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Commutator estimates in W^* -algebras

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Abstract

Let \mathcal{M} be a W^* -algebra and let $LS(\mathcal{M})$ be the algebra of all locally measurable operators affiliated with \mathcal{M} . It is shown that for any self-adjoint element $a \in LS(\mathcal{M})$ there exists a self-adjoint element c_0 from the center of $LS(\mathcal{M})$, such that for any $\varepsilon > 0$ there exists a unitary element u_ε from \mathcal{M} , satisfying $\|[a, u_\varepsilon]\| \geq (1 - \varepsilon)|a - c_0|$. A corollary of this result is that for any derivation δ on \mathcal{M} with the range in a (not necessarily norm-closed) ideal $I \subseteq \mathcal{M}$, the derivation δ is inner, that is $\delta(\cdot) = \delta_a(\cdot) = [a, \cdot]$, and $a \in I$. Similar results are also obtained for inner derivations on $LS(\mathcal{M})$.

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1. Introduction

Let \mathcal{M} be a W^* -algebra and let $Z(\mathcal{M})$ be the center of \mathcal{M} . Fix $a \in \mathcal{M}$ and consider the inner derivation δ_a on \mathcal{M} generated by the element a , that is $\delta_a(\cdot) := [a, \cdot]$. Obviously, δ_a is a linear bounded operator on $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$, where $\|\cdot\|_{\mathcal{M}}$ is a C^* -norm on \mathcal{M} . It is well known (see e.g. [22, Theorem 4.1.6]) that there exists $c \in Z(\mathcal{M})$ such that the following estimate holds: $\|\delta_a\| \geq \|a - c\|_{\mathcal{M}}$. In view of this result, it is natural to ask whether there exists an element $y \in \mathcal{M}$ with $\|y\| \leq 1$ and $c \in Z(\mathcal{M})$ such that $\|[a, y]\| \geq |a - c|$?

The following estimate easily follows from the main result of the present article: for every self-adjoint element $a \in \mathcal{M}$ there exists an element $c \in Z(\mathcal{M})$ and the family $\{u_\varepsilon\}_{\varepsilon>0}$ of unitary

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operators from \mathcal{M} such that

$$|\delta_a(u_\varepsilon)| \geq (1 - \varepsilon)|a - c|, \quad \forall \varepsilon > 0. \quad (1)$$

The estimate above is actually sharp and, with its aid, we shall easily show (see Corollary 3 below) that every derivation δ on \mathcal{M} taking its values in a (not necessary closed in the norm $\|\cdot\|_{\mathcal{M}}$) two-sided ideal $I \subset \mathcal{M}$ has the form $\delta = \delta_a$, where $a \in I$. This result can be further reformulated in two equivalent forms (Corollaries 4 and 5) yielding generalizations and complements to classical results of J. Calkin [5] and M.J. Hoffman [12] obtained originally for the special case when \mathcal{M} coincides with the algebra $B(H)$ of all bounded linear operators on a Hilbert space H . It should be pointed out that our approach to the proof of Corollaries 4 and 5 is based on the estimate (1) and appears to be more direct than those employed in [5] and [12]. In Section 3 below we present a number of other extensions of results from [5,12], in particular to ideals of measurable (unbounded) operators, which significantly extend recent results from [1,2] obtained under an additional assumption that \mathcal{M} is of type I .

Further, we recall the following well-known problem, which is also somewhat relevant to our discussion of derivations taking values in ideals.

Let \mathcal{N} be a von Neumann subalgebra of the von Neumann algebra \mathcal{M} and let I be an arbitrary (two-sided) ideal in \mathcal{M} . What conditions should be imposed on \mathcal{M} , \mathcal{N} , I to guarantee that for every derivation $\delta : \mathcal{N} \rightarrow I$ there exists $a \in I$ so that the equality $\delta = \delta_a$ holds?

Different partial solutions of this problem can be found in [13,17,21]. In the present paper we present a positive solution in the special case when \mathcal{M} is an arbitrary von Neumann algebra and when $\mathcal{N} = \mathcal{M}$. Let us note that if, in addition, we assume that the ideal I is closed with respect to the norm $\|\cdot\|_{\mathcal{M}}$, then the positive solution can be obtained directly from the Dixmier approximation theorem (see e.g. [22, Theorem 2.1.16]). However, the latter theorem is inapplicable when the ideal I is not closed in the C^* -norm on \mathcal{M} .

Analogous (but much harder) questions to those discussed above can be also reformulated in a more general setting of the theory of non-commutative integration initiated by I.E. Segal [25] (for alternative approach to this theory, see E. Nelson's paper [20]). In these reformulations (see e.g. [2,3]), the W^* -algebra \mathcal{M} is replaced with a larger algebra of 'measurable' operators affiliated with \mathcal{M} and the ideal I in \mathcal{M} is replaced with an ideal of measurable operators. The most general algebra considered in the theory of non-commutative integration to date is the classical algebra $LS(\mathcal{M})$ (see [23]) of all locally measurable operators affiliated with \mathcal{M} (all necessary definitions are given in Section 2 below). It is important to emphasize that our methods are totally different from the methods employed in [5,12,13,17,21,1,2] and are strong enough to enable us (see Theorem 1 and Corollaries 8, 11 below) to resolve all these questions also in the setting of the algebra $LS(\mathcal{M})$. A number of such applications to symmetric spaces of measurable operators (some of them are in spirit of papers [5,12,17,1,2]) are presented in Section 3.

Our main result in this paper is the following theorem.

Theorem 1. *Let \mathcal{M} be a W^* -algebra and let $a = a^* \in LS(\mathcal{M})$.*

- (i) *If \mathcal{M} is a finite W^* -algebra or else a purely infinite σ -finite W^* -algebra, then there exist $c_0 = c_0^* \in Z(LS(\mathcal{M}))$ and $u_0 = u_0^* \in U(\mathcal{M})$, such that*

$$|[a, u_0]| = u_0^*|a - c_0|u_0 + |a - c_0|, \quad (2)$$

where $U(\mathcal{M})$ is the group of all unitary elements in \mathcal{M} .

(ii) There exists $c_0 = c_0^* \in Z(LS(\mathcal{M}))$, so that for any $\varepsilon > 0$ there exists $u_\varepsilon = u_\varepsilon^* \in U(\mathcal{M})$ such that

$$|[a, u_\varepsilon]| \geq (1 - \varepsilon)|a - c_0|. \quad (3)$$

The main ideas of the proof of Theorem 1 are firstly demonstrated in Section 4 and then presented in full in Section 5. The proof of Theorem 1 in the special case when \mathcal{M} is a factor can be also found in our earlier paper [3]. We point out that the proof in the general case has required a significant new technical insight and that our direct approach may be of interest in its own right (not in the least due to the fact that we do not use any complicated technical devices such as Boolean-valued analysis and direct integrals).

Remark 2. Observe that the equality (2) trivially yields the estimate (3) even for the case $\varepsilon = 0$. Nevertheless, the result of Theorem 1(ii) is still sharp. Indeed, if \mathcal{M} is an infinite semi-finite σ -finite factor, then there exists a self-adjoint element $a \in LS(\mathcal{M})$ such that for every $\lambda \in \mathbb{C}$ and $u \in U(\mathcal{M})$ the inequality $|[a, u]| \geq |a - \lambda \mathbf{1}|$ fails [3]. Hence, the multiplier $(1 - \varepsilon)$ in the part (ii) of Theorem 1 cannot be omitted.

2. Preliminaries

For details on von Neumann algebra theory, the reader is referred to e.g. [7,16,22] or [28]. General facts concerning measurable operators may be found in [20,25] (see also [29, Chapter IX]). For the convenience of the reader, some of the basic definitions are recalled.

Let \mathcal{M} be a von Neumann algebra on a Hilbert space H . The set of all self-adjoint projections (respectively, all unitary elements) in \mathcal{M} is denoted by $P(\mathcal{M})$ (respectively, $U(\mathcal{M})$). We use the notation $\mathbf{s}(x)$, $\mathbf{l}(x)$, $\mathbf{r}(x)$, $\mathbf{c}(x)$ to denote the support, left support, right support, central support respectively of an element $x \in \mathcal{M}$.

Let $p, q \in P(\mathcal{M})$. The projections p and q are said to be equivalent, if there exists a partial isometry $v \in \mathcal{M}$, such that $v^*v = p$, $vv^* = q$. In this case, we write $p \sim q$. The fact that the projections p and q are not equivalent is recorded as $p \not\sim q$. If there exists a projection $q_1 \in P(\mathcal{M})$ such that $q_1 \leq p$, $q_1 \sim q$, then we write $q \preceq p$. If $q \preceq p$ and $p \not\sim q$, then we employ the notation $q \prec p$.

A linear operator $x : \mathfrak{D}(x) \rightarrow H$, where the domain $\mathfrak{D}(x)$ of x is a linear subspace of H , is said to be *affiliated* with \mathcal{M} if $yx \subseteq xy$ for all $y \in \mathcal{M}'$ (which is denoted by $x \eta \mathcal{M}$). A linear operator $x : \mathfrak{D}(x) \rightarrow H$ is termed *measurable* with respect to \mathcal{M} if x is closed, densely defined, affiliated with \mathcal{M} and there exists a sequence $\{p_n\}_{n=1}^\infty$ in $P(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subseteq \mathfrak{D}(x)$ and p_n^\perp is a finite projection (with respect to \mathcal{M}) for all n . It should be noted that the condition $p_n(H) \subseteq \mathfrak{D}(x)$ implies that $xp_n \in \mathcal{M}$. The collection of all measurable operators with respect to \mathcal{M} is denoted by $S(\mathcal{M})$, which is a unital $*$ -algebra with respect to strong sums and products (denoted simply by $x + y$ and xy for all $x, y \in S(\mathcal{M})$).

Let a be a self-adjoint operator affiliated with \mathcal{M} . We denote its spectral measure by $\{e^a\}$. It is known if x is a closed operator in H with the polar decomposition $x = u|x|$ and $x \eta \mathcal{M}$, then $u \in \mathcal{M}$ and $e \in \mathcal{M}$ for all projections $e \in \{e^{|x|}\}$. Moreover, $x \in S(\mathcal{M})$ if and only if x is closed, densely defined, affiliated with \mathcal{M} and $e^{|x|}(\lambda, \infty)$ is a finite projection for some $\lambda > 0$. It follows immediately that in the case when \mathcal{M} is a von Neumann algebra of type III or a type I factor, we have $S(\mathcal{M}) = \mathcal{M}$. For type II von Neumann algebras, this is no longer true.

An operator $x \in S(\mathcal{M})$ is called τ -measurable if there exists a sequence $\{p_n\}_{n=1}^\infty$ in $P(\mathcal{M})$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subseteq \mathfrak{D}(x)$ and $\tau(p_n^\perp) < \infty$ for all n . The collection $S(\tau)$ of all τ -measurable operators is a unital $*$ -subalgebra of $S(\mathcal{M})$ denoted by $S(\mathcal{M}, \tau)$. It is well known that a linear operator x belongs to $S(\mathcal{M}, \tau)$ if and only if $x \in S(\mathcal{M})$ and there exists $\lambda > 0$ such that $\tau(e^{|x|}(\lambda, \infty)) < \infty$.

A closed operator $x \eta \mathcal{M}$ is called locally measurable if there exists a sequence $\{z_n\}_{n=1}^\infty$ of central projections in \mathcal{M} such that $z_n \uparrow \mathbf{1}$ and $xz_n \in S(\mathcal{M})$ for any $n \in \mathbb{N}$. The collection of all locally measurable operators with respect to \mathcal{M} is denoted by $LS(\mathcal{M})$, which is a unital $*$ -algebra with respect to strong sums and products (denoted simply by $x + y$ and xy for all $x, y \in LS(\mathcal{M})$).

It follows directly from the definition of local measurability, that every measurable operator c with respect to $Z(\mathcal{M})$ is a locally measurable operator and so $S(Z(\mathcal{M})) \subset Z(LS(\mathcal{M}))$. On the other hand, if $c \in Z(LS(\mathcal{M}))$ and $c = v|c|$ is a polar decomposition for c , then $v \in Z(\mathcal{M})$ and the spectral family $e_\lambda^{|c|}$ of $|c|$ belongs to $Z(\mathcal{M})$ as well. This means that $|c| \in S(Z(\mathcal{M}))$. Hence, $c = v|c| \in S(Z(\mathcal{M}))$. So, we have

$$Z(LS(\mathcal{M})) = S(Z(\mathcal{M})).$$

In particular, $*$ -algebra $Z(LS(\mathcal{M}))$ is $*$ -isomorphic to the commutative $*$ -algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable complex-valued functions on a measurable space with a locally finite measure [25].

Suppose now that two von Neumann algebras \mathcal{M} and \mathcal{N} are $*$ -isomorphic. In this case, it follows from [6] that the algebras $LS(\mathcal{M})$ and $LS(\mathcal{N})$ of locally measurable operators affiliated with von Neumann algebras \mathcal{M} and \mathcal{N} respectively are also $*$ -isomorphic. Assuming now that \mathcal{M} is an arbitrary W^* -algebra and \mathcal{N} is a von Neumann algebra $*$ -isomorphic with \mathcal{M} , we see that the algebra $LS(\mathcal{N})$ is uniquely defined up to $*$ -isomorphism. In view of this observation, the algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with an arbitrary W^* -algebra \mathcal{M} is well defined.

3. Applications to derivations and essential commutants

Recall that a *derivation* on a complex algebra A is a linear map $\delta : A \rightarrow A$ such that

$$\delta(xy) = \delta(x)y + x\delta(y), \quad x, y \in A.$$

If $a \in A$, then the map $\delta_a : A \rightarrow A$, given by $\delta_a(x) = [a, x]$, $x \in A$, is a derivation. A derivation of this form is called *inner*. In this section we demonstrate a number of corollaries from Theorem 1 to inner derivations on the algebra A (in the setting when $A = \mathcal{M}$ or $A = LS(\mathcal{M})$) taking value in some two-sided ideal in A .

3.1. Applications to ideals in W^* -algebras

Our results here extend those in [5,12]. We are also motivated by results in [13,17,21]. For convenience of the reader, we now quickly list a few important properties of ideals in von Neumann algebras, which play a role in the subsequent exposition. For more detailed account of such properties, we refer to [16, Section 6.8].

The set \mathcal{F} of all operators with finite range projection in a von Neumann algebra \mathcal{M} is a two-sided ideal in \mathcal{M} . Each non-zero two-sided ideal in a factor \mathcal{M} contains this ideal [16, Theorem 6.8.3]. Each finite (respectively, σ -finite type III) factor is simple [16, Corollary 6.8.4] (respectively, [16, Corollary 6.8.5]). If \mathcal{M} is a σ -finite factor of type I_∞ or II_∞ , the norm closure $\overline{\mathcal{F}}$ of the two-sided ideal \mathcal{F} of operators with finite range projection is the only proper, norm-closed, two-sided ideal [16, Theorem 6.8.7]. Each two-sided ideal in a von Neumann algebra is self-adjoint [16, Proposition 6.8.9]. On the other hand, every semi-finite non-finite von Neumann algebra contains infinitely many non-trivial two-sided ideals (see the end of this section).

