle h \rangle L$. The last assertion follows easily as in [9, p. 178–182].

Part (iii) may be proved as in (ii) and (iv) follows by a direct comparision of $|\overline{E_{\epsilon,\sigma}}|$, $|C_{\overline{E_{\epsilon,\sigma}}}(t_{35})|$ and $|C_{\overline{E_{\epsilon,\sigma}}}(t_0)|$.

We turn next to a study of $C_{\overline{E_{\epsilon,\sigma}}}(t_{35})$, which will henceforth be identified with $H_\epsilon^* = C_{G_\epsilon^*}(z^*)$ in view of (1.1) (iv).

LEMMA 1.2. (i) *There are precisely two noncentral conjugacy classes of involutions in* $L = (H_\epsilon^*)'$ *with representatives* t_{16} *and* t_0.

(ii) $C_{H_\epsilon^*}(t_{16}) = \langle h, h' \rangle K$ *where* $K = \langle L_{\alpha_4}, L_{\alpha_2}, L_{\alpha_0}, L_\beta \rangle_\sigma$ *is normal in* $C_{H_\epsilon^*}(t_{16})$ *and is isomorphic to* Spin(8, q). *Also* $(C_{H_\epsilon^*}(t_{16}))' = K$.

(iii) $C_{H_\epsilon^*}(t_{146}) = \langle h_{\alpha_2}(\kappa) \rangle \langle JL_0L_b, h \rangle$ *where* $J = \langle L_{\alpha_3}, L_{\alpha_4}, L_{\alpha_5} \rangle_\sigma$ *and is isomorphic to* $SL_\epsilon(4, q)$. *Moreover* $[J, L_0] = [J, L_b] = [L_0, L_b] = 1$ *and* $\langle JL_0L_b, h \rangle$ *is a subgroup of index* 2 *in* $C_{H_\epsilon^*}(t_0)$.

Proof. When $\epsilon = 1$, Iwahori-Ree's method [6, p. 280] shows that L contains two conjugacy classes of involutions with representatives t_{16}, t_0. Since $C_L(t_{16})$ and $C_L(t_0)$ are not isomorphic, therefore t_{16} and t_0 do not fuse in H_ϵ^*. The other assertions in this case follow immediately by direct computation as in [6, p. 280]

Assume that $\epsilon = -1$. Applying (1.1), we have $H_\epsilon^* = \langle h \rangle (\tilde{L})_\sigma$ where $\tilde{L} = \langle L_{\alpha_0}, L_{\alpha_2}, L_{\alpha_3}, L_{\alpha_4}, L_{\alpha_5} \rangle$. By [6, p. 275] and [10, p. 177], it follows that $L = \tilde{L}_\sigma$ satisfies (i).

To prove (ii), we note that $C_{\bar{E}_\epsilon}(t_{35}, t_{16}) \cap \tilde{L} = \langle h_{\alpha_3}(\lambda) \rangle \tilde{K}$ where $\tilde{K} = \langle L_{\alpha_4}, L_{\alpha_2}, L_{\alpha_0}, L_\beta \rangle$. As in (1.1) (ii), we show that $\sigma(x_{\pm\beta}(t)) = x_{\pm\beta}(\bar{t})$. Thus σ acts as the field automorphism of order 2 on $\tilde{K}$. It follows from the uniqueness of Bruhat factorization of element in $\tilde{K}$, that $K = \tilde{K}_\sigma = \langle L_4, L_2, L_0, L_b \rangle \simeq$ Spin(8, q). Again as in (1.1) (ii), we conclude that $C_{H_\epsilon^*}(t_{16}) = \langle h, h' \rangle K$. The other assertions of (ii) are obvious (iii) is proved in a similar manner as in (ii).

LEMMA 1.3. *Let u be the unique involution in* $\langle h \rangle$. *Then* $C_{H_\epsilon^*}(u) = \langle h \rangle \times N$ *where* $N \simeq GL_\epsilon(5, q)$ *and* $\langle t_{35}, t_{16} \rangle = \Omega_1(Z(S))$ *for some Sylow 2-subgroup of* N.

