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Abstract

Let $G=(V,E)$ be a graph. A set $S \subseteq V$ is a dominating set if every vertex of $V-S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set D is a least dominating set if $\gamma(\langle D \rangle) \leqslant \gamma(\langle S \rangle)$ for any dominating set S, and $\gamma_\ell(G)$ is the minimum cardinality of a least dominating set. Sampathkumar (Discrete Math. 86 (1990) 137–142) conjectured that $\gamma_\ell(G) \leqslant 3n/5$ for every connected graph on $n \geqslant 2$ vertices. This conjecture was proven by Favaron (Discrete Math. 150 (1996) 115–122). We shall characterise graphs G of order n that are edge-minimal with respect to satisfying G connected and $\gamma_\ell(G)=3n/5$. Furthermore, we construct a family of graphs G of order n that are not cycles and are edge-minimal with respect to satisfying G connected, $\delta(G) \geqslant 2$ and $\gamma_\ell(G)=3n/5$. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

Let $G=(V,E)$ be a graph with vertex set V and edge set E, and let v be a vertex in V. The *open neighbourhood* of v is $N(v)=\{u \in V \mid uv \in E\}$ and the *closed neighbourhood* of v is $N[v]=\{v\} \cup N(v)$. A path (cycle) on n vertices is denoted by P_n (C_n, respectively). For a subset S of V, the subgraph of G induced by the vertices in S is denoted by $\langle S \rangle$. The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)$ (respectively, $\Delta(G)$). For disjoint subsets A and B of V, we define $[A,B]$ to be the set of all edges that join a vertex of A and a vertex of B. Furthermore, for $a \in A$, we define the *private neighbourhood* $pn(a,A,B)$ of a in B

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to be the set of vertices in B that are adjacent to a but to no other vertex of A; that is, $pn(a,A,B) = \{b \in B \mid N(b) \cap A = \{a\}\}$. For other graph theory terminology, we follow [1].

A set $D \subseteq V$ is a *dominating set* if every vertex in $V - D$ is adjacent to a vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. For disjoint subsets X and Y of V, we say X *dominates* Y if every vertex of Y is adjacent to some vertex of X. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The book by Chartrand and Lesniak [1] includes a chapter on domination. For a more thorough study of domination in graphs, see [3,4].

Various authors have investigated upper bounds on the domination number of a connected graph in terms of the minimum degree and order of the graph. The earliest such result is due to Ore [6], who showed that if G is a graph of order n with no isolated vertex, then $\gamma(G) \leqslant n/2$. McCraig and Shepherd [5] investigated upper bounds on the domination number of a connected graph with minimum degree at least 2.

Theorem 1 (McCraig and Shepherd [5]). *If G is a connected graph of order n with $\delta(G) \geqslant 2$, and if G is not one of seven exceptional graphs* (*one of order* 4 *and six of order* 7), *then* $\gamma(G) \leqslant 2n/5$.

McCraig and Shepherd [5] also characterised those connected graphs G of order n which are edge-minimal with respect to the satisfying $\delta(G) \geqslant 2$ and $\gamma(G) \geqslant 2n/5$.

Sampathkumar [7] introduced the concept of least domination in graphs. A *least dominating set* (*l.d.s.*) of a graph G is defined in [7] as a dominating set D satisfying $\gamma(\langle D \rangle) \leqslant \gamma(\langle S \rangle)$ for any dominating set S. The *least domination number* $\gamma_\ell(G)$ is the minimum cardinality of a least dominating set. We refer to a l.d.s. of G of cardinality $\gamma_\ell(G)$ as a γ_ℓ-set of G. Least domination in graphs has been studied by among others, Favaron [2], Sampathkumar [7], and Zverovich [8]. Results on least domination in graphs can also be found in the two books on domination by Haynes et al. [3,4]. An application for the concept of a least dominating set includes the following. A desirable property for a committee from a collection of people might be that every nonmember know at least one member of the committee, for ease of communication. Furthermore, among all such committees we may wish to select a subcommittee of smallest size from the committee with the desirable property that every committee member not on the subcommittee know at least one member of the subcommittee. A committee with a smallest such subcommittee is a least dominating set of the acquaintance graph of the set of people.

The least domination number of a path and a cycle is established in [7].

Proposition 2 (Sampathkumar [7]). *For the path P_n and cycle C_n,*

$$\gamma_\ell(P_n) = \gamma_\ell(C_n) = n - 2\left\lceil \frac{n}{5} \right\rceil.$$

When $n \equiv 0 \pmod 5$, Proposition 2 implies that $\gamma_\ell(P_n) = \gamma_\ell(C_n) = 3n/5$. Sampathkumar [7] conjectured that the least domination number of a connected nontrivial graph is at most three-fifths its order. This conjecture was proven by Favaron [2] and independently by Zverovich [8].

Theorem 3 (Favaron [2], Zverovich [8]). *If G is a graph of order n with no isolated vertex, then*

$$\gamma_\ell(G) \leqslant \frac{3n}{5}.$$

Proposition 2 shows that the bound of Theorem 3 is sharp even if we restrict the minimum degree to be at least 2. It appears a difficult problem to characterise connected graphs of order at least 2 with least domination number three-fifths their order. Hence, following McCraig and Shepherd [5] and others, we shall restrict our attention to edge-minimal graphs. More precisely, we will refer to a graph G of order $n \geqslant 2$ that is edge-minimal with respect to satisfying G connected and $\gamma_\ell(G) = 3n/5$ as a $\frac{3}{5}$*-minimal graph*. Furthermore, we will refer to G as a $\frac{3}{5}$*-minimal* 2*-graph* if G is edge-minimal with respect to satisfying the following three conditions:

(i) $\delta(G) \geqslant 2$,
(ii) G is connected, and
(iii) $\gamma_\ell(G) = 3n/5$.

In this paper we study graphs with least domination number three-fifths their order. We have two aims: first to characterise $\frac{3}{5}$-minimal graphs, and second to construct a family of $\frac{3}{5}$-minimal 2-graphs that are not cycles.

2. A family of $\frac{3}{5}$-minimal graphs

In order to characterise $\frac{3}{5}$-minimal graphs, we introduce a family $\mathscr{T}$ of $\frac{3}{5}$-minimal graphs. Let F be a forest that consists of $k \geqslant 1$ (disjoint) paths P_5. Colour the end-vertices in F with the colour *blue*, colour the vertices adjacent to an end-vertex with the colour *green*, and colour the central vertex of each path with the colour *red*. Hence each vertex in F is coloured either blue, green, or red. If $k \geqslant 2$, then we construct a tree G from the forest F by adding $k-1$ edges such that each added edge joins vertices of the same colour. If $k=1$, then we let $G=F$. We refer to the forest F as the *underlying forest* of G. The collection of all such trees G of order $5k$ we denote by $\mathscr{T}_k$ and the union of all the families $\mathscr{T}_k$ we denote by $\mathscr{T}$.

Before proceeding further, we introduce some additional notation. Let $G \in \mathscr{T}_k$. We let $\mathscr{H}_G = \{H_1, H_2, \ldots, H_k\}$, where $H_1, H_2, \ldots, H_k$ denote the k paths in the underlying forest of G. Let D_G denote the set of all green and red vertices in G. Then D_G is a dominating set of G of cardinality $3k = 3n/5$. Let R_G denote the set of red vertices in G. Then R_G is a dominating set of $\langle D_G \rangle$ of cardinality k, and so $\gamma(\langle D_G \rangle) \leqslant k$.

We shall prove:

Theorem 4. *If $G \in \mathscr{T}$, then D_G is the unique l.d.s. of G and R_G is the unique γ-set of $\langle D_G \rangle$.*

Proof. We proceed by induction on $k \geqslant 1$. If $k = 1$, then G is a path P_5 and it is straightforward to verify that the statement of the theorem is true. Suppose the result is true for all trees in $\mathscr{T}_{k'}$ where $1 \leqslant k' < k$. Let $G \in \mathscr{T}_k$ and let F denote the underlying forest of G. Let S be a γ_ℓ-set of G and let S' be a γ-set of $\langle S \rangle$. Since $\gamma(\langle D_G \rangle) \leqslant k$, we know that $|S'| = \gamma(\langle S \rangle) \leqslant k$.