Corollary 3. *Let \mathcal{M} be a W^* -algebra and let I be an ideal in \mathcal{M} . Let $\delta : \mathcal{M} \rightarrow I$ be a derivation. Then there exists an element $a \in I$, such that $\delta = \delta_a = [a, \cdot]$.*

Proof. Since δ is a derivation on a W^* -algebra, it is necessarily inner [22, Theorem 4.1.6]. Thus, there exists an element $d \in \mathcal{M}$, such that $\delta(\cdot) = \delta_d(\cdot) = [d, \cdot]$. It follows from our hypothesis that $[d, \mathcal{M}] \subseteq I$.

Using [16, Proposition 6.8.9] (or [3, Lemma 7]), we obtain $[d^*, \mathcal{M}] = -[d, \mathcal{M}]^* \subseteq I^* = I$ and $[d_k, \mathcal{M}] \subseteq I$, $k = 1, 2$, where $d = d_1 + id_2$, $d_k = d_k^* \in \mathcal{M}$, for $k = 1, 2$. It follows now from Theorem 1, that there exist $c_1, c_2 \in Z(\mathcal{M})$ and $u_1, u_2 \in U(\mathcal{M})$, such that $\|[d_k, u_k]\| \geq 1/2|d_k - c_k|$ for $k = 1, 2$. Again applying [3, Lemma 7], we obtain $d_k - c_k \in I$, for $k = 1, 2$. Setting $a := (d_1 - c_1) + i(d_2 - c_2)$, we deduce that $a \in I$ and $\delta = [a, \cdot]$. \square

Corollary 3 can be restated as follows.

Corollary 4. *Let \mathcal{M} be a W^* -algebra, let I be an ideal in \mathcal{M} and let $\pi : \mathcal{M} \rightarrow \mathcal{M}/I$ be a canonical epimorphism. Then $\pi^{-1}(\text{center}(\mathcal{M}/I)) = Z(\mathcal{M}) + I$.*

Proof. Let $a \in \pi^{-1}(\text{center}(\mathcal{M}/I))$. Then $[a, x] = ax - xa \in I$ for any $x \in \mathcal{M}$. Therefore the inner derivation δ_a satisfies the condition of Corollary 3. Then there exists an element $c \in Z(\mathcal{M})$ such that $a + c \in I$. Therefore $a \in Z(\mathcal{M}) + I$, that is $\pi^{-1}(\text{center}(\mathcal{M}/I)) \subset Z(\mathcal{M}) + I$. The converse inclusion is trivial. \square

In the special case when $\mathcal{M} = B(H)$, where H is a separable Hilbert space the result of Corollary 4 coincides with that of [5, Theorem 2.9]. Thus, we can view Corollary 4 as an extension of the classical Calkin theorem to arbitrary W^* -algebras. This result has the following “lifting” interpretation. Let ϕ be an epimorphism from W^* -algebra \mathcal{M} on an arbitrary algebra \mathcal{A} and let a be an element from the center of the algebra \mathcal{A} . Then there exists an element $z \in Z(\mathcal{M})$ such that $\phi(z) = a$.

Let us give another important corollary of Theorem 1 related to the notions of essential commutant and multiplier ideals [12]. To this end, let us fix a W^* -algebra \mathcal{M} and two self-adjoint ideals I, J in \mathcal{M} . We set

$$I : J = \{x \in \mathcal{M} : xJ \subset I\}$$

and

$$D(J, I) = \{x \in \mathcal{M} : [x, y] \in I, \forall y \in J\}.$$

Observe that $I : J$ is an ideal in \mathcal{M} . In particular, $(I : J)^* = I : J = \{x \in \mathcal{M} : Jx \subset I\}$.

Corollary 5. For any W^* -algebra \mathcal{M} and any ideals I, J in \mathcal{M} we have

$$D(J, I) = I : J + Z(\mathcal{M}).$$

Proof. Let $a \in I : J, c \in Z(\mathcal{M}), b \in J$. Then $[a + c, b] = [a, b] = ab - ba \in I - I \subset I$, that is $a + c \in D(J, I)$. Therefore $I : J + Z(\mathcal{M}) \subset D(J, I)$.

In order to prove that $D(J, I) \subset I : J + Z(\mathcal{M})$, fix an element $a \in D(J, I)$. For an arbitrary $x \in J, y \in \mathcal{M}$, we have $x[a, y] + [a, x]y = xay - xya + axy - xay = [a, xy] \in [a, J] \subset I, [a, x]y \in Iy \subset I$. Hence, $x[a, y] \in I$. Since x is an arbitrary element from J , we obtain $[a, y] \in I : J$ and so $[a, \mathcal{M}] \subset I : J$. Now, it follows from Corollary 3 that $a + c \in I : J$ for some $c \in Z(\mathcal{M})$. Consequently, $a = (a + c) - c \in I : J + Z(\mathcal{M})$. \square

In the special case, when $\mathcal{M} = B(H)$, the result of Corollary 5 coincides with that of [12, Theorem 1.1]. The proof given there does not extend to arbitrary von Neumann algebras and is spelt out only for the case of a separable Hilbert space H (the case of a nonseparable H requires a substantial effort outlined in [12]). The proof given above works for an arbitrary W^* -algebra \mathcal{M} and hence for an arbitrary von Neumann algebra represented on an arbitrary Hilbert space.

Classical examples of normed ideals I satisfying the assumptions of Corollary 3 above are given by symmetric operator ideals [9,10,24,26].

Definition 6. A linear subspace \mathcal{I} in the von Neumann algebra \mathcal{M} equipped with a norm $\|\cdot\|_{\mathcal{I}}$ is said to be a symmetric operator ideal if

- (1) $\|S\|_{\mathcal{I}} \geq \|S\|$ for all $S \in \mathcal{I}$,
- (2) $\|S^*\|_{\mathcal{I}} = \|S\|_{\mathcal{I}}$ for all $S \in \mathcal{I}$,
- (3) $\|ASB\|_{\mathcal{I}} \leq \|A\| \|S\|_{\mathcal{I}} \|B\|$ for all $S \in \mathcal{I}, A, B \in \mathcal{M}$.

Observe, that every symmetric operator ideal \mathcal{I} is a two-sided ideal in \mathcal{M} , and therefore by [7, I.1.6, Proposition 10], it follows from $0 \leq S \leq T$ and $T \in \mathcal{I}$ that $S \in \mathcal{I}$ and $\|S\|_{\mathcal{I}} \leq \|T\|_{\mathcal{I}}$.

Corollary 7. Let \mathcal{M} be a W^* -algebra, let I be a symmetric operator ideal in \mathcal{M} and let $\delta : \mathcal{M} \rightarrow I$ be a self-adjoint derivation. Then there exists an element $a \in I$, satisfying the inequality $\|a\|_I \leq \|\delta\|_{\mathcal{M} \rightarrow I}$ and such that $\delta = \delta_a = [a, \cdot]$.

Proof. Firstly, we observe that $\|\delta\|_{\mathcal{M} \rightarrow I} < \infty$. Indeed, we have $\delta = \delta_a, a \in I$ and therefore $\|\delta(x)\|_I = \|ax - xa\|_I \leq \|ax\|_I + \|xa\|_I \leq 2\|a\|_I \|x\|_{\mathcal{M}}$, that is $\|\delta\|_{\mathcal{M} \rightarrow I} \leq 2\|a\|_I < \infty$.

Let now δ be a self-adjoint derivation on \mathcal{M} , that is $\delta(\cdot) = \delta_d(\cdot) = [d, \cdot]$ for some $d \in \mathcal{M}$, such that $[d, x]^* = [d, x^*]$ for all $x \in \mathcal{M}$. We have $x^*d^* - d^*x^* = dx^* - x^*d$, that is, $x^*(d^* + d) = (d^* + d)x^*$ for all $x \in \mathcal{M}$. This immediately implies $\Re(d) \in Z_h(\mathcal{M})$ and so, we can safely assume that $\delta = \delta_{i \cdot d} = [i \cdot d, \cdot]$, where d is a self-adjoint operator from \mathcal{M} . Fix $\varepsilon > 0$ and let $c_0 \in Z_h(\mathcal{M}), u_\varepsilon \in U(\mathcal{M})$ be such that

$$|[d, u_\varepsilon]| \geq (1 - \varepsilon)|d - c_0|.$$

The assumption on $(I, \|\cdot\|_I)$ guarantees that $(1 - \varepsilon)\|d - c_0\|_I \leq \|\delta(u_\varepsilon)\|_I \leq \|\delta\|_{\mathcal{M} \rightarrow I}$. Since ε was chosen arbitrarily, we conclude that $\|d - c_0\|_I \leq \|\delta\|_{\mathcal{M} \rightarrow I}$. Setting $a = i(d - c_0)$ completes the proof. \square

If the von Neumann algebra \mathcal{M} is equipped with a faithful normal semi-finite trace τ , then the set

$$\mathcal{L}_p(\mathcal{M}) = \{S \in \mathcal{M}: \tau(|S|^p) < \infty\}$$

equipped with a standard norm

$$\|S\|_{\mathcal{L}_p(\mathcal{M})} = \max\{\|S\|_{B(H)}, \tau(|S|^p)^{1/p}\}$$

is called Schatten–von Neumann p -class. In the type I setting these are the usual Schatten–von Neumann ideals of compact operators [9,10,24,26]. The result of Corollary 7 complements results given in [17, Section 6], where derivations from some subalgebras of \mathcal{M} into Schatten–von Neumann p -classes were studied.

3.2. Applications to ideals in $LS(\mathcal{M})$

We begin by proving an analogue of Corollary 3 for ideals of (unbounded) locally measurable operators. The following result significantly strengthens [2, Proposition 6.17] where a similar assertion was established under additional assumptions that $\mathcal{M} = L_\infty(\nu) \bar{\otimes} B(H)$ (here, $L_\infty(\nu)$ is an algebra of all bounded measurable functions on a measure space) and \mathcal{A} is an absolutely solid algebra such that $\mathcal{M} \subset \mathcal{A}$. Similarly, the result below complements the main result of [1], which considered the case of an arbitrary von Neumann algebra \mathcal{M} of type I and algebras $\mathcal{A} = S(\mathcal{M}, \tau), S(\mathcal{M}), LS(\mathcal{M})$. Our approach here is completely different from techniques used in [1,2] which crucially exploited structural results describing type I von Neumann algebras.

Corollary 8. *Let \mathcal{M} be a W^* -algebra and let \mathcal{A} be a linear subspace in $LS(\mathcal{M})$, such that $\mathcal{A}^* = \mathcal{A}$, $x \in \mathcal{A} \Leftrightarrow |x| \in \mathcal{A}$, $0 \leq x \leq y \in \mathcal{A} \Rightarrow x \in \mathcal{A}$. Fix $a \in LS(\mathcal{M})$ and consider inner derivation $\delta = \delta_a$ on the algebra $LS(\mathcal{M})$ given by $\delta(x) = [a, x]$, $x \in LS(\mathcal{M})$. If $\delta(\mathcal{M}) \subseteq \mathcal{A}$, then there exists an element $d \in \mathcal{A}$ such that $\delta(x) = [d, x]$.*

Proof. Let $a = a_1 + ia_2$, where $a_1 = \Re a$ and $a_2 = \Im a$. We have $2[a_1, x] = [a + a^*, x] = [a, x] - [a, x^*]^* = \mathcal{A} - \mathcal{A}^* \subseteq \mathcal{A}$ for any $x \in \mathcal{M}$. Analogously, $[a_2, x] \in \mathcal{A}$ for any $x \in \mathcal{M}$. By Theorem 1, there is an element $c_k \in Z_h(LS(\mathcal{M}))$ and a unitary element $u_k \in U(\mathcal{M})$, such that $|[a_k, u_k]| \geq 1/2|a_k - c_k|$ for $k = 1, 2$. The assumption on \mathcal{A} guarantees that $a_k - c_k \in \mathcal{A}$, for $k = 1, 2$. Setting $d = (a_1 - c_1) + i(a_2 - c_2)$, we deduce that $d \in \mathcal{A}$ and $\delta = [d, \cdot]$. \square

Consider the following classical example (see e.g. [13, Lemmas 3.1 and 3.2]): $\mathcal{M} = B(H)$ and \mathcal{A} is the algebra of all compact operators on H . Suppose that an element $a = a^* \in \mathcal{M}$ is such that $\delta_a : \mathcal{M} \rightarrow \mathcal{A}$. Then the result of Corollary 8 asserts that there exists $\lambda \in \mathbb{R}$ such that $a - \lambda \mathbf{1}$ is a compact operator. An important example extending this classical result can be obtained as follows. Let a semi-finite von Neumann algebra \mathcal{M} be equipped with a faithful normal semi-finite trace τ . Let $x \in S(\mathcal{M}, \tau)$. The set $S_0(\mathcal{M}, \tau)$ of all τ -compact operators in $LS(\mathcal{M})$ is defined as the subset of all $x \in S(\mathcal{M}, \tau)$ such that $\lim_{t \rightarrow \infty} \mu_t(x) = 0$ (see the definition of the generalized singular value function μ below). The result of Corollary 8 asserts, in particular, that for any $a \in LS(\mathcal{M})$ such that $\delta_a : LS(\mathcal{M}) \rightarrow S_0(\mathcal{M}, \tau)$ there exists an element $c \in LS(Z(\mathcal{M}))$ such that $a - c \in S_0(\mathcal{M}, \tau)$.

Numerous examples of absolutely solid subspaces \mathcal{A} in $LS(\mathcal{M})$ satisfying the assumptions of the preceding corollary are given by \mathcal{M} -bimodules of $LS(\mathcal{M})$.

Definition 9. A linear subspace E of $LS(\mathcal{M})$, is called an \mathcal{M} -bimodule of local measurable operators if $uxv \in E$ whenever $x \in E$ and $u, v \in \mathcal{M}$. If an \mathcal{M} -bimodule E is equipped with a (semi-)norm $\|\cdot\|_E$, satisfying

$$\|uxv\|_E \leq \|u\|_{\mathcal{M}}\|v\|_{\mathcal{M}}\|x\|_E, \quad x \in E, u, v \in \mathcal{M}, \quad (4)$$

then E is called a (semi-)normed \mathcal{M} -bimodule of local measurable operators.

We omit a straightforward verification of the fact that every \mathcal{M} -bimodule of locally measurable operators satisfies the assumption of Corollary 8.

The best-known examples of normed \mathcal{M} -bimodules of $LS(\mathcal{M})$ are given by the so-called symmetric operator spaces (see e.g. [8,27,18]). We briefly recall relevant definitions (for more detailed information we refer to [18] and references therein).

Let L_0 be a space of Lebesgue measurable functions either on $(0, 1)$ or on $(0, \infty)$, or on \mathbb{N} finite almost everywhere (with identification m -a.e.). Here m is a Lebesgue measure or else counting measure on \mathbb{N} . Define S as the subset of L_0 which consists of all functions x such that $m(\{|x| > s\})$ is finite for some s .

Let E be a Banach space of real-valued Lebesgue measurable functions either on $(0, 1)$ or $(0, \infty)$ (with identification m -a.e.) or on \mathbb{N} . The space E is said to be *absolutely solid* if $x \in E$ and $|y| \leq |x|$, $y \in L_0$ implies that $y \in E$ and $\|y\|_E \leq \|x\|_E$.