Proof. First let $\epsilon = 1$. In view of the uniqueness of Bruhat factorization of elements of $L = \langle L_{\alpha_0}, L_{\alpha_2}, L_{\alpha_3}, L_{\alpha_4}, L_{\alpha_5}\rangle$ and the fact $u = h_{\alpha_1}(-1)$ normalizes x_α, α a root of Φ with $X_\alpha \subseteq \tilde{L}$, it follows $C^*_{H_\epsilon}(u) = \langle h\rangle \times N$ where $N = \langle h_{\alpha_3}(\kappa), L_{\alpha_0}, L_{\alpha_2}, L_{\alpha_4}, L_{\alpha_5}\rangle$ and $N \simeq GL(5, q)$. The remaining assertion follows from the structure of $GL(5, q)$.

Next assume $\epsilon = -1$. Apply the previous paragraph to $C_{E_\epsilon}(h_{\alpha_1}(-1)) \cap C_{\bar{E}_\epsilon}(t_{35}) = C$, we see that $C = \langle h_{\alpha_1}(y)\rangle \times \langle h_{\alpha_3}(\lambda), \tilde{N}\rangle$ where $\tilde{N} = \langle L_{\alpha_0}, L_{\alpha_2}, L_{\alpha_4}, L_{\alpha_5}\rangle$. Therefore $yxCx^{-1}y^{-1} = C_{E_\epsilon}(u) \cap C_{E_\epsilon}(t_{35})$ where x, y are the elements used to define h earlier; whence $C_{H^*_\epsilon}(u) = \langle h\rangle \times \langle h', (y\, x\tilde{N}x^{-1}y^{-1})\sigma\rangle$.

We claim that $(y\, x\tilde{N}x^{-1}y^{-1})\sigma \simeq$ the set of fixed points of $x^{-1}y^{-1}\sigma yx$ in $\tilde{N}$. Set $\theta_1 = (-\alpha_2)$; $\theta_2 = (-\alpha_4 - \alpha_5)$; $\theta_3 = \alpha_5$ and

$$\theta_4 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6).$$

We note that the θ_i's form a simple system of $\{\alpha \in \Phi \mid L_\alpha \subset N\}$. As $\sigma' = x^{-1}y^{-1}\sigma yx = \sigma w_\beta(t)\, w_\gamma(t)$ where $t = \lambda^{\frac{1}{2}(q+1)}$, it follows from $(\overline{*})$,

$$\sigma' x_{\theta_1}(t')\sigma' = x_{\theta_4}(ct^2t'),$$
$$\sigma' x_{\theta_2}(t')\sigma' = x_{\theta_3}(c't^{-1}t'),$$

where $c, c' = \pm 1$.

Let $h_4 \in \langle h_{\theta_4}(\lambda)\rangle$ such that $h_4^{-1}x_{\theta_4}(ct^2t')h_4 = x_{\theta_4}(t')$. (Such element exists because ct^2 is a square in $\langle\lambda\rangle$). Since $\langle h_{\alpha_3}(\lambda), L_{\alpha_4+\alpha_5}\rangle \simeq \mathrm{GL}(2, q^2)$, there is $h_3 \in \langle h_{\alpha_3}(\lambda)\rangle$ such that $h_3 x_{\theta_2}(t')h_3^{-1} = x_{\theta_2}(c'tt')$. Set $\sigma^* = h_3^{-1}h_4^{-1}\sigma\, h_4 h_3$. We see that σ^* acts as the twisting automorphism on $\tilde{N}$. Thus the set of fixed point of σ^* in $\tilde{N}$ is isomorphic to $SU(5q)$; whence $(yx\tilde{N}x^{-1}y^{-1})\sigma \simeq SU(5, q)$. Now $\langle h', (yx\tilde{N}x^{-1}y^{-1})\sigma\rangle \simeq GU(5, q)$ follows from direct computations.