Lemma 5. *If G contains an edge joining two green vertices, then $S = D_G$ and $S' = R_G$.*

Proof. Suppose $bb' \in E(G)$, where b and b' are two green vertices. By construction, bb' is a bridge of G and the two components of $G - bb'$ both belong to $\mathscr{T}$. Let G_1 and G_2 be the two components of $G - bb'$ where G_1 contains the vertex b. For $i = 1, 2$, let $S_i = S \cap V(G_i)$ and let $S_i' = S' \cap V(G_i)$. For $i = 1, 2$, we may assume G_i has order $5k_i$.

Claim 6. $b, b' \in S$.

Proof. If $b, b' \notin S$, then S_i is a dominating set of G_i for $i = 1, 2$. Applying the inductive hypothesis to G_i, D_{G_i} is the unique l.d.s. of G_i. Since $S_i \neq D_{G_i}$, $\gamma(\langle S_i \rangle) \geqslant k_i + 1$. Since $b, b' \notin S$, $\gamma(\langle S \rangle) = \gamma(\langle S_1 \rangle) + \gamma(\langle S_2 \rangle)$. Consequently, $\gamma(\langle S \rangle) \geqslant k_1 + k_2 + 2 = k + 2$, a contradiction. Hence we may assume that $b' \in S$.

Suppose $b \notin S$. If $\gamma(\langle S_2 \rangle) \geqslant k_2 + 1$, then $S_1 \cup D_{G_2}$ is a dominating set of G satisfying $\gamma(\langle S_1 \cup D_{G_2} \rangle) = \gamma(\langle S_1 \rangle) + \gamma(\langle D_{G_2} \rangle) < \gamma(\langle S_1 \rangle) + \gamma(\langle S_2 \rangle) = \gamma(\langle S \rangle)$, contradicting the fact that S is a l.d.s. of G. Hence $\gamma(\langle S_2 \rangle) \leqslant k_2$. Thus, applying the inductive hypothesis to G_2, $S_2 = D_{G_2}$, $\gamma(\langle S_2 \rangle) = k_2$ and S_2' consists of the red vertices of G_2.

If S_1 dominates b, then S_1 is a dominating set of G_1. However $b \notin S_1$, and so, applying the inductive hypothesis to G_i, $\gamma(\langle S_1 \rangle) \geqslant k_1 + 1$. Hence $\gamma(\langle S \rangle) = \gamma(\langle S_1 \rangle) + \gamma(\langle S_2 \rangle) \geqslant k_1 + k_2 + 1 = k + 1$, a contradiction. Thus S_1 cannot dominate b, i.e., no neighbour of b in G_1 belongs to S_1.

Let $b \in V(H)$ where $H \in \mathscr{H}_G$ denotes the path a, b, c, d, e. Then $a, c \notin S_1$. If b is adjacent to a green vertex in G_1, then there exists two adjacent green vertices that do not belong to S. As shown earlier, this produces a contradiction. Hence a and c are the only neighbours of b in G_1.

Since S_1 must dominate a, and $a, b \notin S_1$, a has degree at least 2 in G_1, and so $k_1 \geqslant 2$. Let G_a be the component of $G - ab$ containing a. By construction, G_a has order $5k_a + 1$ for some $k_a \geqslant 1$. Let $S_a = S \cap V(G_a)$.

Claim 6.1. $|S_a| \leqslant 3k_a$.

Proof. Since $b \notin S$, S_a is a dominating set of G_a. Since b is dominated by b' in S, S_a must be a γ_ℓ-set of G_a, for otherwise we could add the vertices in a γ_ℓ-set of G_a to the vertices in $S - S_a$ to produce a dominating set S^* of G satisfying either $\gamma(\langle S^* \rangle) < \gamma(\langle S \rangle)$ or $\gamma(\langle S^* \rangle) = \gamma(\langle S \rangle)$ and $|S^*| < |S|$, contradicting our choice of S. Hence, by Theorem 3, $|S_a| = \gamma_\ell(G_a) \leqslant 3(5k_a + 1)/5$, i.e., $|S_a| \leqslant 3k_a$. □

Let $D_a = D_G \cap V(G_a)$.

Claim 6.2. $\gamma(\langle S_a \rangle) \geqslant \gamma(\langle D_a \rangle) + 1$.

Proof. Let G' be obtained from G_a by attaching a path a, a_1, a_2, a_3, a_4 to a. Then $Y = S_a \cup \{a_2, a_3\}$ is a dominating set of G'. By construction, $G' \in \mathscr{T}$ and G' has order less than $5k$. Applying the inductive hypothesis to G', $D_{G'}$ is the unique γ_ℓ-set of G'. Note that $\gamma(\langle D_{G'} \rangle) = \gamma(\langle D_a \rangle) + 1$ and $\gamma(\langle Y \rangle) = \gamma(\langle S_a \rangle) + 1$. Furthermore, $|Y| = |S_a| + 2$ while $|D_{G'}| = |D_a| + 3$. Since $Y \neq D_{G'}$, Y cannot be a γ_ℓ-set of G'. Hence either $\gamma(\langle Y \rangle) \geqslant \gamma(\langle D_{G'} \rangle) + 1$, in which case $\gamma(\langle S_a \rangle) \geqslant \gamma(\langle D_a \rangle) + 1$, or $\gamma(\langle Y \rangle) = \gamma(\langle D_{G'} \rangle)$ and $|Y| > |D_{G'}|$, in which case $\gamma(\langle S_a \rangle) = \gamma(\langle D_a \rangle)$ and $|S_a| \geqslant |D_a| + 2 = 3k_a + 2$. However, by Claim 6.1, $|S_a| \leqslant 3k_a$. Consequently, $\gamma(\langle S_a \rangle) \geqslant \gamma(\langle D_a \rangle) + 1$. □

Let G_c be the component of $G - bc$ containing c. By construction, G_c has order $5k_c + 3$ for some $k_c \geqslant 0$. Let $S_c = S \cap V(G_c)$. Since $b, c \notin S$, $b, c \notin S_c$.

Claim 6.3. $|S_c| \leqslant 3k_c + 1$.

Proof. Since $b \notin S$, S_c is a dominating set of G_c. Since b is dominated by b' in S, S_c must be a γ_ℓ-set of G_c. Hence, by Theorem 3, $|S_c| = \gamma_\ell(G_c) \leqslant 3(5k_c + 3)/5$, i.e., $|S_c| \leqslant 3k_c + 1$. □

Let $D_c = D_{G_1} - D_a$.

Claim 6.4. $\gamma(\langle S_c \rangle) \geqslant \gamma(\langle D_c \rangle)$.