The absolutely solid space $E \subseteq S$ is said to be *symmetric* if for every $x \in E$ and every y the assumption $y^* = x^*$ implies that $y \in E$ and $\|y\|_E = \|x\|_E$ (see e.g. [19]).

Here, x^* denotes the non-increasing right-continuous rearrangement of x given by

$$x^*(t) = \inf\{s \geq 0: m(\{|x| \geq s\}) \leq t\}.$$

In the case when x is a sequence we denote by x^* the usual decreasing rearrangement of the sequence $|x|$.

If $E = E(0, 1)$ is a symmetric space on $(0, 1)$, then

$$L_\infty \subseteq E \subseteq L_1.$$

If $E = E(0, \infty)$ is a symmetric space on $(0, \infty)$, then

$$L_1 \cap L_\infty \subseteq E \subseteq L_1 + L_\infty.$$

If $E = E(\mathbb{N})$ is a symmetric space on \mathbb{N} , then

$$\ell_1 \subseteq E \subseteq \ell_\infty,$$

where ℓ_1 and ℓ_∞ are classical spaces of all absolutely summable and bounded sequences respectively.

Let a semi-finite von Neumann algebra \mathcal{M} be equipped with a faithful normal semi-finite trace τ . Let $x \in S(\mathcal{M}, \tau)$. The generalized singular value function of x is $\mu(x) : t \rightarrow \mu_t(x)$, where, for $0 \leq t < \tau(\mathbf{1})$

$$\mu_t(x) = \inf\{s \geq 0 \mid \tau(e^{|x|}(s, \infty)) \leq t\}.$$

Consider $\mathcal{M} = L^\infty([0, \infty))$ as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(0, \infty)$, with the trace given by integration with respect to Lebesgue measure m . It is easy to see that the set of all τ -measurable operators affiliated with \mathcal{M} consists of all measurable functions on $[0, \infty)$ which are bounded except on a set of finite measure, that is $S(\mathcal{M}, \tau) = S$ and that the generalized singular value function $\mu(f)$ is precisely the decreasing rearrangement f^* .

If $\mathcal{M} = B(H)$ (respectively, $\ell_\infty(\mathbb{N})$) and τ is the standard trace Tr (respectively, the counting measure on \mathbb{N}), then it is not difficult to see that $S(\mathcal{M}, \tau) = \mathcal{M}$. In this case, for $x \in S(\mathcal{M}, \tau)$ we have

$$\mu_n(x) = \mu_t(x), \quad t \in (n - 1, n], \quad n = 0, 1, 2, \dots$$

For $\mathcal{M} = B(H)$ the sequence $\{\mu_n(T)\}_{n=1}^\infty$ is just the sequence of singular values $(s_n(T))_{n=1}^\infty$.

Definition 10. Let \mathcal{E} be a linear subset in $S(\mathcal{M}, \tau)$ equipped with a norm $\|\cdot\|_{\mathcal{E}}$. We say that \mathcal{E} is a *symmetric operator space* (on \mathcal{M} , or in $S(\mathcal{M}, \tau)$) if for any $x \in \mathcal{E}$ and every $y \in S(\mathcal{M}, \tau)$ such that $\mu(y) \leq \mu(x)$, we have $y \in \mathcal{E}$ and $\|y\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}$.

The fact that every symmetric operator space \mathcal{E} is (an absolutely solid) \mathcal{M} -bimodule of $S(\mathcal{M}, \tau)$ is well known (see e.g. [27,18] and references therein). In the special case, when $\mathcal{M} = B(H)$ and τ is a standard trace Tr , the notion of symmetric operator space introduced in Definition 10 coincides with the notion of symmetric operator ideal given in Definition 6.

There exists a strong connection between symmetric function and operator spaces recently exposed in [18] (see earlier results in [24,9,10,26]).

Let E be a symmetric function space on the interval $(0, 1)$ (respectively, on the semi-axis or on \mathbb{N}) and let \mathcal{M} be a type II_1 (respectively, II_∞ or type I) von Neumann algebra. Define

$$E(\mathcal{M}, \tau) := \{S \in S(\mathcal{M}, \tau) : \mu_t(S) \in E\}, \quad \|S\|_{E(\mathcal{M}, \tau)} := \|\mu_t(S)\|_E.$$

Main results of [18] assert that $(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M}, \tau)})$ is a symmetric operator space. If $E = L_p$, $1 \leq p < \infty$, then $(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M}, \tau)})$ coincides with the classical non-commutative L_p -space associated with the algebra (\mathcal{M}, τ) . If \mathcal{M} is a semi-finite atomless von Neumann algebra, then the converse result also holds [27]. That is, if \mathcal{E} is a symmetric operator space on \mathcal{M} , then

$$E(0, \infty) := \{f \in S_0((0, \infty)) : f^* = \mu(x) \text{ for some } x \in \mathcal{E}\}, \quad \|f\|_E := \|x\|_{\mathcal{E}}$$

is a symmetric function space on $(0, \tau(\mathbf{1}))$. It is obvious that $\mathcal{E} = E(\mathcal{M}, \tau)$. Similarly, if $E = E(\mathbb{N})$ is a symmetric sequence space on \mathbb{N} , and the algebra \mathcal{M} is a type I factor with standard trace, then (see [18]) setting

$$\mathcal{E} := \{S \in \mathcal{M} : (s_n(S))_{n=1}^\infty \in E\}, \quad \|S\|_{\mathcal{E}} := \|(s_n(S))_{n=1}^\infty\|_E$$

yields a symmetric operator ideal. Conversely, every symmetric operator ideal \mathcal{E} in \mathcal{M} defines a unique symmetric sequence space $E = E(\mathbb{N})$ by setting

$$E := \{a = (a_n)_{n=1}^\infty \in \ell_\infty : a^* = (s_n(S))_{n=1}^\infty \text{ for some } S \in \mathcal{E}\}, \quad \|a\|_E := \|S\|_{\mathcal{E}}.$$

We are now fully equipped to provide a full analogue of Corollaries 3 and 7.

Corollary 11. *Let \mathcal{M} be a semi-finite W^* -algebra and let \mathcal{E} be a symmetric operator space. Fix $a = a^* \in S(\mathcal{M})$ and consider inner derivation $\delta = \delta_a$ on the algebra $LS(\mathcal{M})$ given by $\delta(x) = [a, x]$, $x \in LS(\mathcal{M})$. If $\delta(\mathcal{M}) \subseteq \mathcal{E}$, then there exists $d \in \mathcal{E}$ satisfying the inequality $\|d\|_{\mathcal{E}} \leq \|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}}$ and such that $\delta(x) = [d, x]$.*

Proof. The existence of $d \in \mathcal{E}$ such that $\delta(x) = [d, x]$ follows from Corollary 8. Now, if $u \in U(\mathcal{M})$, then $\|\delta(u)\|_{\mathcal{E}} = \|du - ud\|_{\mathcal{E}} \leq \|du\|_{\mathcal{E}} + \|ud\|_{\mathcal{E}} = 2\|d\|_{\mathcal{E}}$. Hence, if $x \in \mathcal{M}_1 = \{x \in \mathcal{M} : \|x\| \leq 1\}$, then $x = \sum_{i=1}^4 \alpha_i u_i$, where $u_i \in U(\mathcal{M})$ and $|\alpha_i| \leq 1$ for $i = 1, 2, 3, 4$, and so $\|\delta(x)\|_{\mathcal{E}} \leq \sum_{i=1}^4 \|\delta(\alpha_i u_i)\|_{\mathcal{E}} \leq 8\|d\|_{\mathcal{E}}$, that is $\|\delta\|_{\mathcal{M} \rightarrow \mathcal{E}} \leq 8\|d\|_{\mathcal{E}} < \infty$.

The final assertion is established exactly as in the proof of Corollary 7. \square

An illustration of the result above complementing the example given after Corollary 8, can be obtained when the space E is given by the norm closure of the subspace $L_1 \cap L_\infty$ in the space $L_1 + L_\infty$. In this case, the space $\mathcal{E} = E(\mathcal{M}, \tau)$ can be equivalently described as the set of all $x \in L_1 + L_\infty(\mathcal{M}, \tau)$ such that $\lim_{t \rightarrow \infty} \mu_t(x) = 0$. This space is a normed counterpart of the space $S_0(\mathcal{M}, \tau)$ of all τ -compact operators in $LS(\mathcal{M})$.

4. The outline of the proof of Theorem 1

The next two examples demonstrate respectively two main ideas behind the proof of Theorem 1. The first example yields the proof for the case when \mathcal{M} coincides with algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices.

Example 12. The assertion of Theorem 1 holds for the case $\mathcal{M} = M_n(\mathbb{C})$, $n \in \mathbb{N}$.

Proof. Fix $a = a^* \in M_n(\mathbb{C})$ and select a unitary matrix $v \in M_n(\mathbb{C})$ such that

$$v^* a v = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \in M_n(\mathbb{C}),$$

where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$.

Let the unitary matrix $u \in M_n(\mathbb{C})$ be counter-diagonal, that is

$$u = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

and observe that

$$u^*v^*avu = \begin{vmatrix} \lambda_n & 0 & \dots & 0 \\ 0 & \lambda_{n-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_1 \end{vmatrix}.$$

Therefore,

$$|[a, vu]| = |u^*v^*avu - a| = \begin{vmatrix} |\lambda_n - \lambda_1| & 0 & \dots & 0 \\ 0 & |\lambda_{n-1} - \lambda_2| & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |\lambda_1 - \lambda_n| \end{vmatrix}.$$

If n is odd, then for all $1 \leq k \leq n$ we have

$$|\lambda_k - \lambda_{n+1-k}| = |\lambda_k - \lambda_0| + |\lambda_{n+1-k} - \lambda_0|$$

for $\lambda_0 = \lambda_{(n+1)/2}$.

If n is even, then for all $1 \leq k \leq n$ we have

$$|\lambda_k - \lambda_{n+1-k}| = |\lambda_k - \lambda_0| + |\lambda_{n+1-k} - \lambda_0|$$

for every $\lambda_0 \in [\lambda_{n/2}, \lambda_{n/2+1}]$.

Therefore, for every $n \in \mathbb{N}$, we have

$$|[v^*av, u]| = u^*v^*|a - \lambda_0\mathbf{1}|vu + v^*|a - \lambda_0\mathbf{1}|v.$$

Then $|[a, vu v^*]| = |(vuv^*)^*a(vuv^*) - a| = v|u^*v^*avu - v^*av|v^* = v|[v^*av, u]|v^* = (vuv^*)^*|a - \lambda_0\mathbf{1}|(vuv^*) + |a - \lambda_0\mathbf{1}|, (vuv^*)^2 = \mathbf{1}$.

This completes the proof of the equality (2). The second assertion of Theorem 1 trivially follows from the first one (see Remark 2). \square

The idea of the proof of the second part of Theorem 1 is demonstrated in Example 12.

Example 13. Let \mathcal{M} be a σ -finite factor of type I_∞ or II_∞ . Then \mathcal{M} contains a family of pairwise orthogonal and pairwise equivalent projections $\{p_n\}_{n=1}^\infty$, such that $\sum_{n=1}^\infty p_n = \mathbf{1}$. Let $\mathbb{R}_+ \ni \lambda_n \downarrow 0$. Set $a = \sum_{n=1}^\infty \lambda_n p_n$ (this series converges in the strong operator topology). Using the same arguments as in [3], it is easy to show that there is no $\lambda_0 \in \mathbb{C}$ and $u \in U(\mathcal{M})$ such that $|[a, u]| \geq |a - \lambda_0\mathbf{1}|$. Nevertheless, the part (ii) of Theorem 1 holds.

Proof. We refer the reader to [3] for the proof of the first assertion and pass to the proof of the second one.

Fix $\varepsilon > 0$. For every $n \in \mathbb{N}$ there exist infinitely many $m \in \mathbb{N}$ such that $\lambda_m < \varepsilon\lambda_n$. Hence, the set \mathbb{N} can be split up into the set of all pairs $\{n_k, m_k\}_{k=1}^\infty$ such that $\lambda_{m_k} < \varepsilon\lambda_{n_k}$. For every $k \geq 1$, select a partial isometry $v_{n_k m_k}$ such that $v_{n_k m_k}^* v_{n_k m_k} = p_{m_k}$, $v_{n_k m_k} v_{n_k m_k}^* = p_{n_k}$. Clearly,

the projections p_{n_k}, p_{m_k} and the partial isometry $v_{n_k m_k}$ generate a $*$ -subalgebra in \mathcal{M} which is $*$ -isomorphic to $M_2(\mathbb{C})$. Without loss of generality, under this $*$ -isomorphism, we equate

$$p_{n_k} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \quad p_{m_k} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}, \quad v_{n_k m_k} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}.$$

Next, consider a W^* -subalgebra of \mathcal{M} generated by the elements $p_{n_k}, p_{m_k}, v_{n_k m_k}, k \geq 1$ which we identify with $\bigoplus_{k=1}^\infty M_2(\mathbb{C})$. We have

$$a = \bigoplus_{k=1}^\infty \begin{vmatrix} \lambda_{n_k} & 0 \\ 0 & \lambda_{m_k} \end{vmatrix}.$$

Set

$$u = \bigoplus_{k=1}^\infty \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

It clearly follows that

$$|[u, a]| = \bigoplus_{k=1}^\infty \begin{vmatrix} |\lambda_{n_k} - \lambda_{m_k}| & 0 \\ 0 & |\lambda_{n_k} - \lambda_{m_k}| \end{vmatrix} \geq (1 - \varepsilon) \bigoplus_{k=1}^\infty \begin{vmatrix} |\lambda_{n_k}| & 0 \\ 0 & |\lambda_{m_k}| \end{vmatrix} = (1 - \varepsilon)|a|. \quad \square$$

The idea of the proof of Theorem 1 consists in the splitting of the identity in \mathcal{M} into the sum of three pairwise orthogonal central projections p_0, p_-, p_+ with certain properties. For the reduced algebra $\mathcal{M}p_0$ the method from Example 12 will be applied, and for the reduced algebras $\mathcal{M}p_-$ and $\mathcal{M}p_+$ the method from Example 13 will be adjusted. It is of interest to observe that in the special case when \mathcal{M} is a σ -finite purely infinite W^* -algebra and a is bounded, the assertion of Theorem 1 can also be obtained from the Kadison’s result [14]. We quickly outline this approach here (we indebted to the referee for this observation).

For any self-adjoint element a in a von Neumann algebra \mathcal{M} , there exists a central element c_0 in \mathcal{M} such that for any central projection z in \mathcal{M} the equality

$$\|(a - c_0)z\| = \inf \|az - c\|$$

holds, where the inf is taken over all central elements c [14, Lemma 4]. Without loss of generality, we may assume that $c_0 = 0$. Then $\|az\| \leq \|az - c\|$ for any central projection z and any central element c .