2. Fusion of Involutions

We shall identify $H_\epsilon = C_G(z)$ with $H_\epsilon{}^*$ and hence z with t_{35} and use only those relations of $(\overline{*})$ which invlove only element of H_ϵ.

Lemma 2.1. *A Sylow 2-subgroup of H_ϵ is a Sylow 2-subgroup of G.*

Proof. Let S be a Sylow 2-subgroup of H_ϵ. Note that $z \in S'$ and $S' \subseteq L$ since H_ϵ/L is cyclic. (See (1.2) for notation.) By way of contradiction, suppose there exists $x \in G - H_\epsilon$ which normalizes S; whence $z^x \in S' \subseteq L$. By (1.2) z is the only involution in the center of a Sylow 2-subgroup of L. It follows $z^x = z$ contradicting $x \notin H_\epsilon$; whence the result follows.

Lemma 2.2. $z \not\sim_G t_0$.

Proof. Assume the contrary, i.e., $z \sim_G t_0$. Then there is an $x \in G - H_\epsilon$ such that $(t_0)^x = z$. Since $z \in C_G(t_0)'$, $(z)^x \in C_G(z)'$. By (1.2), we may suppose that $z^x = t_0$, whence x normalizes $C_G(z, t_0)' = JL_0L_b$. It follows that $J^x = J$ [7]; whence $Z(J)^x = Z(J)$, i.e., $z^x = z$, a contradiction.

LEMMA 2.3. *Either $G = O(G)H_\epsilon$ or $z \sim_G t_{16}$.*

Proof. If z is conjugate to another involution of L, then we are done by (1.2) and (2.2). Suppose then z is not conjugate to another involution in L. Let S be a Sylow 2-subgroup of H_ϵ containing that of $\langle h\rangle$. Suppose the unique involution of $\langle h\rangle$ is not conjugate to an involution of L then by the repeated use of Harada–Gorenstein–Thompson's fusion lemma [4], G contains a subgroup G_0 of index $O(h)_2$, not containing $\langle h\rangle \cap S$; whence $S \cap L$ is a Sylow 2-subgroup of G_0 and z is not conjugate to another involution of $S \cap L$. Now Glauberman's theorem states that $G_0 = O(G_0)(H_\epsilon \cap G_0)$; whence $G = O(G)H_\epsilon$ since $O(G_0) = O(G)$.

Finally suppose u is conjugate to an involution of L. A comparison of the orders of $C_{H_\epsilon}(u)$, $C_{H_\epsilon}(z)$, $C_{H_\epsilon}(t_{16})$ and $C_{H_\epsilon}(t_0)$ shows that if T is a Sylow 2-subgroup of $C_{H_\epsilon}(u)$ there exists $g \in C_G(u) - C_{H_\epsilon}(u)$ normalizing T. By (1.3), we may choose T such that $\Omega_1(Z(T)) = \langle u, z, t_{16}\rangle$ and $\langle z, t_{16}\rangle = \Omega_1(Z(T \cap N))$. Since $\langle t_{16}\rangle = \Omega_1(Z(T)) \cap T'$, g centralizes t_{16}; whence either $z^g = uz$ or $z^g = ut_{16}z$. Now $u \sim_{H_\epsilon} ut_{16}$ (conjugation by $\omega_\beta\omega_\gamma$ when $\epsilon = 1$; by $yx\omega_\gamma x^{-1}y^{-1}$ when $\epsilon = -1$ where x, y have the same meaning as before.) If $z^g = uz$, then $(t_{16}z)^g = ut_{16}z \sim_{H_\epsilon} uz$. It follows that $z \sim_G t_{16}z \sim_{H_\epsilon} t_{16}$ contradicting our assumption that z is not conjugate to another involution of L. The other case $(z)^g = ut_{16}z$ leads to the same contradiction in a similar way. This complete the proof.

3. CONSTRUCTION OF G_0

In view of (2.3), we shall assume from now on $z \sim_G t_{16}$. This will enable us to show that $G_0 = \langle C_G(z), C_G(t_0)\rangle \cong G_\epsilon^*$.