Proof. Let G' be obtained from G_c by attaching a path c, f, g to c. By construction, $G' \in \mathscr{T}$ and G' has order less than $5k$. Applying the inductive hypothesis to G', $D_{G'} = D_c$ is the unique γ_ℓ-set of G'. Let $Y = S_c \cup \{g\}$. Then Y is a dominating set of G'. Furthermore, $\gamma(\langle Y \rangle) = \gamma(\langle S_c \rangle) + 1$ and $|Y| = |S_c| + 1$. Since $Y \neq D_{G'}$, Y cannot be a γ_ℓ-set of G'. Hence either $\gamma(\langle Y \rangle) \geqslant \gamma(\langle D_{G'} \rangle) + 1$, in which case $\gamma(\langle S_c \rangle) \geqslant \gamma(\langle D_c \rangle)$, or $\gamma(\langle Y \rangle) = \gamma(\langle D_{G'} \rangle)$ and $|Y| > |D_{G'}|$, in which case $\gamma(\langle S_c \rangle) = \gamma(\langle D_c \rangle) - 1$ and $|S_c| \geqslant |D_c| = 3k_c + 3$. However, by Claim 6.3, $|S_c| \leqslant 3k_c + 1$. Consequently, $\gamma(\langle S_c \rangle) \geqslant \gamma(\langle D_c \rangle)$. □

By Claims 6.2 and 6.4, $\gamma(\langle S_1 \rangle) = \gamma(\langle S_a \rangle) + \gamma(\langle S_c \rangle) \geqslant \gamma(\langle D_a \rangle) + \gamma(\langle D_c \rangle) + 1 = \gamma(\langle D_{G_1} \rangle) + 1 = k_1 + 1$. Furthermore, $\gamma(\langle S_2 \rangle) = k_2$ as observed earlier. Hence, $\gamma(\langle S \rangle) =$

$\gamma(\langle S_1\rangle)+\gamma(\langle S_2\rangle)\geqslant k_1+k_2+1=k+1$, a contradiction. Hence we must have $b\in S$. This completes the proof of Claim 6. □

By Claim 6, $b,b'\in S$. Thus S_i is a dominating set of G_i for $i=1,2$. Applying the inductive hypothesis to G_i, $\gamma(\langle S_i\rangle)\geqslant k_i$ with equality if and only if $S_i=D_{G_i}$.

Suppose $S_i\neq D_{G_i}$ for $i=1,2$. Then $\gamma(\langle S_i\rangle)\geqslant k_i+1$ for each i. If $b,b'\in S'$, then S_i' dominates S_i, whence $|S_i'|\geqslant\gamma(\langle S_i\rangle)\geqslant k_i+1$. But then $|S'|=|S_1'|+|S_2'|\geqslant k+2$, a contradiction. So we may assume that $b\notin S'$. Then $S_1'\cup\{b\}$ dominates S_1, and so $|S_1'|+1\geqslant\gamma(\langle S_1\rangle)\geqslant k_1+1$, i.e., $|S_1'|\geqslant k_1$. If $b'\in S'$, then $|S_2'|\geqslant k_2+1$, and so $|S'|=|S_1'|+|S_2'|\geqslant k+1$, a contradiction. Hence $b'\notin S'$. Since $b,b'\notin S'$, S_i' dominates S_i for $i=1,2$, and so $|S_i'|\geqslant k_i+1$. Thus, $|S'|=|S_1'|+|S_2'|\geqslant k+2$, a contradiction. Hence we may assume that $S_1=D_{G_1}$.

Since $S_1=D_{G_1}$, $\gamma(\langle S_1\rangle)=k_1$ and the k_1 red vertices in G_1 form a unique γ-set of $\langle S_1\rangle$. If S_1' does not dominate S_1, then $S_1'\cup\{b\}$ dominates S_1. However, $S_1'\cup\{b\}$ is not the unique γ-set of $\langle S_1\rangle$, and so $|S_1'|+1\geqslant k_1+1$, i.e., $|S_1'|\geqslant k_1$. On the other hand, if S_1' does dominate S_1, then $|S_1'|\geqslant k_1$ with equality if and only if S_1' consists of the red vertices of G_1. In any event, $|S_1'|\geqslant k_1$.

Suppose $S_2\neq D_{G_2}$. Then $\gamma(\langle S_2\rangle)\geqslant k_2+1$. If $b'\in S'$, then S_2' dominates S_2 and therefore $|S_2'|\geqslant k_2+1$. But then $|S'|=|S_1'|+|S_2'|\geqslant k_1+k_2+1=k+1$, a contradiction. Hence $b'\notin S'$. Suppose S_2' does not dominate S_2. Then $b\in S_1'$, and so S_1' is not the unique γ-set of $\langle S_1\rangle$. Thus $|S_1'|\geqslant k_1+1$. Furthermore, $S_2'\cup\{b'\}$ dominates S_2. Consequently, $|S_2'|+1\geqslant\gamma(\langle S_2\rangle)\geqslant k_2+1$, i.e., $|S_2'|\geqslant k_2$. Thus $|S'|=|S_1'|+|S_2'|\geqslant k_1+k_2+1=k+1$, a contradiction. Hence S_2' dominates S_2, and so $|S_2'|\geqslant\gamma(\langle S_2\rangle)\geqslant k_2+1$. Thus $|S|=|S_1'|+|S_2'|\geqslant k_1+k_2+1=k+1$, a contradiction. Hence $S_2=D_{G_2}$.

We have now established that $S=S_1\cup S_2=D_{G_1}\cup D_{G_2}=D_G$. Furthermore, $\gamma(\langle S_i\rangle)=k_i$ and the k_i red vertices in G_i form a unique γ-set of $\langle S_i\rangle$. As observed earlier, $|S_i'|\geqslant k_i$ for $i=1,2$. If $b\in S'$, then, as observed earlier, $|S_1'|\geqslant k_1+1$, and so $|S|=|S_1'|+|S_2'|\geqslant k_1+k_2+1=k+1$, a contradiction. Hence $b\notin S'$. Similarly, $b'\notin S'$. Thus S_i' dominates S_i for $i=1,2$. By induction, $|S_i'|\geqslant k_i$ with equality if and only if S_i' consists of the red vertices of G_i. Since $k=|S'|=|S_1'|+|S_2'|\geqslant k_1+k_2=k$, it follows that $|S_i'|=k_i$ and S_i' consists of the red vertices of G_i for $i=1,2$. Thus $S'=R_G$. This completes the proof of Lemma 5. □

By Lemma 5, if G contains an edge joining two green vertices, then $S=D_G$ and $S'=R_G$, i.e., D_G is the unique l.d.s. of G and R_G is the unique γ-set of $\langle D_G\rangle$. Hence in what follows, we assume that there is no edge joining two green vertices.

Lemma 7. *If G contains an edge joining two red vertices, then $S=D_G$ and $S'=R_G$.*

Proof. The proof is similar to that of Lemma 5 and some of the details are therefore omitted. Suppose $cc'\in E(G)$, where c and c' are two red vertices. By construction, cc' is a bridge of G and the two components of $G-cc'$ both belong to $\mathscr{T}$. Let G_1 and

G_2 be the two components of $G-cc'$ where G_1 contains the vertex c. For $i=1,2$, let $S_i=S\cap V(G_i)$ and let $S_i'=S'\cap V(G_i)$. For $i=1,2$, we may assume G_i has order $5k_i$.

As in the proof of Claim 6, at least one of c or c' belongs to S. We may assume $c'\in S$. Suppose $c\notin S$. Then (as in the proof of Claim 6) $S_2=D_{G_2}$ and $\gamma(\langle S_2\rangle)=k_2$. Furthermore no neighbour of c in G_1 belongs to S_1. Let $c\in V(H)$ where $H\in\mathscr{H}_G$ denotes the path a,b,c,d,e. Then $b,d\notin S_1$. If c is adjacent to a red vertex in G_1, then there exists two adjacent red vertices that do not belong to S. This, however, produces a contradiction. Hence b and d are the only neighbours of c in G_1.

Let G_b be the component of $G-bc$ containing b. By construction, G_b has order $5k_b+2$ for some $k_b\geqslant 0$. Let $S_b=S\cap V(G_b)$ and let $D_b=(D_G\cap V(G_b))\cup\{c,d\}$. Then $|S_b|\leqslant 3k_b+1$ while $|D_b|=3k_b+3$. Let G' be obtained from G_b by attaching a path b,c,d,e to b. Then $Y=S_b\cup\{d\}$ is a dominating set of G'. By construction, $G'\in\mathscr{T}$ and G' has order less than $5k$. Applying the inductive hypothesis to G', $D_{G'}=D_b$ is the unique γ_ℓ-set of G'. Note that $\gamma(\langle Y\rangle)=\gamma(\langle S_b\rangle)+1$ while $|Y|=|S_b|+1$. Since $Y\neq D_{G'}$, Y cannot be a γ_ℓ-set of G'. Hence either $\gamma(\langle Y\rangle)\geqslant\gamma(\langle D_{G'}\rangle)+1$, in which case $\gamma(\langle S_b\rangle)\geqslant\gamma(\langle D_b\rangle)$, or $\gamma(\langle Y\rangle)=\gamma(\langle D_{G'}\rangle)$ and $|Y|>|D_{G'}|$, in which case $\gamma(\langle S_b\rangle)=\gamma(\langle D_b\rangle)-1$ and $|S_b|\geqslant|D_b|=3k_b+3$. However, as observed earlier, $|S_b|\leqslant 3k_b+1$. Consequently, $\gamma(\langle S_b\rangle)\geqslant\gamma(\langle D_b\rangle)$.