Let $p := s(a_+), q := s(a_-)$. We shall show that $p \sim q$. By [16, Theorem 6.2.7] and due to the assumption that \mathcal{M} is a σ -finite purely infinite W^* -algebra, it follows that there exist 3 central projections e, f, g such that $e + f + g = \mathbf{1}, pe = 0, pf \sim qf, qg = 0$. Assuming ag is non-zero, we would obtain by restricting to $\mathcal{M}g$ that ag is positive, so $\|ag - (\|ag\|/2)g\| = \|ag\|/2 < \|ag\|$, which contradicts the preceding inequality. Similarly, we see that $ae = 0$ and so $p \sim q$.

Let v be a partial isometry satisfying $vv^* = p, v^*v = q$. Setting $u = v + v^* + (\mathbf{1} - p - q)$, we obtain a unitary operator satisfying $[[a, u]] = u^*|a|u + |a|, u^* = u$.

It should be emphasized though that such an argument does not hold in the general semi-finite setting.

5. The proof of Theorem 1

For better readability, we break the proof into the following series of lemmas.

Until the end of the proof, we fix an arbitrary element $a \in LS_h(\mathcal{M})$.

For projections $p, q \in P(\mathcal{M})$ we assume $p \ll q$, if $pz \prec qz$ for every $0 < z \leq \mathbf{c}(p) \vee \mathbf{c}(q)$, $z \in P(Z(\mathcal{M}))$. Set $p \gg q$, if $q \ll p$.

We recall the following comparison result:

Theorem 14. (See [16, Theorem 6.2.7].) *Let e and f be projections in a von Neumann algebra \mathcal{M} . There are unique orthogonal central projections p and q maximal with respect to the properties $qe \sim qf$, and, if p_0 is a non-zero central subprojection of p , then $p_0e \prec p_0f$. If r_0 is a non-zero central subprojection of $\mathbf{1} - p - q$, then $r_0f \prec r_0e$.*

The following form of the preceding theorem will be more convenient for our purposes.

Theorem 15. *Let \mathcal{M} be a W^* -algebra and $p, q \in P(\mathcal{M})$. Then there exists a unique triple of pairwise orthogonal projections $z_-, z_0, z_+ \in P(Z(\mathcal{M}))$, such that $z_- + z_0 + z_+ = \mathbf{1}$, $z_-p \ll z_-q$, $z_0p \sim z_0q$, $z_+p \gg z_+q$.*

Corollary 16. *Let \mathcal{M} be a W^* -algebra and $p_1, p_2, \dots, p_n \in P(\mathcal{M})$. Then there exists a family $\{z_\sigma\}_{\sigma \in S_n}$ (here, S_n is the permutation group of n elements) of pairwise orthogonal projections from $P(Z(\mathcal{M}))$ such that $\sum_{\sigma \in S_n} z_\sigma = \mathbf{1}$ $z_\sigma p_{\sigma(1)} \preceq z_\sigma p_{\sigma(2)} \preceq \dots \preceq z_\sigma p_{\sigma(n)}$ for every $\sigma \in S_n$.*

Proof. For every pair of projections $p_i, p_j, i < j$ we denote by z_{ij}^1 the largest central projection such that $z_{ij}^1 p_i \preceq z_{ij}^1 p_j$. Then for $z_{ij}^2 := \mathbf{1} - z_{ij}^1$ we have $z_{ij}^2 p_i \gg z_{ij}^2 p_j$. Let \mathfrak{B} be a Boolean algebra generated by all central projections z_{ij}^1 and $z_{ij}^2, 1 \leq i < j \leq n$. Every atom $z \in \mathfrak{B}$ may be uniquely written as

$$z = \prod_{i=1}^{n-1} \prod_{j=i+1}^n z_{ij}^{\varepsilon_{ij}}, \tag{5}$$

where $\varepsilon_{ij} \in \{1, 2\}$. Observe that the sum of all elements as in (5) is equal to $\mathbf{1}$.

Fix an atom $z \in \mathfrak{B}$ and define on the set $\{1, \dots, n\}$ a (linear) order \leq_z . To this end, we set for any $i, j, i \neq j$,

$$i \leq_z j \iff zp_i \preceq zp_j.$$

This definition is correct, since we have $z \leq z_{ij}^{\varepsilon_{ij}}$ for $i < j$ or $z \leq z_{ji}^{\varepsilon_{ji}}$ for $i > j$. So, we have $zp_i \preceq zp_j$ or $zp_i \succ zp_j$.

Let $\{i_1, \dots, i_n\} = \{1, \dots, n\}$, $i_1 \leq_z i_2 \leq_z \dots \leq_z i_n$ and set $\sigma(k) = i_k, \sigma \in S_n$. Then $z = z_\sigma$. \square

Let Λ be a lattice. Subset $I \subset \Lambda$ is called a \vee -ideal (\wedge -ideal) in Λ if for every $s, t \in I$ and $u \in \Lambda, u \leq s$ ($u \geq s$) we have $u \in I$ and $s \vee t \in I$ ($s \wedge t \in I$) [4].

Lemma 17. *Let ∇ be a complete Boolean algebra, let I be a non-zero \vee -ideal in ∇ . Then there exists a family $\{s_\alpha\}_{\alpha \in \Omega} \subset I$ of pairwise disjoint non-zero elements such that $\bigvee_{\alpha \in \Omega} s_\alpha = \bigvee I$.*

Proof. It follows from Zorn’s lemma that among all families of pairwise disjoint non-zero elements from I there exists a maximal one with respect to the inclusion. Let $\{s_\alpha\}_{\alpha \in \Omega}$ be one of these maximal families. It is clear that $s := \bigvee_{\alpha \in \Omega} s_\alpha \leq \bigvee I = t$. Suppose $s < t$. Then there exists $v \in I$ such that $v \not\leq s$. In this case $v \wedge s' \neq 0$ (by s' we denote the complement to s). Then $\{v \wedge s'\} \cup \{s_\alpha\}_{\alpha \in \Omega}$ is a family of pairwise disjoint non-zero elements from I . Thus, we have found a contradiction to the assumption that $\{s_\alpha\}_{\alpha \in \Omega}$ is maximal. So, our assumption $s < t$ fails and we have $s = t$. \square

The algebra (over reals) of all self-adjoint elements from the center of the algebra $LS(\mathcal{M})$ will be denoted by $Z_h(LS(\mathcal{M}))$. The latter algebra is a lattice with respect to the order induced from $LS(\mathcal{M})$. As we have already explained in Section 2 the $*$ -algebra $Z(LS(\mathcal{M}))$ is $*$ -isomorphic to the $*$ -algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable complex-valued functions on a measurable space with a locally finite measure. This isomorphism yields the assertions of Lemmas 18 and 19 below.

Lemma 18. *Let $\Omega := \{c_j\}_{j \in J}$ be a family of pairwise disjoint elements from $Z_h(LS(\mathcal{M}))$. Then there exists an element $c \in Z_h(LS(\mathcal{M}))$ such that $cs(c_j) = c_j$ for every $j \in J$.*

Lemma 19. *A lattice $Z_h(LS(\mathcal{M}))$ is conditionally complete.*

Lemma 20. *Let $p, q, r \in P(\mathcal{M})$, $p < q$, $p < r \ll q$. Then there exists an element $r_1 \in P(\mathcal{M})$ such that $r_1 \sim r$ and $p < r_1 < q$.*

Proof. Due to the assumptions, there exists an element $p_1 \in P(\mathcal{M})$ such that $p \sim p_1 < r$. Suppose that $(r - p_1)z \not\geq (q - p)z$ for some $0 \neq z \in P(Z(\mathcal{M}))$, $z \leq \mathbf{c}(q)$. Then $rz = (r - p_1)z + p_1z \not\geq (q - p)z + pz = qz$. This fact contradicts the assumption $r \ll q$, which yields the estimate $rz < qz$ for every $0 < z \leq \mathbf{c}(q) = \mathbf{c}(r) \vee \mathbf{c}(q)$ (the previous equality follows from the implication $r < q \Rightarrow \mathbf{c}(r) \leq \mathbf{c}(q)$). Hence, from Theorem 15 we have that $r - p_1 < q - p$. So, there exists an element $0 \neq p_2 \in P(\mathcal{M})$, such that $r - p_1 \sim p_2 < q - p$. Then $p < p + p_2 < q$ and $p + p_2 \sim p_1 + (r - p_1) = r$, where the equivalence $p + p_2 \sim p_1 + (r - p_1)$ is guaranteed by the implication

$$p \sim p_1, \quad p_2 \sim r - p_1, \quad pp_2 = 0, \quad p_1(r - p_1) = 0 \quad \Rightarrow \quad p + p_2 \sim p_1 + (r - p_1).$$

We are done, by letting $r_1 := p + p_2$. \square

Lemma 21. *Let p be a properly infinite projection in \mathcal{M} . Then we have:*

- (i) *If $P(\mathcal{M}) \ni q_1, \dots, q_n, \dots \preceq p$, $q_n q_m = 0$ for $n \neq m$, then $\bigvee_{n=1}^\infty q_n \preceq p$.*
- (ii) *If $P(\mathcal{M}) \ni q_1, \dots, q_n \ll p$, $q_i q_j = 0$ for $i \neq j$, then $\bigvee_{i=1}^n q_i \ll p$.*
- (iii) *If $p \not\geq \mathbf{1} - p$, then $p \sim \mathbf{1}$.*
- (iv) *If $P(\mathcal{M}) \ni q \ll p$, $qp = pq$, then $p(\mathbf{1} - q) \sim p$.*
- (v) *If $q \in P(\mathcal{M})$ is a finite projection, $P(\mathcal{M}) \ni p_n \uparrow p$ and $z_n \in P(Z(\mathcal{M}))$ are such that $(\mathbf{1} - z_n)p_n \preceq (\mathbf{1} - z_n)q$ for all $n \in \mathbb{N}$, then $\bigvee_{n=1}^\infty z_n \geq \mathbf{c}(p)$.*

(vi) If $P(\mathcal{M}) \ni q_n \uparrow q \gg p$, $z_n \in P(Z(\mathcal{M}))$ and $(\mathbf{1} - z_n)p \succcurlyeq (\mathbf{1} - z_n)q_n$ for all $n \in \mathbb{N}$, then $\bigvee_{n=1}^\infty z_n \geq \mathbf{c}(p)$.

Proof. (i) Since p is a properly infinite projection then there are pairwise disjoint projections $p_1, \dots, p_n, \dots \in P(\mathcal{M})$ such that $p = \bigvee_{n=1}^\infty p_n$, $p_n \sim p$ for all $n \in \mathbb{N}$. Then $p_n \succcurlyeq q_n$ for every $n \in \mathbb{N}$. Hence, $p = \bigvee_{n=1}^\infty p_n \succcurlyeq \bigvee_{n=1}^\infty q_n$.

(ii) It follows from Corollary 16 that there exists a family $\{z_\sigma\}_{\sigma \in S_n}$ of pairwise orthogonal projections in $P(Z(\mathcal{M}))$ such that $\sum_{\sigma \in S_n} z_\sigma = \mathbf{c}(p)$, $z_\sigma q_{\sigma(1)} \preccurlyeq z_\sigma q_{\sigma(2)} \preccurlyeq \dots \preccurlyeq z_\sigma q_{\sigma(n)}$ for all $\sigma \in S_n$. We shall consider only those elements $\sigma \in S_n$, for which $z_\sigma \neq 0$. If $z_\sigma q_{\sigma(n)}$ is a finite projection, then $\bigvee_{j=1}^n z_\sigma q_{\sigma(j)}$ is also finite and $\bigvee_{j=1}^n z_\sigma q_{\sigma(j)} \ll p z_\sigma$. Indeed, it follows from (i) that $\bigvee_{j=1}^n z_\sigma q_{\sigma(j)} \preccurlyeq p z_\sigma$. Moreover, we have $p z_\sigma$ is a properly infinite projection. So, there does not exist central projection $0 < z \leq z_\sigma$, such that $\bigvee_{j=1}^n z q_{\sigma(j)} \neq 0$ that $\bigvee_{j=1}^n z q_{\sigma(j)} \sim p z$ since, in this case, we would have that $0 \neq p z$ is finite. If $z_\sigma q_{\sigma(n)}$ is a properly infinite projection, then, according to (i), $\bigvee_{j=1}^n z_\sigma q_{\sigma(j)} \preccurlyeq z_\sigma q_{\sigma(n)} \ll p z_\sigma$. Hence, $\bigvee_{i=1}^n q_i = \bigvee_{i=1}^n \sum_{\sigma \in S_n} z_\sigma q_i = \sum_{\sigma \in S_n} \bigvee_{j=1}^n z_\sigma q_{\sigma(j)} \ll \sum_{\sigma \in S_n} p z_\sigma = p$.

(iii) Since, $\mathbf{1} = p + (\mathbf{1} - p)$, the estimate $\mathbf{1} \preccurlyeq p$ follows from (i).

(iv) We have $p = qp + (\mathbf{1} - q)p$, $qp \leq q \ll p$. Suppose $(\mathbf{1} - q)pz \ll pz$ for some $0 \neq z \in P(Z(\mathcal{M}))$. Then, according to (ii), $p \ll p$, which is false. Hence, $(\mathbf{1} - q)p \succcurlyeq p$. So, since $(\mathbf{1} - q)p \leq p$, we conclude $(\mathbf{1} - q)p \sim p$.

(v) Let $z_0 := \bigwedge_{n=1}^\infty (\mathbf{1} - z_n)$. Then, it is clear, $z_0 \leq \mathbf{1} - z_n$. We shall now use a well-known implication

$$e \preccurlyeq f, \quad z \in P(Z(\mathcal{M})) \quad \Rightarrow \quad ze \preccurlyeq zf. \tag{6}$$

By the assumption, we have $(\mathbf{1} - z_n)p_n \preccurlyeq (\mathbf{1} - z_n)q$ and therefore, by (6) $z_0 p_n = z_0(\mathbf{1} - z_n)p_n \preccurlyeq z_0(\mathbf{1} - z_n)q = z_0 q$ for all $n \in \mathbb{N}$. Then all projections $z_0 p_n$ are finite and it follows from [28, Chapter V, Lemma 2.2] that $z_0 p \preccurlyeq z_0 q$. In this case $z_0 p$ is a finite projection. Since p is a properly infinite projection, we conclude $z_0 p = 0$. The latter trivially implies that $p(\mathbf{1} - z_0) = p$ and since $\mathbf{c}(p) \leq \mathbf{1} - z_0$, we also obtain $z_0 \mathbf{c}(p) = 0$. So,

$$(\mathbf{1} - \mathbf{c}(p)) \vee \bigvee_{n=1}^\infty z_n = (\mathbf{1} - \mathbf{c}(p)) \vee (\mathbf{1} - z_0) = \mathbf{1},$$

in particular, $\bigvee_{n=1}^\infty z_n \geq \mathbf{c}(p)$.