LEMMA 3.1. *There exists an element $g \in N(L_0) \cap C(t_2)$ such that g interchanges t_β, t_4 and z, t_{16} by conjugation.*

Proof. By (2.3), there exists $g \in G$ such that $t_{16}^g = z$ and we can further assume that $z^g = t_{16}$ as in (2.3); whence g normalizes $C = C_{H_\epsilon}(t_{16})'$ which is isomorphic to Spin(8, q). Replacing g with gg' for some suitable g' in C, we conclude by [9, p. 156–160] that

$$(X_{\pm\alpha_1})^g = X_{\pm\beta}, \qquad (X_{\pm\alpha_2})^g = X_{\pm\alpha_2}, \qquad (X_{\pm\alpha_0})^g = X_{\pm\alpha_3}$$

as we already know that $t_{16}^g = z$ and $z^g = t_{16}$ and $Z(C) = \langle t_{16}, z \rangle$. In particular, we get $t_2{}^g = t_2$; $t_0{}^g = t_0$.

Next we compute that $(C_G(t_0) \cap C)' = L_0 L_b L_1 L_c$. It follows that $g \in N(L_0)$ [7].

LEMMA 3.2. *We have* $N(L_0) = \langle h_{\alpha_2}(\kappa) \rangle L_0\, C(L_0)$, $[N(L_0) \colon L_0 C(L_0)] = 2$, $L_0 \cap C(L_0) = \langle t_0 \rangle$ *and* $C(L_0) = SL_\epsilon(6, q)/Z$ *where* Z *to the unique subgroup of order* $d = (3, q - \epsilon)$ *in* $Z(SL_\epsilon(6, q))$.

Proof. We claim first $N(L_0)$ contains a subgroup N_0 of index 2 not containing $h_{\alpha_2}(\kappa)$. Clearly $C_{H_\epsilon}(t_0) \subseteq N(L_0)$ and a Sylow 2-subgroup T of $C_{H_\epsilon}(t_0)$ is a Sylow 2-subgroup of $N(L_0)$; otherwise we would get $t_0 \sim_G z$ as $| C(z) \colon C(t_0)|_2 = 2$, in contradiction to (2.2). Now $h_{\alpha_2}(\kappa)$ induces on outer automorphism on L_0 ; whereas all elements of order $q - \epsilon$ in $\langle h, JL_0 L_b \rangle$ act as inner automorphisms. The claim follows immediately from Gorenstein–Harada–Thompson's fusion lemma.

Let $\bar{N}_0 = N_0/L_0$ and we shall use the 'bar' convention for homomorphic images of subsets of N_0 . Since $t_\beta L_0$ contains precisely three involution, i.e., t_β , t_0 , $t_\beta t_0 = z$. As $t_\beta \not\sim_G z$ by (2.2), it follows that

$$C_{\bar{N}_0}(\bar{t}_\beta) = \overline{C_{N_0}(t_\beta)}$$

and so $C_{\bar{N}_0}(\bar{t}_\beta) = \langle \bar{h}, \overline{JL_b} \rangle$.

In view of (3.1), we conclude from [7; 8] that $\bar{N}_0 = PSL_\epsilon(6, q)$. By the uniqueness of composition factors of N_0 and the structure of $\mathrm{Aut}(L_0)$, it follows that $C(L_0)/Z(L_0) = PSL_\epsilon(6, q)$. From H_ϵ , we see that the central extension of $C(L_0)/Z(L_0)$ is nontrivial. As the universal covering group of $PSL_\epsilon(6, q)$ is $SL_\epsilon(6, q)$, we conclude that $C(L_0) \simeq SL_\epsilon(6, q)/Z$ where $Z \subseteq Z(SL_\epsilon(6, q))$ and $| Z | = (3, q - \epsilon)$.

LEMMA 3.3. $C(t_0) = N(L_0)$.

Proof. It is obvious that $N(L_0) \subseteq C(t_0) = C$. From the structure of $C(L_0)$, we conclude that there are precisely two conjugacy classes of involutions in $C(L_0)$ with representatives z, t_β and $C_G(g) \cap C_G(t_0) \subseteq N(L_0)$ for all involution g of $C(L_0)$.