Let G_d be the component of $G-cd$ containing d. By construction, G_d has order $5k_d+2$ for some $k_d\geqslant 0$. Let $S_d=S\cap V(G_d)$ and let $D_d=(D_G\cap V(G_d))\cup\{b,c\}$. Then $\gamma(\langle S_d\rangle)\geqslant\gamma(\langle D_d\rangle)$.

Now $\gamma(\langle S_1\rangle)=\gamma(\langle S_b\rangle)+\gamma(\langle S_d\rangle)\geqslant\gamma(\langle D_b\rangle)+\gamma(\langle D_d\rangle)=\gamma(\langle D_{G_1}\rangle)+1=k_1+1$. Furthermore, $\gamma(\langle S_2\rangle)=k_2$ as observed earlier. Hence, $\gamma(\langle S\rangle)=\gamma(\langle S_1\rangle)+\gamma(\langle S_2\rangle)\geqslant k_1+k_2+1=k+1$, a contradiction. Hence we must have $c\in S$.

Since $c,c'\in S$, S_i is a dominating set of G_i for $i=1,2$. Continuing now as in the last four paragraphs of the proof of Lemma 5 (with 'b' and 'b'' replaced by 'c' and 'c''), respectively, we can show that $S=D_G$ and $S'=R_G$. This completes the proof of Lemma 7. $\square$

By Lemma 7, if G contains an edge joining two red vertices, then D_G is the unique l.d.s. of G and R_G is the unique γ-set of $\langle D_G\rangle$. Hence in what follows, we assume that there is no edge joining two red vertices. Thus all $k-1\geqslant 1$ edges added to the underlying forest of G to construct G join blue vertices.

Suppose a and a' are two adjacent blue vertices of G. Let G_1 and G_2 be the two components of $G-aa'$ where G_1 contains the vertex a. For $i=1,2$, let $S_i=S\cap V(G_i)$ and let $S_i'=S'\cap V(G_i)$. By construction, each of G_1 and G_2 belong to $\mathscr{T}$. For $i=1,2$, we may assume $G_i\in\mathscr{T}_{k_i}$. Applying the inductive hypothesis to G_i, D_{G_i} is the unique l.d.s. of G_i and the red vertices in G_i form a unique γ-set of $\langle D_{G_i}\rangle$ for $i=1,2$.

Suppose $a,a'\in S$. Then $S_i\neq D_{G_i}$ and $\gamma(\langle S_i\rangle)\geqslant k_i+1$ for $i=1,2$. If $a,a'\in S'$, then S_i' dominates S_i, whence $|S_i'|\geqslant\gamma(\langle S_i\rangle)\geqslant k_i+1$. But then $|S'|=|S_1'|+|S_2'|\geqslant k+2$, a contradiction. So we may assume that $a\notin S'$. Then $S_1'\cup\{a\}$ dominates S_1, and so $|S_1'|+1\geqslant\gamma(\langle S_1\rangle)\geqslant k_1+1$, i.e., $|S_1'|\geqslant k_1$. If $a'\in S'$, then $|S_2'|\geqslant k_2+1$, and so $|S'|=|S_1'|+|S_2'|\geqslant k+1$, a contradiction. Hence $a'\notin S'$. Since $a,a'\notin S'$, S_i' dominates S_i for

$i=1,2$, and so $|S_i'|\geqslant k_i+1$. Thus, $|S'|=|S_1'|+|S_2'|\geqslant k+2$, a contradiction. Hence we may assume $a\notin S$.

Suppose $a'\in S$. Then $S_2\neq D_{G_2}$ and $\gamma(\langle S_2\rangle)\geqslant k_2+1$. If S_1 does not dominate G_1, then $S_1\cup\{a\}$ dominates S_1. Since $S_1\cup\{a\}\neq D_{G_1}$, $\gamma(\langle S_1\rangle)+1=\gamma(\langle S_1\cup\{a\}\rangle)\geqslant k_1+1$, and so $\gamma(\langle S_1\rangle)\geqslant k_1$. On the other hand, if S_1 dominates G_1, then $\gamma(\langle S_1\rangle)\geqslant k_1$. In any event, $\gamma(\langle S_1\rangle)\geqslant k_1$. Hence, since $a\notin S$, $\gamma(\langle S\rangle)=\gamma(\langle S_1\rangle)+\gamma(\langle S_2\rangle)\geqslant k_1+k_2+1=k+1$, a contradiction. Hence $a'\notin S$.

Since $a,a'\notin S$, S_i is a dominating set of G_i and S_i' dominates S_i for $i=1,2$. By induction, $|S_i'|\geqslant k_i$ with equality if and only if $S_i=D_{G_i}$ and S_i' consists of the red vertices of G_i. Since $k\geqslant|S'|=|S_1'|+|S_2'|\geqslant k_1+k_2=k$, it follows that $S_i=D_{G_i}$, $|S_i'|=k_i$ and S_i' consists of the red vertices of G_i for $i=1,2$. Thus $S=D_G$ and $S'=R_G$. This completes the proof of Theorem 4. □

By Theorem 4, D_G is the unique γ_ℓ-set of G. In particular, $\gamma_\ell(G)=|D_G|=3k=3n/5$. Furthermore, G is edge-minimal with respect to satisfying G connected. Hence we have the following result.

Proposition 8. *Each graph in the family $\mathscr{T}$ is a $\frac{3}{5}$-minimal graph.*

3. A characterization of $\frac{3}{5}$-minimal graphs

We shall prove:

Theorem 9. *A graph G is a $\frac{3}{5}$-minimal graph if and only if $G\in\mathscr{T}$.*

The sufficiency of Theorem 9 follows from Proposition 8. To prove the necessity of Theorem 9, we first present a proof of Theorem 3. The proof follows that of Favaron [2] and Zverovich [8].

Proof of Theorem 3. Let $G=(V,E)$ be a graph of order n with no isolated vertex. Let D be a γ_ℓ-set of G with the minimum number of isolated vertices in $\langle D\rangle$. Let I be the set of isolated vertices in $\langle D\rangle$. Let X be a minimum dominating set of $\langle D-I\rangle$, and let $Y=D-(I\cup X)$. Then $\mathrm{pn}(x,X,Y)\neq\emptyset$ for every $x\in X$. Let $X_1=\{x\in X\colon |\mathrm{pn}(x,X,Y)|=1\}$ and let $X_2=X-X_1$.

Claim 10. $\mathrm{pn}(v,D,V-D)\neq\emptyset$ *for every* $v\in D-X_2$.