(vi) Let $z_0 := \bigwedge_{n=1}^\infty (\mathbf{1} - z_n)$. Then, from the conditions $\mathbf{1} - z_n \geq z_0$, $(\mathbf{1} - z_n)p \succcurlyeq (\mathbf{1} - z_n)q_n$ we have $z_0 p \succcurlyeq z_0 q_n$ for all $n \in \mathbb{N}$. So, it follows from (i) that $z_0 p = 0$ or $z_0 p \succcurlyeq z_0 q_1 + \bigvee_{n=1}^\infty (z_0 q_{n+1} - z_0 q_n) = z_0 q$. However, due to the assumption $q \gg p$, the estimate $z_0 p \succcurlyeq z_0 q$ can hold only when $z_0 \mathbf{c}(q) = 0$. Since $\mathbf{c}(p) \leq \mathbf{c}(q)$, we have $z_0 \mathbf{c}(p) = 0$ in any case. Hence, $\bigvee_{n=1}^\infty z_n = \mathbf{1} - z_0 \geq \mathbf{c}(p)$. \square

Let $c \in Z_h(LS(\mathcal{M}))$. We set

$$\begin{aligned} e_z^a(-\infty, c) &:= \mathbf{s}((c - a)_+), & e_z^a(c, +\infty) &:= \mathbf{s}((a - c)_+), \\ e_z^a(-\infty, c] &:= \mathbf{1} - e_z^a(c, +\infty), & e_z^a[c, +\infty) &:= \mathbf{1} - e_z^a(-\infty, c). \end{aligned}$$

Observe that all the projections defined above belong to a commutative W^* -subalgebra of \mathcal{M} , generated by $Z(\mathcal{M})$ and spectral projections of the element a . Having this observation in mind, for any $c_1, c_2 \in Z_h(LS(\mathcal{M}))$, such that $c_1 \leq c_2$, we set

$$e_z^a[c_1, c_2] := e_z^a(-\infty, c_2)e_z^a[c_1, +\infty), \quad e_z^a(c_1, c_2] := e_z^a(-\infty, c_2)e_z^a(c_1, +\infty),$$

$$e_z^a[c_1, c_2] := e_z^a(-\infty, c_2]e_z^a[c_1, +\infty), \quad e_z^a(c_1, c_2) := e_z^a(-\infty, c_2)e_z^a(c_1, +\infty).$$

Finally, we set

$$e_z^a\{c\} = e_z^a[c, c].$$

Observe that our spectral measure is analogous to the construction given in [11, Definition 2.4] (for the case of unbounded locally measurable self-adjoint operator).

Lemma 22. Fix $c \in Z_h(LS(\mathcal{M}))$.

- (i) If $c_1, c_2 \in Z_h(LS(\mathcal{M}))$, $c_1 \leq c_2$ then $e_z^a(-\infty, c_1) \leq e_z^a(-\infty, c_2)$ and $e_z^a(c_1, +\infty) \geq e_z^a(c_2, +\infty)$.
- (ii) If $z \in P(Z(\mathcal{M}))$ then $e_z^a(-\infty, c)z = e_z^a(-\infty, cz)z = e_z^{az}(-\infty, cz)z$ and $e_z^a(c, +\infty)z = e_z^a(cz, +\infty)z = e_z^{az}(cz, +\infty)z$.
- (iii) $ae_z^a(-\infty, c] \leq ce_z^a(-\infty, c]$.
- (iv) $ae_z^a[c, +\infty) \geq ce_z^a[c, +\infty)$.
- (v) $ae_z^a\{c\} = ce_z^a\{c\}$.
- (vi) If $\{c_\alpha\}_{\alpha \in I} \subset Z_h(LS(\mathcal{M}))$ and $c = \bigvee_{\alpha \in I} c_\alpha$ then $\bigvee_{\alpha \in I} e_z^a(-\infty, c_\alpha) = e_z^a(-\infty, c)$.
- (vii) If $\{c_\alpha\}_{\alpha \in I} \subset Z_h(LS(\mathcal{M}))$ and $c = \bigwedge_{\alpha \in I} c_\alpha$ then $\bigvee_{\alpha \in I} e_z^a(c_\alpha, +\infty) = e_z^a(c, +\infty)$.

Proof. (i) Since $c_1 - a \leq c_2 - a$, then $\mathbf{s}((c_1 - a)_+) \leq \mathbf{s}((c_2 - a)_+)$ and $\mathbf{s}((a - c_1)_+) \geq \mathbf{s}((a - c_2)_+)$.

(ii) $e_z^a(-\infty, c)z = \mathbf{s}((c - a)_+)z = \mathbf{s}((cz - a)_+)z = \mathbf{s}((cz - az)_+)z$. The second set of equalities is proven in the same way.

(iii) $(c - a)e_z^a(-\infty, c] = (c - a)(\mathbf{1} - e_z^a(c, +\infty)) = (c - a) + (a - c)\mathbf{s}((a - c)_+) = (c - a) + (a - c)_+ = (a - c)_- \geq 0$.

The proof of (iv) is the same.

(v) From (iii) and (iv) we have that $ae_z^a\{c\} \leq ce_z^a\{c\}$ and $ae_z^a\{c\} \geq ce_z^a\{c\}$. Hence, $ae_z^a\{c\} = ce_z^a\{c\}$.

(vi) Since, $(\mathbf{1} - e_z^a(-\infty, c))(c_\alpha - a) \leq (\mathbf{1} - e_z^a(-\infty, c))(c - a) \leq 0$ then $\mathbf{1} - e_z^a(-\infty, c) \leq \mathbf{1} - e_z^a(-\infty, c_\alpha)$ or $e_z^a(-\infty, c_\alpha) \leq e_z^a(-\infty, c)$ for every $\alpha \in I$. Let $q = e_z^a(-\infty, c) - \bigvee_{\alpha \in I} e_z^a(-\infty, c_\alpha)$. Then $(c - a)q \geq 0$. On the other hand, $(c_\alpha - a)q \leq 0$ for every $\alpha \in I$. Hence, $(c - a)q = \bigvee_{\alpha \in I} (c_\alpha - a)q \leq 0$. Then $(c - a)q = 0$. So, $q = 0$ since $q \leq \mathbf{s}((c - a)_+)$ and q commute with $c - a$.

(vii) Since $c \leq c_\alpha$, then $e_z^a(c_\alpha, +\infty) \leq e_z^a(c, +\infty)$ for every $\alpha \in I$ (see the beginning of the proof of (vi)). Let $q = e_z^a(c, +\infty) - \bigvee_{\alpha \in I} e_z^a(c_\alpha, +\infty)$. Then $q(a - c) \geq 0$. On the other hand, $q(a - c_\alpha) \leq 0$ for every $\alpha \in I$. So, $q(a - c) = \bigvee_{\alpha \in I} q(a - c_\alpha) \leq 0$. Hence, $q(a - c) = 0$ and $q = 0$. \square

Lemma 23. There exists an element $c \in Z_h(LS(\mathcal{M}))$ such that

$$pe_z^a(-\infty, c) < pe_z^a(c, +\infty), \quad \forall p \in P(Z(\mathcal{M})), \quad p > 0. \tag{7}$$

Proof. Since $a \in LS_h(\mathcal{M})$, there exists a set of pairwise disjoint projections $\{p_n\}_{n=1}^\infty$ from $P(Z(\mathcal{M}))$ such that $\bigvee_{n=1}^\infty p_n = \mathbf{1}$ and $ap_n \in S_h(\mathcal{M})$ for every $n \in \mathbb{N}$. Without loss of generality, we may assume $a \in S_h(\mathcal{M})$. Indeed, if for every ap_n , $n \geq 1$ there exists an element $c^{(n)} \in Z_h(LS(\mathcal{M}p_n))$, satisfying (7), then, by Lemma 18, there exists an element $c \in Z_h(LS(\mathcal{M}))$ satisfying $cp_n = c^{(n)}p_n$, $n \geq 1$ and (7). So, we may (and shall) consider only the case $a \in S_h(\mathcal{M})$. The latter assumption guarantees that there exists a scalar $\lambda_1 \in \mathbb{R}$ such that $e^a(-\infty, \lambda_1)$ is a finite projection. Let $\{\lambda_n\}_{n=1}^\infty \subset \mathbb{R}$, $\lambda_n \downarrow -\infty$. By the spectral theorem, we have $e^a(-\infty, \lambda_n) \downarrow 0$. For every $n \in \mathbb{N}$ set

$$q_n := \bigvee \{r \in P(Z(\mathcal{M})) : re^a(-\infty, \lambda_n) \succ re^a(\lambda_n, +\infty)\}, \quad q := \bigwedge_{n=1}^\infty q_n.$$

Fix n and let k tend to infinity. Then $qe^a(-\infty, \lambda_{n+k}) \succ qe^a(\lambda_{n+k}, +\infty) \geq qe^a(\lambda_n, +\infty)$. Since all projections $qe^a(-\infty, \lambda_{n+k})$ are finite and $qe^a(-\infty, \lambda_{n+k}) \downarrow 0$, we have $qe^a(\lambda_n, +\infty) = 0$ for every $n \in \mathbb{N}$ [2, Lemma 6.11]. On the other hand, $e^a(\lambda_n, +\infty) \uparrow \mathbf{1}$. So, $qe^a(\lambda_n, +\infty) \uparrow q$. Hence, $q = 0$. Then $\bigvee_{n=1}^\infty (\mathbf{1} - q_n) = \mathbf{1}$.

Let

$$r_1 := \mathbf{1} - q_1, \quad r_{n+1} := \bigvee_{k=1}^{n+1} (\mathbf{1} - q_k) - \bigvee_{k=1}^n (\mathbf{1} - q_k), \quad n \geq 1.$$

Then projections r_n , $n \geq 1$ are pairwise disjoint, $\bigvee_{n=1}^\infty r_n = \mathbf{1}$, $r_n \leq \mathbf{1} - q_n$ for all $n \in \mathbb{N}$. It follows from Lemma 18 that there exists an element $c \in Z_h(LS(\mathcal{M}))$ such that $cr_n = \lambda_n r_n$.

Let $0 \neq p \in P(Z(\mathcal{M}))$. It follows from Theorem 15 that $p = e_1 + e_2$, where $e_1, e_2 \in P(Z(\mathcal{M}))$, $e_1 e_z^a(-\infty, c) < e_1 e_z^a(c, +\infty)$, $e_2 e_z^a(-\infty, c) \succ e_2 e_z^a(c, +\infty)$. In this case, using Lemma 22(ii), we obtain

$$\begin{aligned} e_2 r_n e^a(-\infty, \lambda_n) &= e_2 r_n e^a(-\infty, \lambda_n r_n) = e_2 r_n e_z^a(-\infty, cr_n) \\ &= e_2 r_n e_z^a(-\infty, c) \succ e_2 r_n e_z^a(c, +\infty) \\ &= e_2 r_n e_z^a(cr_n, +\infty) = e_2 r_n e^a(\lambda_n r_n, +\infty) \\ &= e_2 r_n e^a(\lambda_n, +\infty) \end{aligned}$$

for any $n \in \mathbb{N}$. Due to the definition of the projection q_n , we obtain $e_2 r_n \leq q_n$. Hence, $e_2 r_n = 0$ (since $e_2 r_n \leq r_n \leq \mathbf{1} - q_n$) for every $n \in \mathbb{N}$. So, $e_2 = \bigvee_{n=1}^\infty e_2 r_n = 0$. Hence, $p = e_1$ and $p e_z^a(-\infty, c) < p e_z^a(c, +\infty)$. \square

Set

$$\begin{aligned} \Lambda_- &:= \{c \in Z_h(LS(\mathcal{M})) : p e_z^a(-\infty, c) < p e_z^a(c, +\infty), \forall p \in P(Z(\mathcal{M})), p > 0\}, \\ \Lambda_+ &:= \{c \in Z_h(LS(\mathcal{M})) : p e_z^a(-\infty, c) \succ p e_z^a(c, +\infty), \forall p \in P(Z(\mathcal{M})), p > 0\}. \end{aligned}$$

It follows from Lemma 23 that $\Lambda_- \neq \emptyset$. The proof that $\Lambda_+ \neq \emptyset$ is similar.

Lemma 24. $c(e_z^a(c, +\infty)) = \mathbf{1}$ for every $c \in \Lambda_-$ and $c(e_z^a(-\infty, c)) = \mathbf{1}$ for every $c \in \Lambda_+$.

Proof. Let $c \in \Lambda_-$. Suppose $p = \mathbf{1} - \mathbf{c}(e_z^a(c, +\infty)) > 0$. Then $0 = pe_z^a(c, +\infty) \succ pe_z^a(-\infty, c) \geq 0$ which means $0 > 0$. This contradiction shows $p = 0$.

The proof of the second part is similar. \square

Lemma 25. Λ_- is a \vee -ideal and Λ_+ is a \wedge -ideal in the lattice $Z_h(LS(\mathcal{M}))$.

Proof. We will show only that Λ_- is a \vee -ideal in $Z_h(LS(\mathcal{M}))$. The proof of the second part is analogous.

Let $c_1, c_2 \in \Lambda_-$. If $c_1 \geq c \in Z_h(LS(\mathcal{M}))$ then from Lemma 22(i) and the definition of Λ_- we have $pe_z^a(-\infty, c) \leq pe_z^a(-\infty, c_1) < pe_z^a(c_1, +\infty) \leq pe_z^a(c, +\infty)$ for every $0 \neq p \in P(Z(\mathcal{M}))$. So, $c \in \Lambda_-$.

We set $q := \mathbf{s}((c_2 - c_1)_+)$ and observe that $q \in P(Z(\mathcal{M}))$, and that $qc_1 \leq qc_2$, $(\mathbf{1} - q)c_1 \geq (\mathbf{1} - q)c_2$. Then $c_1 \vee c_2 = (\mathbf{1} - q)c_1 + qc_2$. Let $0 \neq p \in P(Z(\mathcal{M}))$. Then, from Lemma 22(ii) we have $pe_z^a(-\infty, c_1 \vee c_2) = p(\mathbf{1} - q)e_z^a(-\infty, c_1) + pqe_z^a(-\infty, c_2)$. In this case $p(\mathbf{1} - q) = 0$ or $p(\mathbf{1} - q)e_z^a(-\infty, c_1) < p(\mathbf{1} - q)e_z^a(c_1, +\infty)$. So, $pq = 0$ or $pqe_z^a(-\infty, c_2) < pqe_z^a(c_2, +\infty)$. Hence, in both cases we have $pe_z^a(-\infty, c_1 \vee c_2) = p(\mathbf{1} - q)e_z^a(-\infty, c_1) + pqe_z^a(-\infty, c_2) < p(\mathbf{1} - q)e_z^a(c_1, +\infty) + pqe_z^a(c_2, +\infty) = pe_z^a(c_1 \vee c_2, +\infty)$. Hence, $c_1 \vee c_2 \in \Lambda_-$. \square

Lemma 26. Let $c_1 \in \Lambda_-$, $c_2 \in \Lambda_+$. Then $pc_1 < pc_2$ for every $0 \neq p \in P(Z(\mathcal{M}))$.

Proof. Suppose a contrary. Then $pc_1 \geq pc_2$ for some $0 \neq p \in P(Z(\mathcal{M}))$. In this case,

$$\begin{aligned} pe_z^a(-\infty, c_1) &= pe_z^a(-\infty, pc_1) && \text{(by Lemma 22(ii))} \\ &\geq pe_z^a(-\infty, pc_2) && \text{(by Lemma 22(i))} \\ &= pe_z^a(-\infty, c_2) && \text{(by Lemma 22(i))} \\ &\succ pe_z^a(c_2, +\infty) && \text{(by the definition)} \\ &= pe_z^a(pc_2, +\infty) && \text{(by Lemma 22(ii))} \\ &\geq pe_z^a(pc_1, +\infty) && \text{(by Lemma 22(i))} \\ &= pe_z^a(c_1, +\infty) && \text{(by Lemma 22(ii))} \end{aligned}$$

which is not true since $c_1 \in \Lambda_-$. \square

Due to Lemmas 19 and 26, we can now set

$$c_0 := \bigvee \Lambda_- \in Z_h(LS(\mathcal{M}))$$

and apply again Lemma 26 to infer that $c_0 \leq c$ for any $c \in \Lambda_+$.