As in (3.2), we know that $C(z, t_0)$ contains a Sylow 2-subgroup of C. Let v be an involution in $N(L_0) - C_0$ where $C_0 = L_0 C(L_0)$ and suppose $v \sim_C z$. We note that $C(v)$ contains an element, say i, in the conjugacy class of z in $N(L_0)$. This follows from the fact that $C(z) \cap N(L_0)$ contains a Sylow 2-subgroup of $N(L_0)$ and therefore the order of the class is odd; whence v fixes at least one element of this class. Similarly there is an j in the conjugacy class of t_β , which centralizes v. Let $g \in C$ such that $v^g = z$. Since

$C(z) \cap C(t_0) \subseteq N(L_0)$, i^g and j^g lie in $N(L_0)$. As $i \not\sim_G j$, $\langle ij \rangle$ contains a unique involution k. Thus one of i^g, j^g, k^g, say $l^g \in L_0C(L_0)$.

Now every involution w in $L_0C(L_0) - C(L_0)$ has the form w_1w_2, $w_1 \in L_0$, $w_2 \in C(L_0)$ and $w_1^2 = (w_2^{-1})^2 = t_{146}$. As $C(w_2) \cap C(L_0)/\langle t_0 \rangle$ involves one of the following groups [7; 8]:

$PSL_\epsilon(3, q) \times PSL_\epsilon(3, q)$; $PSL_\epsilon(5, q)$; $PSL_\epsilon(3, q^2)$ we see that i, j, k cannot be conjugate in C to an involution of $L_0C(L_0) - C(L_0)$. It follows that $l^g \in C(L_0)$; whence there exists $g' \in C(L_0)$ such that $l^{gg'} = l$, as two involutions of $C(L_0)$ conjugate in G are already conjugate in $C(L_0)$. On the other hand, $C(l) \cap C(t_0) \subseteq N(L_0)$ and so $g \in N(L_0)$. But $v \not\sim_{N(L_0)} z$, a contradiction. Similarly $v \not\sim_C t_\beta$; whence no involution of $N(L_0) - L_0C(L_0)$ is conjugate in C to one in $C(L_0)$.

The above paragraph also proves that no involution in $L_0C(L_0) - C(L_0)$ is conjugate to one in $C(L_0)$.

Let $c \in C$. Since z and $(t_\beta)^c$ are not conjugate in G, $\langle z(t_\beta)^c \rangle$ contains an involution t such that z, $(t_\beta)^c \in C(t)$ and either $zt \sim z$ or $zt \sim t_\beta$. From above, t must lie in $C(L_0)$; whence $C(t) \cap C(t_{146}) \subseteq N(L_0)$. But t_β and $(t_\beta)^c$ are already conjugate in $C(L_0)$. Hence there is an $c' \in N(L_0)$ such that $cc' \in C(t_\beta) \cap C(t_0) \subseteq N(L_0)$, proving our result.

LEMMA 3.4. *Let* $G_0 = \langle C_G(z), C_G(t_0) \rangle$. *Then* $G_0 \cong G_\epsilon^*$.

Proof. We prove the result in a number of steps.

(i) $uz \sim z$ and $u \sim t_0$.

Let $\bar{X} = C(L_0)/Z(L_0)$. Then $C_{\bar{X}}(\bar{t}_\beta) = \langle \bar{h}, \bar{L}_b, \bar{N} \rangle$ and we have $\bar{v} \sim_{\bar{X}} \bar{t}_\beta$. Hence $\langle u, t_0 \rangle$ is conjugate to $\langle t_0, t_\beta \rangle$. In the later group only one involution can be conjugate to z namely $t_0t_\beta = z$ (by (2.2). Relabelling u by uz, if necessary, (i) follows since $ut_0 \sim_{H_\epsilon} uz$ (conjugation by $(\omega_{\alpha_3}\omega_{\alpha_5})^{\omega_{\alpha_4}\omega_{\alpha_3}\omega_{\alpha_2}\omega_{\alpha_4}\omega_{\alpha_0}\omega_{\alpha_2}}$).