Proof. If $\mathrm{pn}(v,I,V-D)=\emptyset$ and $v'\in N(v)$, then $D'=(D-\{v\})\cup\{v'\}$ is a γ_ℓ-set of G with fewer isolated vertices in $\langle D'\rangle$ than in $\langle D\rangle$, contrary to our choice of D. Hence $\mathrm{pn}(v,I,V-D)\neq\emptyset$ for every $v\in I$. Clearly, the minimality of D implies that $\mathrm{pn}(y,Y,V-D)\neq\emptyset$ for every $y\in Y$. Finally, if $x\in X_1$ and $\mathrm{pn}(x,X_1,Y)=\{y\}$, then $\mathrm{pn}(x,X_1,V-D)\neq\emptyset$, for otherwise $D-\{x\}$ is a dominating set of G and

$\gamma(\langle D-\{x\}\rangle)=|(I\cup X)-\{x\}\cup\{y\}|=|I\cup X|=\gamma(\langle D\rangle)$, which contradicts the minimality of D. □

By Claim 10, $|V-D|\geqslant|D-X_2|$, and so $n-\gamma_\ell(G)\geqslant\gamma_\ell(G)-|X_2|$, or, equivalently, $\gamma_\ell(G)\leqslant(n+|X_2|)/2$. Furthermore, by definition of X_2, $|Y|\geqslant 2|X_2|$. Hence $n=|V-D|+|D|\geqslant|D-X_2|+|D|=2|D|-|X_2|\geqslant 2(|Y|+|X_2|)-|X_2|=2|Y|+|X_2|\geqslant 5|X_2|$, and so $|X_2|\leqslant n/5$. Thus $\gamma_\ell(G)\leqslant(n+|X_2|)/2\leqslant 3n/5$. This completes the proof of Theorem 3. □

We are now in a position to prove the necessity of Theorem 9. We proceed by induction on the order $n=5k$, where $k\geqslant 1$ is an integer, of a $\frac{3}{5}$-minimal graph. It is straightforward to check that the only $\frac{3}{5}$-minimal graph on five vertices is $P_5\in\mathscr{T}$. Hence the result is true if $k=1$. Let $k\geqslant 2$, and assume the result is true for all $\frac{3}{5}$-minimal graphs of order less than n. Let $G=(V,E)$ be a $\frac{3}{5}$-minimal graph of order $n=5k$. If $G\cong P_n$, then the result follows. So we may assume that G is not a path. Since C_n is not a $\frac{3}{5}$-minimal graph, we must have $\Delta(G)\geqslant 3$. In what follows, we shall use the notation employed in the proof of Theorem 3 presented above.

Since $\gamma_\ell(G)=3n/5$, all the inequalities in the last paragraph of the proof of Theorem 3 must be equalities. In particular, $|D|=|Y|+|X_2|$ (and so $I=\emptyset$ and $X=X_2$), $X=\{x\in X\colon |\mathrm{pn}(x,X,Y)|=2\}$, and $|\mathrm{pn}(y,Y,V-D)|=1$ for every $y\in Y$. Let $X=\{x_1,\dots,x_k\}$, $Y=\{y_1,\dots,y_k\}\cup\{w_1,\dots,w_k\}$ and $Z=\{a_1,\dots,a_k\}\cup\{b_1,\dots,b_k\}=V-D$. Then G has the following structure. For each $i=1,\dots,k$, $N(x_i)\cap(V-X)=\{y_i,w_i\}$, $N(y_i)\cap Z=\{a_i\}$, and $N(w_i)\cap Z=\{b_i\}$.

For $i=1,\dots,k$, if y_iw_i is an edge of G, then $D-\{x_i\}$ is a dominating set of G and $\gamma(\langle D-\{x_i\}\rangle)=|X-\{x_i\}\cup\{y_i\}|=|X|=\gamma(\langle D\rangle)$, which contradicts the minimality of D. Hence y_iw_i cannot be an edge of G. Let $H_i=\langle\{x_i,y_i,w_i,a_i,b_i\}\rangle$, and let $\mathscr{H}_G=\{H_1,\dots,H_k\}$.

Before proceeding further, we prove a few results that will be useful in what follows.

Claim 11. *If $e\in E$ and $e\notin[X,Y]\cup[Y,Z]$, then e is a bridge of G.*

Proof. Suppose $G-e$ is connected. Let S be a γ_ℓ-set of $G-e$. Since $G-e$ has no isolated vertex, Theorem 3 implies that $|S|=\gamma_\ell(G-e)\leqslant 3n/5$. Furthermore, since S is a dominating set of G, $\gamma(\langle D\rangle)\leqslant\gamma(\langle S\rangle)$. On the other hand, since $e\notin[X,Y]\cup[Y,Z]$, D is a dominating set of $G-e$, and so $\gamma(\langle S\rangle)\leqslant\gamma(\langle D\rangle)$. Consequently, $\gamma(\langle S\rangle)=\gamma(\langle D\rangle)$. Thus S is a l.d.s. of G, and so $3n/5=|D|=\gamma_\ell(G)\leqslant|S|\leqslant 3n/5$. Hence we must have $|S|=|D|=3n/5$. Thus, $G-e$ is a connected graph satisfying $\gamma_\ell(G-e)=|S|=3n/5$. This contradicts the minimality of G. Hence $G-e$ is disconnected. □

By Claim 11, $a_ib_i\notin E(G)$ for all $i=1,\dots,k$. Hence $H\cong P_5$ for each $H\in\mathscr{H}_G$.

Claim 12. *If there is a vertex in $X\cup Y$ of degree at least 3, then $G\in\mathscr{T}$.*

Proof. Suppose $\deg v \geqslant 3$ for some $v \in X \cup Y$. Suppose $v \in V(H)$, where $H \in \mathscr{H}_G$. Since v has degree 2 in H, v must be adjacent to a vertex u not in H. From the structure of G we know that either $u, v \in X$ or $u, v \in Y$. In any event, $\deg u \geqslant 3$ and $uv \notin [X, Y] \cup [Y, Z]$. By Claim 11, uv is a bridge of G. Thus $G - uv$ contains two components, namely a component G_1 containing u and a component G_2 containing v. For $i = 1, 2$, let G_i have order n_i. Since the vertices of each graph in $\mathscr{H}_G$ all belong to the same component of $G - uv$, $n_i \equiv 0 \pmod 5$. Suppose G_i contains k_i of the subgraphs of $\mathscr{H}_G$. Then $n_i = 5k_i$. Furthermore, $k = k_1 + k_2$.

For $i = 1, 2$, let $D_i = D \cap V(G_i)$. Then $|D_i| = 3k_i$ and $\gamma(\langle D_i \rangle) = k_i$. For $i = 1, 2$, let S_i be a γ_ℓ-set of G_i. If $\gamma(\langle S_1 \rangle) < k_1$, then $S_1 \cup D_2$ would be a dominating set of G satisfying $\gamma(\langle S_1 \cup D_2 \rangle) < k = \gamma(\langle D \rangle)$, contradicting our choice of D. Hence $\gamma(\langle S_1 \rangle) \geqslant k_1$. However, since D_1 is a dominating set of G_1 and $\gamma(\langle D_1 \rangle) = k_1$, $\gamma(\langle S_1 \rangle) = k_1$. If $|S_1| < 3k_1$, then $S_1 \cup D_2$ would be a dominating set of G satisfying $\gamma(\langle S_1 \cup D_2 \rangle) = k$ and $|S_1 \cup D_2| < 3k = |D|$, contradicting our choice of D. Hence $|S_1| = 3k_1$. Thus D_1 is a γ_ℓ-set of G_1. Similarly, D_2 is a γ_ℓ-set of G_2. Thus, for $i = 1, 2$, G_i is a connected graph satisfying $\gamma_\ell(G_i) = 3n_i/5$. By the inductive hypothesis, $G_i \in \mathscr{T}_{k_i}$ for $i = 1, 2$. Furthermore, since D_i is a γ_ℓ-set of G_i, $D_i = D_{G_i}$ by Theorem 4. Thus the vertices of X, Y, and Z in G_i are coloured red, green, and blue, respectively. If $u, v \in X$, then u and v are both coloured red. On the other hand, if $u, v \in Y$, then u and v are both coloured green. In any event, $G \in \mathscr{T}$. This completes the proof of Claim 12. □

In what follows, we may assume that each vertex in $X \cup Y$ has degree 2 in G, for otherwise $G \in \mathscr{T}$ by Claim 12. Hence for each $i = 1, \ldots, k$, $N(x_i) = \{y_i, w_i\}$, $N(y_i) = \{a_i, x_i\}$, and $N(w_i) = \{b_i, x_i\}$. By Claim 11, each edge in $\langle Z \rangle$ is a bridge of G. Thus G is obtained from $k \geqslant 2$ (disjoint) paths P_5 by adding $k - 1$ edges that join end-vertices from different paths (to produce a connected graph), i.e., $G \in \mathscr{T}$. This completes the proof of Theorem 9. □

4. A family of $\frac{3}{5}$-minimal 2-graphs that are not cycles

Let $\mathscr{C}_5$ denote the family of all cycles of length congruent to 0 modulo 5, that is,

$$\mathscr{C}_5 = \{C_n \mid n \equiv 0 \pmod 5\}.$$

By Proposition 2, each graph in $\mathscr{C}_5$ has least domination number three-fifths its order. Furthermore, each graph in $\mathscr{C}_5$ is clearly edge-minimal with respect to satisfying minimum degree at least 2. Hence we have the following result.