Lemma 27. $c_0 - \varepsilon \mathbf{1} \in \Lambda_-$ for every $\varepsilon > 0$.

Proof. Suppose a contrary. Then there exists a projection $0 < p \in P(Z(\mathcal{M}))$ such that $pe_z^a(-\infty, c_0 - \varepsilon \mathbf{1}) \succ pe_z^a(c_0 - \varepsilon \mathbf{1}, +\infty)$. Choose an arbitrary element $c \in \Lambda_-$. We shall show that $pc \leq p(c_0 - \varepsilon \mathbf{1})$. If it is not the case, then $P(Z(\mathcal{M})) \ni q := \mathbf{s}((p(c_0 - \varepsilon \mathbf{1}) - pc)_-) > 0$, $q \leq p$ i $qc \geq q(c_0 - \varepsilon \mathbf{1})$. Then

$$\begin{aligned}
 qe_z^a(-\infty, c_0 - \varepsilon \mathbf{1}) &= qe_z^a(-\infty, q(c_0 - \varepsilon \mathbf{1})) \quad (\text{by Lemma 22(ii)}) \\
 &\leq qe_z^a(-\infty, qc) \quad (\text{by Lemma 22(i)}) \\
 &= qe_z^a(-\infty, c) \quad (\text{by Lemma 22(ii)}) \\
 &< qe_z^a(c, +\infty) \quad (\text{by the definition}) \\
 &= qe_z^a(qc, +\infty) \quad (\text{by Lemma 22(ii)}) \\
 &\leq qe_z^a(q(c_0 - \varepsilon \mathbf{1}), +\infty) \quad (\text{by Lemma 22(i)}) \\
 &= qe_z^a(c_0 - \varepsilon \mathbf{1}, +\infty) \quad (\text{by Lemma 22(ii)}).
 \end{aligned}$$

On the other hand, $qe_z^a(-\infty, c_0 - \varepsilon \mathbf{1}) \not\geq qe_z^a(c_0 - \varepsilon \mathbf{1}, +\infty)$, since $pe_z^a(-\infty, c_0 - \varepsilon \mathbf{1}) \not\geq pe_z^a(c_0 - \varepsilon \mathbf{1}, +\infty)$ $P(Z(\mathcal{M})) \ni q \leq p$. However the inequalities $qe_z^a(-\infty, c_0 - \varepsilon \mathbf{1}) < qe_z^a(c_0 - \varepsilon \mathbf{1}, +\infty)$ and $qe_z^a(-\infty, c_0 - \varepsilon \mathbf{1}) \not\geq qe_z^a(c_0 - \varepsilon \mathbf{1}, +\infty)$ cannot hold simultaneously. Hence, $q = 0$ and so $pc \leq p(c_0 - \varepsilon \mathbf{1})$. The latter implies $(p(c_0 - \varepsilon \mathbf{1}) - pc)_- = 0$, that is $pc_0 \leq p(c_0 - \varepsilon \mathbf{1})$, since $c_0 = \bigvee \Lambda_-$. Hence, $p = 0$. However, this contradicts with the choice of $p > 0$. \square

Lemma 28. *The inequality $pe_z^a(-\infty, c_0 + \varepsilon \mathbf{1}) \not\geq pe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty)$ holds for all $\varepsilon > 0$ and $0 \neq p \in P(Z(\mathcal{M}))$.*

Proof. Suppose a contrary. Then by Theorem 15 there exists a projection $0 < p \in P(Z(\mathcal{M}))$, such that for every $q \in P(Z(\mathcal{M}))$, $0 < q \leq p$ the inequality $qe_z^a(-\infty, c_0 + \varepsilon \mathbf{1}) < qe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty)$ holds. Let $c \in \Lambda_-$. Put $c_1 = c(\mathbf{1} - p) + (c_0 + \varepsilon \mathbf{1})p$. Then $c_1 \in \Lambda_-$. Hence, there exists a projection $0 < r \in P(Z(\mathcal{M}))$ such that $re_z^a(-\infty, c_1) \not\geq re_z^a(c_1, +\infty)$. Thus, $r(\mathbf{1} - p)e_z^a(-\infty, c) \not\geq r(\mathbf{1} - p)e_z^a(c, +\infty)$ and, therefore, $r(\mathbf{1} - p) = 0$. Also, $rpe_z^a(-\infty, c_0 + \varepsilon \mathbf{1}) \not\geq rpe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty)$ and, therefore, $rp = 0$. It follows that $r = r(\mathbf{1} - p) + rp = 0$. Hence, $c_1 \in \Lambda_-$. However, $c_1 \not\leq c_0$. This contradicts to the definition of c_0 . \square

Let us consider the set

$$\begin{aligned}
 P_0 &:= \{p \in P(Z(\mathcal{M})) : \exists c_p \in Z_h(LS(\mathcal{M})), q_p, r_p \in P(\mathcal{M}) : \\
 &\quad q_p, r_p \leq pe_z^a\{c_p\}, q_p r_p = 0, pe_z^a(-\infty, c_p) + q_p \sim pe_z^a(c_p, +\infty) + r_p\}
 \end{aligned}$$

and set

$$p_0 := \bigvee P_0.$$

Lemma 29. *P_0 is a \vee -ideal in the Boolean algebra $P(Z_h(\mathcal{M}))$ and $p_0 \in P_0$, that is $P_0 = p_0 P(Z_h(\mathcal{M}))$.*

Proof. Let $p \in P_0$, $p > e \in P(Z_h(\mathcal{M}))$ and let c_p, q_p, r_p be exactly the same like in the definition of P_0 . Then by letting $q_e := q_p e$, $r_e := r_p e$, $c_e := c_p$ one can see that these projections satisfy the condition in the definition of P_0 . Indeed, we have $q_e r_e = e q_p r_p e = 0$, $q_e, r_e \leq e p e_z^a\{c_p\} = e e_z^a\{c_e\}$, $e e_z^a(-\infty, c_e) + q_e = e(pe_z^a(-\infty, c_p) + q_p) \sim e(pe_z^a(c_p, +\infty) + r_p) = e e_z^a(c_e, +\infty) + r_e$. Hence, $e \in P_0$.

Let $e_1, e_2 \in P_0$, $e_1e_2 = 0$. Due to the definition of P_0 , we know of the existence of $q_{e_i}, r_{e_i} \in P(\mathcal{M})$, $c_{e_i} \in Z_h(LS(\mathcal{M}))$ such that $q_{e_i}, r_{e_i} \leq e_i e_z^a \{c_{e_i}\}$, $q_{e_i} r_{e_i} = 0$, $e_i e_z^a(-\infty, c_{e_i}) + q_{e_i} \sim e_i e_z^a(c_{e_i}, +\infty) + r_{e_i}$ $i = 1, 2$. Let $c_{e_1+e_2} = c_{e_1}e_1 + c_{e_2}e_2$. Then $c_{e_1+e_2}e_i = c_i e_i$, $i = 1, 2$. So, we have $q_{e_1} + q_{e_2}, r_{e_1} + r_{e_2} \leq (e_1 + e_2) e_z^a \{c_{e_1+e_2}\}$, $(q_{e_1} + q_{e_2})(r_{e_1} + r_{e_2}) = 0$, $(e_1 + e_2) e_z^a(-\infty, c_{e_1+e_2}) + (q_{e_1} + q_{e_2}) \sim (e_1 + e_2) e_z^a(c_{e_1+e_2}, +\infty) + (r_{e_1} + r_{e_2})$. So, $e_1 + e_2 \in P_0$.

Let now $e_1, e_2 \in P_0$. Then $e_1 \vee e_2 = e_1 + e_2(\mathbf{1} - e_1) \in P_0$, since $e_1[e_2(\mathbf{1} - e_1)] = 0$ and $e_2(\mathbf{1} - e_1) \in P_0$. Hence, P_0 is a \vee -ideal in the Boolean algebra $P(Z_h(\mathcal{M}))$.

It follows from Lemma 17 that there exists a family $\{e_i\}_{i \in I}$ of pairwise disjoint projections from P_0 such that $\bigvee_{i \in I} e_i = p_0$. Since, $e_i \in P_0$, there are elements $c_{e_i}, q_{e_i}, r_{e_i}$. It follows from Lemma 18 that there exists $c_{p_0} \in Z_h(LS(\mathcal{M}))$ such that $c_{p_0} s(c_{e_i}) = c_{e_i}$ for every $i \in I$. Let $q_{p_0} = \bigvee_{i \in I} q_{e_i}$, $r_{p_0} = \bigvee_{i \in I} r_{e_i}$. From the definition of projections q_{p_0}, r_{p_0} above, we have

$$q_{p_0}, r_{p_0} \leq \bigvee_{i \in I} e_i e_z^a \{c_{e_i}\} = p_0 e_z^a \{c_{p_0}\}.$$

Furthermore, since $e_i e_j = 0$ for all $i \neq j$, we also have $q_{e_i} q_{e_j} = q_{e_i} r_{e_j} = 0$ and therefore

$$q_{p_0} r_{p_0} = \bigvee_{i \in I} q_{e_i} \bigvee_{i \in I} r_{e_i} = \bigvee_{i \in I} q_{e_i} r_{e_i} = 0.$$

Thus, we have

$$\begin{aligned} p_0 e_z^a(-\infty, c_{p_0}) + q_{p_0} &= \bigvee_{i \in I} [e_i e_z^a(-\infty, c_{e_i}) + q_{e_i}] \\ &\sim \bigvee_{i \in I} [e_i e_z^a(c_{e_i}, +\infty) + r_{e_i}] = p_0 e_z^a(c_{p_0}, +\infty) + r_{p_0}, \end{aligned}$$

in other words, $p_0 \in P_0$. \square

Lemma 30. *Suppose $p_0 = \mathbf{1}$. Then there exists an element $u \in U(\mathcal{M})$ such that $|[a, u]| = u^*|a - c|u + |a - c|$, $u^2 = \mathbf{1}$, where $c = c_{p_0} \in Z_h(LS(\mathcal{M}))$ is from the definition of the set P_0 for the element p_0 .*

Proof. Set $p = q_{p_0}$, $q = r_{p_0}$ and $r := \mathbf{1} - (e_z^a(-\infty, c) + p + e_z^a(c, +\infty) + q)$. Then $p, q, r \leq e_z^a \{c\}$ and so $ap = cp, aq = cq, ar = cr$. We claim that there exists a self-adjoint unitary element u such that $u(e_z^a(-\infty, c) + p) = (e_z^a(c, +\infty) + q)u$, $ur = r$. Indeed, since $e_z^a(-\infty, c) + p \sim e_z^a(c, +\infty) + q$, there exists a partial isometry v such that $v^*v = e_z^a(-\infty, c) + p$, $vv^* = e_z^a(c, +\infty) + q$. Set $u := v + v^* + r$. We have $u^*u = e_z^a(-\infty, c) + p + e_z^a(c, +\infty) + q + r = \mathbf{1}$, $uu^* = e_z^a(c, +\infty) + q + e_z^a(-\infty, c) + p + r = \mathbf{1}$, $u^* = v^* + v + r = u$. This establishes the claim. It now remains to verify that (2) holds.

To this end, first of all observe that the operators a and u^*au commute with the projections $e_z^a(-\infty, c) + p$, $e_z^a(c, +\infty) + q$ and r . This observation guarantees that

$$\begin{aligned} (u^*au - a)(e_z^a(-\infty, c) + p) &= |u^*au - a|(e_z^a(-\infty, c) + p), \\ (a - u^*au)(e_z^a(c, +\infty) + q) &= |u^*au - a|(e_z^a(c, +\infty) + q) \end{aligned}$$

and so

$$\begin{aligned}
 |u^*au - a|(e_z^a(-\infty, c) + p) &= u^*a(e_z^a(c, +\infty) + q)u - a(e_z^a(-\infty, c) + p) \\
 &= u^*a(e_z^a(c, +\infty) + q)u - cu^*(e_z^a(c, +\infty) + q)u \\
 &\quad + c(e_z^a(-\infty, c) + p) - a(e_z^a(-\infty, c) + p) \\
 &= u^*|a(e_z^a(c, +\infty) + q) - c(e_z^a(c, +\infty) + q)|u \\
 &\quad + |c(e_z^a(-\infty, c) + p) - a(e_z^a(-\infty, c) + p)| \\
 &= u^*|a - c|u(e_z^a(-\infty, c) + p) + |a - c|(e_z^a(-\infty, c) + p).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 |u^*au - a|(e_z^a(c, +\infty) + q) &= -u^*a(e_z^a(-\infty, c) + p)u + a(e_z^a(c, +\infty) + q) \\
 &= -(u^*a(e_z^a(-\infty, c) + p)u - cu^*(e_z^a(-\infty, c) + p)u) \\
 &\quad - c(e_z^a(c, +\infty) + q) + a(e_z^a(c, +\infty) + q) \\
 &= u^*|a - c|u(e_z^a(c, +\infty) + q) + |a - c|(e_z^a(c, +\infty) + q).
 \end{aligned}$$

Finally, $(u^*au - a)r = cr - cr = 0$, that is, $|u^*au - a|r = 0$. We now obtain (2) as follows

$$\begin{aligned}
 |u^*au - a| &= |u^*au - a|[(e_z^a(-\infty, c) + p) + (e_z^a(c, +\infty) + q) + r] \\
 &= |u^*au - a|(e_z^a(-\infty, c) + p) + |u^*au - a|(e_z^a(c, +\infty) + q) + |u^*au - a|r \\
 &= (u^*|a - c|u + |a - c|)[(e_z^a(-\infty, c) + p) + (e_z^a(c, +\infty) + q) + r] \\
 &= u^*|a - c|u + |a - c|. \quad \square
 \end{aligned}$$

It follows from Lemma 30 that if $p_0 = \mathbf{1}$, then the part (i) of Theorem 1 holds.