(ii) Let v be an involution conjugate to t_0. Denote the unique normal subgroup of $C_G(v)'$ isomorphic to $SL(2, q)$ by $L(v)$. (See (3.2)). If v', v'', v''' are pairwise commuting involutions in $C(L(v'))$ conjugate to t_0, then $L(v'')$, $L(v''') \subseteq C(L(v'))$ and either $[L(v''), L(v''')] = 1$ or $\langle L(v''), L(v''') \rangle \cong SL_\epsilon(3, q)$ according as $v''v''' \sim t_{35}$ or $v'\ v''' \sim t_0$.

By assumption, we may assume $v' = t_0$ whence $L(v') = L_0$. (See 3.2). It follows then we may suppose $v'' = t_\beta$. In view of (1.2) and because of symmetry, it is clear that $L(v'') = L_b$. Now L_b as subgroup of $C(L(v'))$ is the unique normal subgroup of $C(t_\beta) \cap C(L(v'))$ isomorphic to $SL(2, q)$. From the structure of $C(L(v'))$, there is an $x \in C(L(v'))$ such that $t_\beta^x = v'''$ and so $xL(v'')x^{-1} = L(v''')$ by the uniqueness of $L(v''')$ in $C(v''')$. On the other hand $L(v''')$ is also the unique normal subgroup of $C(L(v')) \cap C(v'')$ isomorphic to $SL(2, q)$. The assertion now follows from the structure of $C(L_0)$.

(iii) Let $\epsilon = 1$; $L_1 = L(u)$; $L_6 = L(ut_\beta t_\gamma)$. Then $[L_1, L_2] = [L_1, L_4] = [L_1, L_5] = [L_1, L_6] = [L_2, L_6] = [L_3, L_6] = [L_4, L_6] = 1$; $\langle L_1, L_3\rangle \cong SL(3, q) \cong \langle L_5, L_6\rangle$ and $G_0 = \langle L_i \mid 1 \leqslant i \leqslant 6\rangle \cong E_6(q)$.

Since $u \sim_{H_\epsilon} ut_\beta$ (See (i)) and $(ut_\beta)^{\omega_{\alpha_3}\omega_{\alpha_4}\omega_{\alpha_5}} = ut_\beta t_\gamma$. By (i), we see that both $L(u)$ and $L(ut_\beta t_\gamma)$ are defined.

We now apply (ii) to t_0, u, t_3 [respectively, t_0, u, t_4 ; t_0 u, t_5 ; t_0, u, $ut_\beta t_\gamma$; t_0, $ut_\beta t_\gamma$, t_3 ; $t_0 ut_\beta t_\gamma$, t_4 ; t_0, $ut_\beta t_\gamma$; t_5] in the roles of v', v'', v'''. From the fact $ut_3 = (u)^{\omega_{\alpha_3}}$ [respectively, $ut_4 = (uz)^{\omega_{\alpha_3}\omega_{\alpha_4}\omega_{\alpha_5}}$; $ut_5 = (uz)^{\omega_{\alpha_3}}$; $u \cdot ut_\beta t_\gamma = t_{16} \sim t_{35}$; $ut_\beta t_\gamma t_3 = (ut_\beta t_\gamma z)^{\omega_{\alpha_5}} \sim uz$; $ut_\beta t_\gamma t_4 = (ut_\beta t_\gamma t_3)^{\omega_{\alpha_4}\omega_{\alpha_3}} \sim uz$; $ut_\beta t_\gamma t_5 = (ut_\beta t_\gamma)^{\omega_{\alpha_5}}$]. It follows $\langle L_1, L_3\rangle \cong SL(3, q)$ [respectively, $[L_1, L_4] = 1$; $[L_1, L_6] = 1$; $[L_6, L_3] = 1$; $[L_6, L_4] = 1$ $\langle L_6, L_5\rangle \cong SL(3, q)$]. Note that we have used the fact $L(t_i) = L_i$ for $i = 3, 4, 5$ as L_i is conjugate to $L_0 = L(t_0)$ in H_ϵ.