Proposition 13. *Each graph in the family $\mathscr{C}_5$ is a $\frac{3}{5}$-minimal 2-graph.*

In this section our aim is to construct a family of $\frac{3}{5}$-minimal 2-graphs, which we call $\mathscr{G}^*$, that is different from the family $\mathscr{C}_5$. For this purpose, let $F_1 = (V, E_1)$ be a forest that consists of $k \geqslant 3$ (disjoint) K_2s, i.e., $F_1 \cong kK_2$. Colour the vertices in F_1 with the

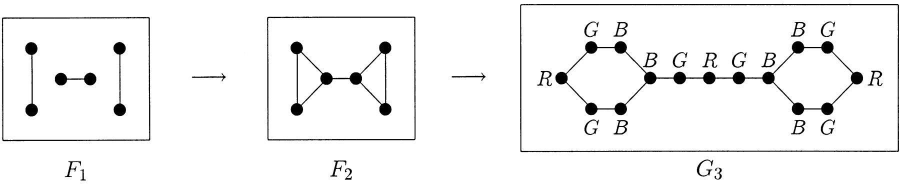

Fig. 1. The construction of the graph $G_3 \in \mathscr{G}$.

colour *blue*. We construct a graph $F_2 = (V, E_1 \cup E_2)$ from the forest F_1 by adding a set E_2 of edges to F_1 in such a way that there are no even cycles that alternate in edges of E_1 and $E_2 - E_1$ and such that F_2 is edge-minimal with respect to satisfying $\delta(F_2) \geqslant 2$ and F_2 connected. We now construct a graph G from F_2 by subdividing each edge of E_1 three times. Each resulting new vertex that is adjacent to a blue vertex we colour with the colour *green*, while each new vertex that is not adjacent to a blue vertex we colour with the colour *red*. We let V_G denote the set of vertices of G that are coloured green or red and are incident with a bridge in G. We refer to the forest F_1 as the *underlying forest* of G and the graph F_2 as the *underlying graph* of G.

By construction, G is a connected graph with minimum degree at least 2 and of order $n = 5k$ for some $k \geqslant 3$. Furthermore, for each edge e of G, $G - e$ is disconnected or $\delta(G - e) = 1$. The collection of all such graphs G of order $5k$ we denote by $\mathscr{G}_k$ and the union of all the families $\mathscr{G}_k$ we denote by $\mathscr{G}$. If $k = 3$, then $\mathscr{G}_k = \{G_3\}$, where G_3 is the graph in $\mathscr{G}$ with underlying forest $F_1 \cong 3K_2$ and with underlying graph F_2 shown in Fig. 1. (The vertices in G_3 coloured blue, green, and red are labelled B, G, and R, respectively.)

To construct the family $\mathscr{G}^*$, let $G_1, \ldots, G_m$ be $m \geqslant 1$ graphs in $\mathscr{G}$. Let G^* be a connected graph obtained from the (disjoint) union $\bigcup_{i=1}^{m} G_i$ by adding a set of $m - 1$ edges E^* such that each added edge joins vertices of the same colour in $\bigcup_{i=1}^{m} V_{G_i}$. If $m = 1$, then $G^* = G_1$. Let E_B denote the set of all edges of G^* that join two blue vertices. By construction, G^* has order congruent to 0 modulo 5 and is edge-minimal with respect to satisfying $\delta(G^*) \geqslant 2$ and G^* connected. The collection of all such graphs G^* we denote by $\mathscr{G}^*$.

Before proceeding further, we present some properties of graphs in the family $\mathscr{G}^*$. Let $G \in \mathscr{G}^*$ have order $5k$. Then, by construction, $G - E^* - E_B$ consists of k (vertex disjoint) P_5 s which we denote by $H_1, H_2, \ldots, H_k$. Let $\mathscr{H}_G = \{H_1, H_2, \ldots, H_k\}$. We refer to $\mathscr{H}_G$ as the path partition of G. Let D_G denote the set of all green and red vertices in G. Then D_G is a dominating set of G of cardinality $3k = 3n/5$. Let R_G denote the set of red vertices in G. Then R_G is a dominating set of $\langle D_G \rangle$ of cardinality k, and so $\gamma(\langle D_G \rangle) \leqslant k$.

We shall prove:

Theorem 14. *If $G \in \mathscr{G}^*$, then D_G is the unique l.d.s. of G and R_G is the unique* γ-set *of* $\langle D_G \rangle$.

Proof. We proceed by induction on $k \geqslant 3$. If $k = 3$, then G is the graph G_3 of Fig. 1 and it is straightforward to verify that the statement of the theorem is true. Suppose the result is true for all graphs in $\mathscr{G}^*$ of order less than $5k$. Let $G \in \mathscr{G}^*$ have order $5k$. Let S be a γ_ℓ-set of G and let S' be a γ-set of $\langle S \rangle$. Since $\gamma(\langle D_G \rangle) \leqslant k$, we know that $|S'| = \gamma(\langle S \rangle) \leqslant k$.

Lemma 15. *If G contains an edge joining two green vertices, then $S = D_G$ and $S' = R_G$.*

Proof. The proof is similar to that of Lemma 5 and some of the details are therefore omitted. Suppose $bb' \in E(G)$, where b and b' are two green vertices. By construction, bb' is a bridge of G and the two components of $G - bb'$ both belong to $\mathscr{G}^*$. Let G_1 and G_2 be the two components of $G - bb'$ where G_1 contains the vertex b. For $i = 1, 2$, let $S_i = S \cap V(G_i)$ and let $S'_i = S' \cap V(G_i)$. For $i = 1, 2$, we may assume G_i has order $5k_i$.

Claim 16. $b, b' \in S$.

Proof. We may assume (as in the proof of Lemma 5) that $b' \in S$. Suppose $b \notin S$. Then (as in the proof of Lemma 5) $S_2 = D_{G_2}$, $\gamma(\langle S_2 \rangle) = k_2$ and S'_2 consists of the red vertices of G_2. Let $b \in V(H)$ where $H \in \mathscr{H}_G$ denotes the path a, b, c, d, e. Then a and c are the only neighbours of b in G_1 and $a, c \notin S_1$. Let G_a be the component of $G - ab$ containing a. By construction, G_a has order $5k_a + 1$ for some $k_a \geqslant 1$. Let $S_a = S \cap V(G_a)$. Then $|S_a| \leqslant 3k_a$. Let $D_a = D_G \cap V(G_a)$.

Claim 17. $\gamma(\langle S_a \rangle) \geqslant \gamma(\langle D_a \rangle) + 1$.