Lemma 31. *Let $p \in P(Z(\mathcal{M}))$ be a finite projection. Then $p \leq p_0$.*

Proof. The case $p = 0$ is trivial. So, we assume $p > 0$. The algebra $\mathcal{M}p$ has a faithful normal center-valued trace τ such that $\tau(p) = p$ [16]. Then $\tau(pe_z^a(-\infty, c)) < \tau(pe_z^a(c, +\infty))$ for every $c \in \Lambda_-$. Since τ is normal, using Lemma 22(vi), (vii) we have $\tau(pe_z^a(-\infty, c_0)) = \tau(pe_z^a(-\infty, \bigvee \Lambda_-)) = \tau(\bigvee pe_z^a(-\infty, \Lambda_-)) = \bigvee \tau(pe_z^a(-\infty, \Lambda_-)) \leq \bigvee \tau(pe_z^a(\Lambda_-, +\infty)) = \tau(\bigvee pe_z^a(\Lambda_-, +\infty)) = \tau(pe_z^a(\bigvee \Lambda_-, +\infty)) = \tau(pe_z^a(c_0, +\infty)) \leq \tau(pe_z^a[c_0, +\infty))$. Hence, $\tau(pe_z^a(-\infty, c_0)) \leq \tau(p)/2 = p/2$. The same arguments with help of Lemma 28 yield $\tau(pe_z^a(c_0 + \varepsilon\mathbf{1}, +\infty)) \leq \tau(p)/2 = p/2$ for every $\varepsilon > 0$. Since $e_z^a(c_0, +\infty) = \bigvee_{\varepsilon > 0} e_z^a(c_0 + \varepsilon\mathbf{1}, +\infty)$, we have

$$\tau(pe_z^a(c_0, +\infty)) \leq p/2.$$

By Theorem 15, there exist projections $p_1, p_2 \in P(Z(\mathcal{M}))$, such that

$$p_1 p_2 = 0, \quad p_1 + p_2 = p,$$

$$p_1 e_z^a(-\infty, c_0) \preceq p_1 e_z^a(c_0, +\infty), \quad p_2 e_z^a(-\infty, c_0) \succcurlyeq p_2 e_z^a(c_0, +\infty).$$

Applying then the inequality above, we have

$$p_1 \geq 2\tau(p_1 e_z^a(c_0, +\infty))$$

$$= \tau(p_1 e_z^a(c_0, +\infty)) + \tau(p_1 e_z^a(-\infty, c_0)) + (\tau(p_1 e_z^a(c_0, +\infty)) - \tau(p_1 e_z^a(-\infty, c_0))),$$

and, immediately,

$$\tau(p_1 e_z^a(c_0, +\infty)) - \tau(p_1 e_z^a(-\infty, c_0)) \leq \tau(p_1(\mathbf{1} - e_z^a(-\infty, c_0) - e_z^a(c_0, +\infty)))$$

$$= \tau(p_1 e_z^a\{c_0\}).$$

Hence,

$$p_1 e_z^a(-\infty, c_0) \preceq p_1 e_z^a(c_0, +\infty) \preceq p_1 e_z^a(-\infty, c_0) + p_1 e_z^a\{c_0\}.$$

Hence, it follows from Lemma 20 that there exists a projection $P(\mathcal{M}) \ni p_{11} \leq p_1 e_z^a\{c_0\}$ such that $p_{11} + p_1 e_z^a(-\infty, c_0) \sim p_1 e_z^a(c_0, +\infty)$.

Analogous arguments show that there exists a projection $P(\mathcal{M}) \ni p_{21} \leq p_2 e_z^a\{c_0\}$, such that $p_{21} + p_2 e_z^a(-\infty, c_0) \sim p_2 e_z^a(c_0, +\infty) + p_{21}$.

Since, $p_1 + p_2 = p$ $p_1 p_2 = 0$ we have $p_{11} + p e_z^a(-\infty, c_0) \sim p e_z^a(c_0, +\infty) + p_{21}$. Hence, $p \in P_0$. \square

In the proof of the following lemma we shall frequently use a well-known fact that $\mathbf{c}(q)z = \mathbf{c}(qz), \forall q \in P(\mathcal{M}), \forall z \in P(Z(\mathcal{M}))$ [15, Proposition 5.5.3].

Lemma 32. *Let $p \in P(Z(\mathcal{M}))$ and let $\mathcal{M}p$ be a σ -finite purely infinite W^* -subalgebra in \mathcal{M} . Then $p \leq p_0$.*

Proof. Consider the projection $p_1 := p(\mathbf{1} - p_0)$. Let $c \in \Lambda_-$, that is $e_z^a(-\infty, c) \ll e_z^a(c, +\infty)$. We shall show $p_1 e_z^a(-\infty, c) = 0$. If this is not the case then for $z = \mathbf{c}(p_1 e_z^a(-\infty, c)) \neq 0$ we would have $\mathbf{c}(p_1 e_z^a(-\infty, c)) = z = zp_1 = \mathbf{c}(p_1 e_z^a(c, +\infty))$, that is $p_1 e_z^a(-\infty, c) \sim p_1 e_z^a(c, +\infty)$ [15, Corollary 6.3.5]. This however contradicts with the assumption $c \in \Lambda_-$. Hence, $p_1 e_z^a(-\infty, c) = 0$. Therefore, $p_1 e_z^a(-\infty, c_0) = \bigvee_{c \in \Lambda_-} p_1 e_z^a(-\infty, c) = 0$.

Let $\varepsilon > 0, c = c_0 + \varepsilon \mathbf{1}$. From Lemma 28, we have $p_1 e_z^a(-\infty, c) \succcurlyeq p_1 e_z^a(c, +\infty)$ (if $P(Z(\mathcal{M})) \ni q \leq p_1$ and $q e_z^a(-\infty, c) \sim q e_z^a(c, +\infty)$ then $q \in P_0$. Hence, $q = 0$). Arguing as above, we obtain $p_1 e_z^a(c, +\infty) = 0$. Thus, $p_1 e_z^a(c_0, +\infty) = \bigvee_{\varepsilon > 0} p_1 e_z^a(c_0 + \varepsilon \mathbf{1}, +\infty) = 0$.

We have thus obtained, that $p_1 e_z^a(-\infty, c_0) = 0 = p_1 e_z^a(c_0, +\infty)$. This means, that $p_1 \leq p_0$, that is $p_1 = 0$. Hence $p \leq p_0$. \square

Let $q \in P(Z(\mathcal{M}))$ and suppose

$$q e_z^a(-\infty, c_0] = q e_z^a(-\infty, c_0) + q e_z^a\{c_0\} \sim q e_z^a(c_0, +\infty).$$

Then setting $c_q := c_0$, $q_q := qe_z^a\{c_0\}$, $r_q := 0$ we see that $q \in P_0$ and therefore $q \leq p_0$. So, there exist projections p_+ $p_- \in P(Z(\mathcal{M}))$ (which may be null projections), such that

$$p_- + p_+ = \mathbf{1} - p_0, \quad p_- p_+ = 0,$$

and

$$p_+ e_z^a(-\infty, c_0] \succ\!\succ p_+ e_z^a(c_0, +\infty), \quad p_- e_z^a(-\infty, c_0] \prec\!\prec p_- e_z^a(c_0, +\infty).$$

By Lemma 31, the following implications hold $p_- \neq 0$ (respectively, $p_+ \neq 0$) $\Rightarrow p_- \mathcal{M}$ (respectively, $p_+ \mathcal{M}$) is a properly infinite W^* -algebra.

Lemma 33. *For every $q \in P(Z(\mathcal{M}))$, $0 < q \leq p_+$, we have $qe_z^a(-\infty, c_0) \succ qe_z^a[c_0, +\infty)$.*

Proof. Firstly, we show that $p_+ e_z^a(-\infty, c_0) \succ p_+ e_z^a(c_0, +\infty)$. If this fails then there exists an element $0 \neq q \in P(Z(\mathcal{M}p_+))$ such that

$$qe_z^a(-\infty, c_0) \prec qe_z^a(c_0, +\infty).$$

On the other hand, we have

$$qe_z^a(-\infty, c_0] \succ\!\succ qe_z^a(c_0, +\infty).$$

Indeed, to see the preceding estimate, let $0 < r \leq q$. Then we have $r \leq p_+ = p_+ e_z^a(-\infty, c_0] \vee p_+ e_z^a(c_0, +\infty)$, and it follows from the definition of the symbol “ $\succ\!\succ$ ” that $re_z^a(-\infty, c_0] \succ re_z^a(c_0, +\infty)$. By Lemma 20, we have that $qe_z^a(c_0, +\infty) \sim qe_z^a(-\infty, c_0) + r$, where $r \in P(Z(\mathcal{M})q)$ and $r \prec qe_z^a\{c_0\}$. In this case, setting $c_q := c_0$, $q_q := r$, $r_q := 0$ we obtain $q \in P_0$ and so $q \leq p_0$. Thus, $q \leq p_0 p_+ = 0$. This contradiction shows that

$$p_+ e_z^a(-\infty, c_0) \succ\!\succ p_+ e_z^a(c_0, +\infty). \tag{8}$$

Let us now consider projections $p_+ e_z^a(-\infty, c_0)$ and $p_+ e_z^a[c_0, +\infty)$ in the algebra $p_+ \mathcal{M}$. In the notation of Theorem 15 applied to the algebra $p_+ \mathcal{M}$, we intend to prove that $z_- = z_0 = 0$ and so $z_+ = p_+$. Suppose that there exists $0 < r \leq p_+$, $r \in P(Z(\mathcal{M}))$ such that $re_z^a(-\infty, c_0) \sim re_z^a[c_0, +\infty) = re_z^a\{c_0\} + re_z^a(c_0, +\infty)$. Then $r \leq p_0$ and therefore $r \leq p_+ p_0 = 0$. This shows that $z_0 = 0$. Next, we shall show the equality $z_- = 0$. Supposing that $z_- > 0$, we have $z_- e_z^a(-\infty, c_0) \prec z_- e_z^a[c_0, +\infty)$. Then $z_- e_z^a[c_0, +\infty) = p_1 + p_2$, where $p_1, p_2 \in P(\mathcal{M})$, $p_1 p_2 = 0$, $p_1 \sim z_- e_z^a(-\infty, c_0)$. From $p_+ e_z^a(-\infty, c_0) \succ\!\succ p_+ e_z^a(c_0, +\infty)$ it follows that $z_- e_z^a(-\infty, c_0) \succ\!\succ z_- e_z^a(c_0, +\infty)$. Then, by Lemma 20, we know that there exists some $q \in P(\mathcal{M}z_-)$, such that $z_- e_z^a(-\infty, c_0) \sim z_- e_z^a(c_0, +\infty) + q$ and $q \prec e_z^a\{c_0\}z_-$. Hence, $z_- \in P_0$, i.e. $z_- \leq p_0$. Therefore, $z_- p_+ = 0$ and $z_- = 0$. This contradiction completes the proof. \square

Lemma 34. *$qe_z^a(c_0, c_0 + \varepsilon\mathbf{1}] \succ qe_z^a(-\infty, c_0] + qe_z^a(c_0 + \varepsilon\mathbf{1}, +\infty)$ for every $\varepsilon > 0$ and every $q \in P(Z(\mathcal{M}))$, $0 < q \leq p_-$.*

Proof. Suppose, there exists $0 < p \in P(Z(\mathcal{M}p_-))$ such that $pe_z^a(c_0, c_0 + \varepsilon\mathbf{1}] \preccurlyeq pe_z^a(-\infty, c_0] + pe_z^a(c_0 + \varepsilon\mathbf{1}, +\infty)$. By the assumption the projection $p_- \neq 0$ and hence it is properly infinite

(see the implication preceding Lemma 33) and since $p \leq p_-$, we conclude that p is also properly infinite. Due to Lemma 21(iii), we have $pe_z^a(-\infty, c_0] + pe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty) \sim p$. However, $pe_z^a(-\infty, c_0] \ll pe_z^a(c_0, +\infty)$. Hence, $pe_z^a(-\infty, c_0] \ll p$, and by Lemma 21(ii) we have $pe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty) \sim p$ (indeed, otherwise, we would have had $pe_z^a(-\infty, c_0] + pe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty) \ll p$). Next, it follows from Lemma 27 that $pe_z^a(-\infty, c_0 + \varepsilon \mathbf{1}) \succ pe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty)$, that is $pe_z^a(-\infty, c_0 + \varepsilon \mathbf{1}) \sim p \sim pe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty)$. Then $p \leq p_0$. Which is a contradiction with the assumption $0 < p \leq p_-$. \square

In the case when $p_+ = \mathbf{1}$, it follows from Lemma 33 that $c_0 \in \Lambda_+$ and therefore in this case $\bigvee \Lambda_- = c_0 = \bigwedge \Lambda_+$, where the first equality is simply the definition of c_0 . Let us explain the second equality. By Lemma 26, we have $c_1 < c_2$ for every $c_1 \in \Lambda_-$, $c_2 \in \Lambda_+$. Therefore, $c_0 \leq c_2$ for any $c_2 \in \Lambda_+$. However, $c_0 \in \Lambda_+$, and hence $c_0 = \bigwedge \Lambda_+$.

In the case when $p_- = \mathbf{1}$, appealing to the definitions of p_- and Λ_- we have $c_0 \in \Lambda_-$. Moreover, in this case we have by Lemma 34 that $e_z^a(c_0 + 2\varepsilon \mathbf{1}, +\infty) \leq e_z^a(c_0 + \varepsilon \mathbf{1}, +\infty) \ll e_z^a(c_0, c_0 + \varepsilon \mathbf{1}) \leq e_z^a(c_0, c_0 + 2\varepsilon \mathbf{1}) \leq e_z^a(-\infty, c_0 + 2\varepsilon \mathbf{1})$, that is $c_0 + 2\varepsilon \mathbf{1} \in \Lambda_+$ for every $\varepsilon > 0$. Then it follows from Lemma 22(v) that in the case $p_- = \mathbf{1}$, we have $\bigvee \Lambda_- = c_0 = \bigwedge \Lambda_+$ as well. Indeed, $c_0 = \bigwedge_{\varepsilon > 0} (c_0 + \varepsilon \mathbf{1}) \geq \bigwedge \Lambda_+ \geq c_0$.

Hence, the cases $p_+ = \mathbf{1}$ and $p_- = \mathbf{1}$ are symmetric, that is, the second case may be obtained from the first case using the substitution $a \rightarrow -a$ and $c_0 \rightarrow -c_0$. Hence, in the sequel we will consider only the case $p_- = \mathbf{1}$. In this case, the algebra \mathcal{M} is properly infinite, since the projection p_- is properly infinite. It follows from Lemma 34 that in this case

$$qe_z^a(c_0, c_0 + \varepsilon \mathbf{1}) \succ qe_z^a(-\infty, c_0] + qe_z^a(c_0 + \varepsilon \mathbf{1}, +\infty) \tag{9}$$

for every $\varepsilon > 0$ and every $0 < q \in P(Z(\mathcal{M}))$.

The following lemma extends the result of [3, Lemma 4].

Lemma 35. *Let $p, q \in P(\mathcal{M})$, $p \succ q$ and let one of the following hold:*

- (i) *q is a finite projection and there exists a non-decreasing sequence $\{p_n\}$ of finite projections in \mathcal{M} such that $p_n \uparrow p$ and $ap_n = p_n a$ for every $n \in \mathbb{N}$;*
- (ii) *q is a properly infinite projection and $ap = pa \in \mathcal{M}$.*

Then there exists a projection $q_1 \leq p$ in \mathcal{M} such that $q_1 \sim q$ and $aq_1 = q_1 a$.

Proof. Assume (i) holds.

Set $\mathcal{A}_1 := \{b \in \mathcal{M} : ba = ab\}$ (that is, \mathcal{A}_1 is the commutant of the family of all spectral projections of a). Since, $p_n \uparrow p$ and $ap_n = p_n a$ for every $n \in \mathbb{N}$, we have $ap = pa$. Let $\mathcal{A} := p\mathcal{A}_1$. Then \mathcal{A} is W^* -subalgebra in \mathcal{M} with identity p .