We can now apply (ii) again to t_5, t_2, u in the roles of v', v'', v''' and since $ut_2 \sim_{H_\epsilon} uz$ (conjugation by $(\omega_{\alpha_3}\omega_{\alpha_5})^{\omega_{\alpha_4}\omega_{\alpha_3}\omega_{\alpha_2}\omega_{\alpha_4}}$), it follows $[L_1, L_2] = 1$. Similarly $[L_2, L_6] = 1$.

An argument of Humphreys [5] shows that the conditions of Curtis' Theorem [2] are satisfied and $\langle L_i \mid 1 \leqslant i \leqslant 6\rangle = G^*$ is isomorphic to a factor group of the universal Chevalley group of type E_6 over F_q. It follows immediately from the order of H_ϵ that G^* is isomorphic to $E_6(q)$. Also from (1.1) we conclude that $C_G(t_0) \subseteq G^*$ and therefore we have $G_0 = G^*$.

(iv) Let $\epsilon = -1$. Then G contains a subgroup L_{16} such that $[L_{16}, L_2] = 1 = [L_{16}, L_4]$; $\langle L_{16}, L_{35}\rangle \cong SL(3, q^2)$ and $G_0 = \langle L_2, L_4, L_{35}, L_{16}\rangle$ which is isomorphic to ${}^2E_6(q)$.

Let g be the element of (3.1). Replacing g with a suitable element in $g\langle L_0, h_{\alpha_2}(\kappa)\rangle$, we may suppose $g \in C(L_0)$ and still have $t_2{}^g = t_2$; $t_\beta{}^g = t_4$, $t_4{}^g = t_\beta$. Since $C(L_0) \cong SU(6, q)/Z$ where $Z \subseteq Z(SU(6, q))$ and $|Z| = (3, q+1)$ and from the structure of $C(L_0)$, it follows that, after replacing g again with a suitable element gg' where $g' \in C(t_\beta, t_4) \cap C(L_0)$, $\langle L_{16}, L_{35}\rangle \cong SL(3, q^2)$ where $L_{16} = (L_{35})^{gg'}$ (This is so because if we let $SU(6, q)$ acts naturally on a six-dimensional hermition vector space over $F_q{}^2$, L_{35} corresponds to the image of a subgroup of $SU(6, q)$ fixing a totally isotopic subspace of dimension 2.)

Now since $[L_b, L_{35}] = 1$, $[L_4, L_{16}] = 1$ as $L_b^{gg'} = L_4$, because $t_2{}^g = t_2$, it follows from the uniqueness L_2 as the normal subgroup of $C(t_2)$ isomorphic to $SL(2, q)$ that $L_2{}^g = L_2$. Finally from the fact $C_G(t_\beta, t_4) \cap C(L_0) = \langle h, h', L_0, L_b, L_4, L_C\rangle$, $L_2^{g'} = L_2$, $[L_2, L_{16}] = [L_2^{gg'}, L_{35}^{gg'}] = [L_2, L_{35}] = 1$.

As in (iii), we conclude from Curtis' theorem [2] that $\langle L_2, L_4, L_{35}, L_{16}\rangle = G_0$ and is isomorphic to ${}^2E_6(q)$.

Conclusion of proof: $G_0 = G$.

Since G_0 has exactly two classes of involutions with representatives z, t_0. It follows from (2.3) that G has precisely two classes of involutions.

Let $x \in G$. Since $t_0 \sim_G z^x$, it follows there is an involution in $\langle t_0 z^x \rangle$ such that t_0, $z^x \in C(v)$. Since $C_G(t_0) \subseteq G_0$, $v \in G_0$; whence $C(v) \subseteq G_0$. Therefore $z^x \in G_0$. But z, z^x are already conjugate in G_0; whence $x \in G_0$ and $G = G_0$.

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