Proof. Let G' be obtained from G_a by attaching a path a, a_1, a_2, a_3, a_4 to a and then attaching a 6-cycle $a_4, v_1, v_2, v_3, v_4, v_5, a_4$ to a_4. Then $Y = S_a \cup \{a_2, a_3, a_4, v_3\}$ is a dominating set of G'. By construction, $G' \in \mathscr{G}^*$ and G' has order less than $5k$. Applying the inductive hypothesis to G', $D_{G'}$ is the unique γ_ℓ-set of G'. Note that $\gamma(\langle D_{G'} \rangle) = \gamma(\langle D_a \rangle) + 2$ and $\gamma(\langle Y \rangle) = \gamma(\langle S_a \rangle) + 2$. Furthermore, $|Y| = |S_a| + 4$ while $|D_{G'}| = |D_a| + 6$. Since $Y \neq D_{G'}$, Y cannot be a γ_ℓ-set of G'. Hence either $\gamma(\langle Y \rangle) \geqslant \gamma(\langle D_{G'} \rangle) + 1$, in which case $\gamma(\langle S_a \rangle) \geqslant \gamma(\langle D_a \rangle) + 1$, or $\gamma(\langle Y \rangle) = \gamma(\langle D_{G'} \rangle)$ and $|Y| > |D_{G'}|$, in which case $\gamma(\langle S_a \rangle) = \gamma(\langle D_a \rangle)$ and $|S_a| \geqslant |D_a| + 3 = 3k_a + 3$. However, as observed earlier, $|S_a| \leqslant 3k_a$. Consequently, $\gamma(\langle S_a \rangle) \geqslant \gamma(\langle D_a \rangle) + 1$. □

Let G_c be the component of $G - bc$ containing c. By construction, G_c has order $5k_c + 3$ for some $k_c \geqslant 1$. Let $S_c = S \cap V(G_c)$. Then $|S_c| \leqslant 3k_c + 1$. Let $D_c = D_{G_1} - D_a$.

Claim 18. $\gamma(\langle S_c \rangle) \geqslant \gamma(\langle D_c \rangle)$.

Proof. Let G' be obtained from G_c by attaching a path c, b, a to c and then attaching a 6-cycle $a, v_1, v_2, v_3, v_4, v_5, a$ to a. Then $Y = S_c \cup \{a, v_3\}$ is a dominating set of G'. By construction, $G' \in \mathscr{G}^*$ and G' has order less than $5k$. Applying the inductive hypothesis to G', $D_{G'}$ is the unique γ_ℓ-set of G'. Note that $\gamma(\langle D_{G'}\rangle)=\gamma(\langle D_c\rangle)+1$ and $\gamma(\langle Y\rangle)=\gamma(\langle S_c\rangle)+2$. Furthermore, $|Y|=|S_c|+2$ while $|D_{G'}|=|D_c|+3$. Since $Y\neq D_{G'}$, Y cannot be a γ_ℓ-set of G'. Hence either $\gamma(\langle Y\rangle)\geqslant\gamma(\langle D_{G'}\rangle)+1$, in which case $\gamma(\langle S_c\rangle)\geqslant\gamma(\langle D_c\rangle)$, or $\gamma(\langle Y\rangle)=\gamma(\langle D_{G'}\rangle)$ and $|Y|>|D_{G'}|$, in which case $\gamma(\langle S_c\rangle)=\gamma(\langle D_c\rangle)-1$ and $|S_c|\geqslant|D_c|+2=3k_c+2$. However, as observed earlier, $|S_c|\leqslant 3k_c+1$. Consequently, $\gamma(\langle S_c\rangle)\geqslant\gamma(\langle D_c\rangle)$. □

By Claims 17 and 18, $\gamma(\langle S_1\rangle)=\gamma(\langle S_a\rangle)+\gamma(\langle S_c\rangle)\geqslant\gamma(\langle D_a\rangle)+\gamma(\langle D_c\rangle)+1=\gamma(\langle D_{G_1}\rangle)+1=k_1+1$. Hence, $\gamma(\langle S\rangle)=\gamma(\langle S_1\rangle)+\gamma(\langle S_2\rangle)\geqslant k_1+k_2+1=k+1$, a contradiction. Hence we must have $b\in S$. This completes the proof of Claim 16. □

By Claim 16, $b,b'\in S$. Proceeding now as in the proof of Lemma 5, we can show that $S=D_G$ and that $S'=R_G$. This completes the proof of Lemma 15. □

By Lemma 15, if G contains an edge joining two green vertices, then $S=D_G$ and $S'=R_G$, i.e., D_G is the unique l.d.s. of G and R_G is the unique γ-set of $\langle D_G\rangle$. Hence in what follows, we assume that there is no edge joining two green vertices.

Lemma 19. *If G contains an edge joining two red vertices, then $S=D_G$ and $S'=R_G$.*

Proof. The proof is similar to that of Lemma 7 and some of the details are therefore omitted. Suppose $cc'\in E(G)$, where c and c' are two red vertices. By construction, cc' is a bridge of G and the two components of $G-cc'$ both belong to $\mathscr{G}^*$. Let G_1 and G_2 be the two components of $G-cc'$ where G_1 contains the vertex c. For $i=1,2$, let $S_i=S\cap V(G_i)$ and let $S_i'=S'\cap V(G_i)$. For $i=1,2$, we may assume G_i has order $5k_i$. We may assume $c'\in S$. Suppose $c\notin S$. Then $S_2=D_{G_2}$ and $\gamma(\langle S_2\rangle)=k_2$. Let $c\in V(H)$, where $H\in\mathscr{H}_G$ denotes the path a,b,c,d,e. Then b and d are the only neighbours of c in G_1 and $b,d\notin S_1$.

Let G_b be the component of $G-bc$ containing b. By construction, G_b has order $5k_b+2$ for some $k_b\geqslant 1$. Let $S_b=S\cap V(G_b)$ and let $D_b=(D_G\cap V(G_b))\cup\{c,d\}$. Then $|S_b|\leqslant 3k_b+1$ while $|D_b|=3k_b+3$. Let G' be obtained from G_b by attaching a path b,c,d,e to b and then attaching a 6-cycle e,v_1,v_2,v_3,v_4,v_5,e to e. Then $Y=S_b\cup\{d,e,v_3\}$ is a dominating set of G'. By construction, $G'\in\mathscr{G}^*$ and G' has order less than $5k$. Applying the inductive hypothesis to G', $D_{G'}$ is the unique γ_ℓ-set of G'. Note that $\gamma(\langle D_{G'}\rangle)=\gamma(\langle D_b\rangle)+1$ and $\gamma(\langle Y\rangle)=\gamma(\langle S_b\rangle)+2$. Furthermore, $|Y|=|S_b|+3$ while $|D_{G'}|=|D_b|+3$. Since $Y\neq D_{G'}$, Y cannot be a γ_ℓ-set of G'. Hence either $\gamma(\langle Y\rangle)\geqslant\gamma(\langle D_{G'}\rangle)+1$, in which case $\gamma(\langle S_b\rangle)\geqslant\gamma(\langle D_b\rangle)$, or $\gamma(\langle Y\rangle)=\gamma(\langle D_{G'}\rangle)$ and $|Y|>|D_{G'}|$, in which case $\gamma(\langle S_b\rangle)=\gamma(\langle D_b\rangle)-1$ and $|S_b|\geqslant|D_b|+1=3k_b+4$. However, as observed earlier, $|S_b|\leqslant 3k_b+1$. Consequently, $\gamma(\langle S_b\rangle)\geqslant\gamma(\langle D_b\rangle)$.

Let G_d be the component of $G-cd$ containing d. By construction, G_d has order $5k_d+2$ for some $k_d\geqslant 1$. Let $S_d=S\cap V(G_d)$ and let $D_d=(D_G\cap V(G_d))\cup\{b,c\}$. Then $\gamma(\langle S_d\rangle)\geqslant\gamma(\langle D_d\rangle)$.

Now $\gamma(\langle S_1\rangle)=\gamma(\langle S_b\rangle)+\gamma(\langle S_d\rangle)\geqslant\gamma(\langle D_b\rangle)+\gamma(\langle D_d\rangle)=\gamma(\langle D_{G_1}\rangle)+1=k_1+1$. Furthermore, $\gamma(\langle S_2\rangle)=k_2$ as observed earlier. Hence, $\gamma(\langle S\rangle)=\gamma(\langle S_1\rangle)+\gamma(\langle S_2\rangle)\geqslant k_1+k_2+1=k+1$, a contradiction. Hence we must have $c\in S$.

Proceeding now as in the proof of Lemma 7, we can show that $S=D_G$ and that $S'=R_G$. This completes the proof of Lemma 19. □

By Lemma 19, if G contains an edge joining two red vertices, then D_G is the unique l.d.s. of G and R_G is the unique γ-set of $\langle D_G\rangle$. Hence in what follows, we assume that there is no edge joining two red vertices. Thus $G\in\mathscr{G}_k\in\mathscr{G}$. If G contains a bridge joining two blue vertices, then, proceeding as in the last four paragraphs of the proof of Theorem 4, we can show that $S=D_G$ and that $S'=R_G$. Hence we assume that there is no bridge in G joining two blue vertices. Thus G is obtained from a path a,b,c,d,e by attaching at least one cycle of length at least 6 and congruent to 1 modulo 5 to each of a and e (by attaching a cycle to a vertex v we mean adding a (disjoint) cycle to the graph and identifying one of its vertices with v). We may assume $\deg a\geqslant\deg e$. Since $k\geqslant 4$, at least one cycle in G has length at least 11 or at least two cycles are attached to a. Let C be a cycle attached to a. Let a' be a neighbour of a on C. Suppose $a'\in V(H)$ where $H\in\mathscr{H}_G$ denotes the path a',b',c',d',e'.

Claim 20. *If C is a* 6*-cycle, then $S=D_G$ and $S'=R_G$.*

Proof. Since C is a 6-cycle, $ae'\in E(G)$. Let G_1 and G_2 be the two components of $G-\{aa',ae'\}$ where G_1 contains the vertex a. For $i=1,2$, let $S_i=S\cap V(G_i)$ and let $S_i'=S'\cap V(G_i)$. By construction, G_1 belongs to $\mathscr{G}_{k-1}$.

Suppose S_1 is not a dominating set of G_1. Then S_1 does not contain a nor any neighbour of a. However, $S_1\cup\{a\}$ is a dominating set of G_1. Since $S_1\cup\{a\}\neq D_{G_1}$, $\gamma(\langle S_1\rangle)+1=\gamma(\langle S_1\cup\{a\}\rangle)\geqslant k$, and so $\gamma(\langle S_1\rangle)\geqslant k-1$. Since a is not dominated by S_1, S_2 must be a dominating set of the 6-cycle a,a',b',c',d',e',a, and so $\gamma(\langle S_2\rangle)\geqslant 2$. Thus, $\gamma(\langle S\rangle)=\gamma(\langle S_1\rangle)+\gamma(\langle S_2\rangle)\geqslant k+1$, a contradiction. Hence S_1 is a dominating set of G_1.

Applying the inductive hypothesis to G_1, $\gamma(\langle S_1\rangle)\geqslant k-1$ with equality if and only if $S_1=D_{G_1}$ and S_1' consists of the red vertices of G_1. Since $k\geqslant|S'|\geqslant|S_1'|+1\geqslant k$, it follows that $S_1=D_{G_1}$, $|S_1'|=k-1$ and S_1' consists of the red vertices of G_1. Furthermore, $S_2=\{b',c',d'\}$ and $S_2'=\{c'\}$. Thus $S=D_G$ and $S'=R_G$. This completes the proof of the claim. □

Claim 21. *If C has length greater than* 6*, then $S=D_G$ and $S'=R_G$.*

Proof. Let a'' be the blue vertex that is adjacent to e' on C. By assumption, C has length at least 11. Let G_1 be the graph obtained from $G-V(H)$ by adding the edge aa''. By construction, G_1 belongs to $\mathscr{G}_{k-1}$. Let $S_1=S\cap V(G_1)$ and let $S_1'=S'\cap V(G_1)$. Further, let $S_2=S\cap V(H)$.

Suppose S_1 is not a dominating set of G_1. Then $a, a'' \notin S_1$ and at least one of a and a'' is not dominated by S_1. However, $S_1 \cup \{a\}$ is a dominating set of G_1. Since $S_1 \cup \{a\} \neq D_{G_1}$, $\gamma(\langle S_1 \rangle) + 1 = \gamma(\langle S_1 \cup \{a\} \rangle) \geqslant k$, and so $\gamma(\langle S_1 \rangle) \geqslant k - 1$. If a is not dominated by S_1, then S_2 must be a dominating set of the path a, a', b', c', d', e'. On the other hand, if a'' is not dominated by S_1, then S_2 must be a dominating set of the path a', b', c', d', e', a''. In any event, $\gamma(\langle S_2 \rangle) \geqslant 2$. Thus, $\gamma(\langle S \rangle) = \gamma(\langle S_1 \rangle) + \gamma(\langle S_2 \rangle) \geqslant k + 1$, a contradiction. Hence S_1 must be a dominating set of G_1.

Applying the inductive hypothesis to G_1, $\gamma(\langle S_1 \rangle) \geqslant k - 1$ with equality if and only if $S_1 = D_{G_1}$ and S_1' consists of the red vertices of G_1. Since $k \geqslant |S'| \geqslant |S_1'| + 1 \geqslant k$, it follows that $S_1 = D_{G_1}$, $|S_1'| = k - 1$ and S_1' consists of the red vertices of G_1. Furthermore, $S_2 = \{b', c', d'\}$ and $S' - S_1' = \{c'\}$. Thus $S = D_G$ and $S' = R_G$. This completes the proof of the claim. □

By Claims 20 and 21, $S = D_G$ and $S' = R_G$. This completes the proof of Theorem 14. □

By Theorem 14, D_G is the unique γ_ℓ-set of G. In particular, $\gamma_\ell(G) = |D_G| = 3k = 3n/5$. Furthermore, G is edge-minimal with respect to satisfying $\delta(G) \geqslant 2$ and G connected. Hence we have the following result.

Proposition 22. *Each graph in the family $\mathscr{G}^*$ is a $\frac{3}{5}$-minimal 2-graph.*

5. Comments

If $G \in \mathscr{C}_5 \cup \mathscr{G}^*$, then, by Propositions 13 and 22, G is a $\frac{3}{5}$-minimal 2-graph. The converse is not true. There are $\frac{3}{5}$-minimal 2-graphs that do not belong to the families $\mathscr{C}_5$ or $\mathscr{G}^*$. For example, the graph G shown in Fig. 2 is a $\frac{3}{5}$-minimal 2-graph that does not belong to $\mathscr{C}_5 \cup \mathscr{G}^*$. Notice, however, that the graph G is obtained from two

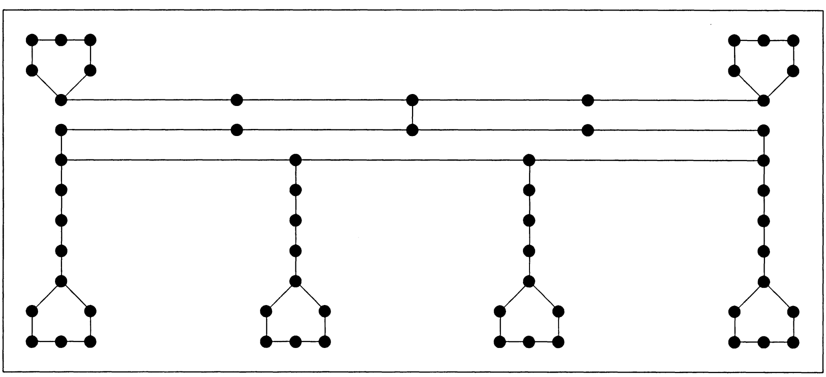

Fig. 2. A $\frac{3}{5}$-minimal 2-graph not in $\mathscr{C}_5 \cup \mathscr{G}^*$.

graphs in $\mathscr{G}^*$ by adding an edge joining two red vertices. It is possible to construct a $\frac{3}{5}$-minimal 2-graph from the (disjoint) union of $m \geqslant 2$ graphs in $\mathscr{G}^*$ (that satisfy certain special properties) by adding a set of $m-1$ edges such that each added edge joins two red vertices or two green vertices at least one of which belongs to a cycle in $\mathscr{G}^*$. However, we have yet to settle which red or green vertices may be used when adding these $m-1$ edges. It remains an open problem to characterize $\frac{3}{5}$-minimal 2-graphs.

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