First of all, let us show that every atom of algebra \mathcal{A} (if it exists) is an atom of the algebra \mathcal{M} . Let e be an atom of the algebra \mathcal{A} , $0 \leq f < e$, $f \in P(\mathcal{M})$. So, if g is a spectral projection of a then $gf = g(p(ef)) = ((gp)e)f \in \{0, e\}f = \{0, ef\} = \{0, f\} \subset \mathcal{M}_h$. In particular, $gf = fg$, that is $f \in \mathcal{A}$. Hence, $f = 0$. Consequently, e is an atom of \mathcal{M} .

Let $z_n = \bigvee \{z \in P(Z(\mathcal{M})) : zq \preceq zp_n\}$. It is easy to see, that $z_n \uparrow \mathbf{1}$ (otherwise $(\mathbf{1} - \bigvee_n z_n)q \succ (\mathbf{1} - \bigvee_n z_n)p$). We will assume $z_n q > 0$ for every $n \in \mathbb{N}$.

Let us construct a non-decreasing sequence of projections $\{f_n\}$ in \mathcal{A} such $p_n z_n \geq f_n \sim qz_n$. Let $f_0 = z_0 = 0$. Suppose that f_0, f_1, \dots, f_{n-1} have been constructed.

The set $\{r \in P(\mathcal{A}): qz_n \preceq r, f_{n-1} \leq r \leq p_n z_n\}$ is non-empty (since, it contains $p_n z_n$) and is contained in the finite W^* -algebra $(p_n \vee q)\mathcal{A}(p_n \vee q)$. Let τ be a center-valued trace on the algebra $(p_n \vee q)\mathcal{A}(p_n \vee q)$. From Zorn’s lemma and the fact that τ is normal, we have that this set has the minimal element r_0 . Suppose $\tau(r_0) > \tau(qz_n)$. Using Zorn’s lemma and the fact that τ is normal again, we obtain that the set $\{r \in P(\mathcal{A}): \tau(r) < \tau(qz_n), 0 \leq r \leq r_0\}$ has a maximal element r_1 . Then $0 \leq \tau(r_1) < \tau(qz_n) < \tau(r_0)$.

We claim that $r_0 - r_1$ is an atom in the algebra \mathcal{A} . Suppose there exists $e \in P(\mathcal{A}), 0 \leq e < r_0 - r_1$. Then there exists a central projection z in the algebra \mathcal{M} such that $z(r_1 + e) \preceq z(qz_n), (\mathbf{1} - z)(r_1 + e) \succ (\mathbf{1} - z)(qz_n)$. If $ze > 0$ then $P(\mathcal{A}) \ni z(r_1 + e) + (\mathbf{1} - z)r_1 > r_1, \tau(z(r_1 + e) + (\mathbf{1} - z)r_1) \leq \tau(z(qz_n) + (\mathbf{1} - z)qz_n) = \tau(qz_n)$ and if $(\mathbf{1} - z)e > 0$ then $P(\mathcal{A}) \ni zr_0 + (\mathbf{1} - z)(r_1 + e) < r_0, \tau(zr_0 + (\mathbf{1} - z)(r_1 + e)) \geq \tau(qz_n)$. The first assumption contradicts the maximality of r_1 , the second assumption contradicts the minimality of r_0 . Hence, $ze = (\mathbf{1} - z)e = 0$ that is $e = 0$. Hence, $r_0 - r_1$ is an atom of algebra \mathcal{A} .

Hence, $r_0 - r_1$ is an atom of algebra \mathcal{M} .

Since, $\tau(r_1) < \tau(qz_n)$, we have $r_1 \prec qz_n$. Hence, there exists a projection $e_1 \in (p_n \vee q)\mathcal{M}(p_n \vee q)$ such that $r_1 \sim e_1 < qz_n$. Then $\tau(qz_n - e_1) = \tau(qz_n) - \tau(r_1) < \tau(r_0 - r_1)$, that is $qz_n - e_1 \prec r_0 - r_1$. Since $r_0 - r_1$ is an atom of $(p_n \vee q)\mathcal{M}(p_n \vee q)$, we infer $qz_n - e_1 = 0$, which is a contradiction to the choice of e_1 . Then $\tau(r_0) = \tau(qz_n)$, that is $r_0 \sim qz_n$. We set $f_n := r_0$.

This completes the construction of the sequence $\{f_n\}_{n \geq 1}$. Let $q_1 := \bigvee_{n=1}^{\infty} f_n$.

Since $f_n \sim qz_n, f_{n+1} \sim qz_{n+1}$ and all four projections are finite we have $f_{n+1} - f_n \sim qz_{n+1} - qz_n$. Indeed, applying the center-valued trace τ on the finite W^* -algebra $(f_{n+1} \vee f_n \vee qz_{n+1} \vee qz_n)\mathcal{M}(f_{n+1} \vee f_n \vee qz_{n+1} \vee qz_n)$ trivially yields $\tau(f_{n+1} - f_n) = \tau(f_{n+1}) - \tau(f_n) = \tau(qz_{n+1}) - \tau(qz_n) = \tau(qz_{n+1} - qz_n)$. The latter implies immediately $f_{n+1} - f_n \sim qz_{n+1} - qz_n$.

Hence, $q_1 = f_1 \vee \bigvee_{n=1}^{\infty} (f_{n+1} - f_n) \sim qz_1 \vee \bigvee_{n=1}^{\infty} (qz_{n+1} - qz_n) = q$.

Assume (ii) holds. By the assumption there exists a projection $q_1^0 \in \mathcal{M}$, such that $q_1^0 \leq p$ and $q_1^0 \sim q$. We set

$$q_1^n := \mathbf{1}(a^n q_1^0), \quad \forall n > 0, \quad q_1 := \bigvee_{k=0}^{\infty} q_1^k.$$

We claim that $q_1 \sim q$. Indeed, since $q_1 \geq q_1^0 \sim q$, we have $q_1 \succcurlyeq q$. On the other hand, we have $q_1^n \sim \mathbf{r}(a^n q_1^0) \leq q_1^0 \sim q$, which implies $q_1^n \preceq q$ for all $n \geq 0$. Now, we shall show that in fact $q_1 \preceq q$. Note that although q is a properly infinite projection we cannot simply refer to Lemma 21(i) since the sequence $\{q_1^k\}_{k \geq 0}$ does not necessarily consist of pairwise orthogonal elements. However, representing the projection q_1 as

$$q_1 = \bigvee_{k=0}^{\infty} q_1^k = \sum_{m=1}^{\infty} \left(\bigvee_{k=0}^m q_1^k - \bigvee_{k=0}^{m-1} q_1^k \right) + q_1^0 = \sum_{m=1}^{\infty} \left(q_1^m \vee \bigvee_{k=0}^{m-1} q_1^k - \bigvee_{k=0}^{m-1} q_1^k \right) + q_1^0,$$

and noting that $q_1^0 \sim q$ and

$$q_1^m \vee \bigvee_{k=0}^{m-1} q_1^k - \bigvee_{k=0}^{m-1} q_1^k \sim q_1^m - \left(q_1^m \wedge \bigvee_{k=0}^{m-1} q_1^k \right) \leq q_1^m \preceq q$$

we infer via Lemma 21(i) that $q_1 \preceq q$. This completes the proof of the claim.

Since $ap = pa$ and $q_1^0 \leq p$, we have $pa^n q_1^0 = a^n p q_1^0 = a^n q_1^0$, and so $q_1^n \leq p$ for all $n > 0$. Hence, $q_1 \leq p$. It remains to show that $aq_1 = q_1a$. The subspace $q_1(H)$ coincides with the closure of linear span of the set $Q := \{a^n q_1^0(H) : n > 0\}$. Indeed, setting $Q_k := a^k q_1^0(H)$, we see that $q_1^k(H) = \overline{Q_k}$ and so $\overline{Q} \supset q_1^k(H)$ for every $k > 0$. Therefore, $q_1(H) = \overline{\bigcup_{k=1}^\infty q_1^k(H)} \subset \overline{Q}$. Conversely, $q_1^k(H) \subset q_1(H)$, for every $k > 0$. Therefore, the closed linear span containing the set $\overline{\bigcup_{k=1}^\infty q_1^k(H)}$ contains \overline{Q} and is contained in $q_1(H)$.

By the assumption the operator ap is bounded, and since $q_1 \leq p$, the operator aq_1 is also bounded. Thus, for every vector $\xi \in Q$, the vector $a\xi = aq_1\xi$ again belongs to Q . Again appealing to the fact that aq_1 is bounded, we infer $q_1aq_1 = aq_1$. From this we conclude that $aq_1 = q_1a$. \square

Lemma 36. *Let $0 < b \in Z(\mathcal{M})$, $\mathbf{s}(b) = \mathbf{1}$; $e_z^a(\mathbf{0}, \infty)$ be a properly infinite projection and $\mathbf{c}(e_z^a(\mathbf{0}, \infty)) = \mathbf{1}$. Let projection $q \in P(\mathcal{M})$ be finite or properly infinite, $\mathbf{c}(q) = \mathbf{1}$ and $q \ll e_z^a(\mathbf{0}, \infty)$. Let $\mathbb{R} \ni \mu_n \downarrow 0$. For every $n \in \mathbb{N}$ we denote by z_n such a projection that $\mathbf{1} - z_n$ is the largest central projection, for which $(\mathbf{1} - z_n)q \succcurlyeq (\mathbf{1} - z_n)e_z^a(\mu_n b, +\infty)$ holds. We have $z_n \uparrow_n \mathbf{1}$ and for*

$$d := \left[\mu_1 z_1 + \sum_{n=1}^\infty \mu_{n+1} (z_{n+1} - z_n) \right] b$$

the following relations hold: $q \ll e_z^a(d, +\infty)$, $0 < d \leq \mu_1 b$ and $\mathbf{s}(d) = \mathbf{1}$. Moreover, if all projections $e_z^a(\mu_n b, +\infty)$, $n \geq 1$ are finite then $e_z^a(d, +\infty)$ is a finite projection as well.

Proof. Since, $e_z^a(\mu_{n+1} b, +\infty) \geq e_z^a(\mu_n b, +\infty)$ (by Lemma 22(i)) we have $(\mathbf{1} - z_{n+1})q \succcurlyeq (\mathbf{1} - z_{n+1})e_z^a(\mu_{n+1} b, +\infty) \geq (\mathbf{1} - z_{n+1})e_z^a(\mu_n b, +\infty)$. Hence, $z_{n+1} \geq z_n$ for every $n \in \mathbb{N}$. In addition, $e_z^a(\mu_n b, +\infty) \uparrow_n e_z^a(\mathbf{0}, +\infty)$ (by Lemma 22(vii)) and $e_z^a(\mathbf{0}, +\infty)$ is properly infinite projection. Hence, in the case when q is finite projection, it follows from Lemma 21(v) that $z_n \uparrow_n \mathbf{1}$. Let us consider the case when q is a properly infinite projection with $\mathbf{c}(q) = \mathbf{1}$ and such that $q \ll e_z^a(\mathbf{0}, \infty)$. In this case, we apply Lemma 21(vi) with $p = q$, $q = e_z^a(\mathbf{0}, +\infty)$, $q_n = e_z^a(\mu_n b, +\infty)$ and deduce $\bigvee_{n=1}^\infty z_n \geq \mathbf{c}(q) = \mathbf{1}$.

All other statements follow from the form of element d . Since, $z_1 d = \mu_1 z_1 b$, $(z_{n+1} - z_n)d = \mu_{n+1}(z_{n+1} - z_n)b$ and $z_n q \ll z_n e_z^a(\mu_n b, +\infty)$ for every $n \in \mathbb{N}$. Observe also that $\mathbf{s}(d) = \mathbf{s}(b)(z_1 + \sum_{n=1}^\infty (z_{n+1} - z_n)) = \mathbf{1}$.

Finally, let all projections $e_z^a(\mu_n b, +\infty)$, $n \geq 1$ be finite. Since $d z_1 = \mu_1 b$, $d(z_{n+1} - z_n) = \mu_{n+1} b(z_{n+1} - z_n)$, we have

$$\begin{aligned} e_z^a(d, +\infty)z_1 &= e_z^a(\mu_1 b, +\infty)z_1, \\ e_z^a(d, +\infty)(z_{n+1} - z_n) &= e_z^a(\mu_{n+1} b, +\infty)(z_{n+1} - z_n) \end{aligned}$$

for every $n \in \mathbb{N}$. There projections standing on the right-hand sides are finite. Hence, $e_z^a(d, +\infty)$ is finite projection as a sum of the left-hand sides [16, Lemma 6.3.6]. \square

In the proof of the following lemma, we shall use a following well-known implication

$$p \ll q \quad \Rightarrow \quad zp \ll zq, \quad \forall z \in P(Z(\mathcal{M})), \quad 0 < z \leq \mathbf{c}(p) \vee \mathbf{c}(q).$$

We supply here a straightforward argument for convenience of the reader. Let $z' \in z \in Z(\mathcal{M})$ be such that $0 < z' \leq \mathbf{c}(pz) \vee \mathbf{c}(qz) = z(\mathbf{c}(p) \vee \mathbf{c}(q))$. Then $z' \leq \mathbf{c}(p) \vee \mathbf{c}(q)$ and therefore $z'(zp) = z'p < z'q = z'(zq)$. This means $zp \ll zq$.

Lemma 37. *Let \mathcal{M} be properly infinite and*

$$e_z^a(\mathbf{0}, t\mathbf{1}] \gg e_z^a(-\infty, \mathbf{0}] + e_z^a(t\mathbf{1}, +\infty)$$

for every $t > 0$. Then for every $\varepsilon > 0$ there exists an element $u_\varepsilon \in U(\mathcal{M})$ such that $\|[a, u_\varepsilon]\| \geq (1 - \varepsilon)|a|, u_\varepsilon^2 = \mathbf{1}$.

Proof. Of course, we may assume $\varepsilon < 1$.

Let us observe that $\mathbf{c}(e_z^a(\mathbf{0}, +\infty)) = \mathbf{1}$. Indeed, $e_z^a(\mathbf{0}, +\infty) \geq e_z^a(\mathbf{0}, t\mathbf{1}] \gg e_z^a(-\infty, \mathbf{0}]$. Observe that the preceding estimate immediately implies that

$$\mathbf{c}(e_z^a(\mathbf{0}, +\infty)) \geq \mathbf{c}(e_z^a(-\infty, \mathbf{0}]).$$

By the definition of elements $e_z^a(-\infty, c]$, we have $e_z^a(\mathbf{0}, +\infty) \vee e_z^a(-\infty, \mathbf{0}] = \mathbf{1}$. Consequently,

$$\mathbf{c}(e_z^a(\mathbf{0}, +\infty)) = \mathbf{c}(e_z^a(\mathbf{0}, +\infty)) \vee \mathbf{c}(e_z^a(-\infty, \mathbf{0}]) \geq e_z^a(\mathbf{0}, +\infty) \vee e_z^a(-\infty, \mathbf{0}] = \mathbf{1}.$$

We want to show that if $d \in Z_+(LS(\mathcal{M}))$ and $\mathbf{s}(d) = \mathbf{1}$ then $e_z^a(\mathbf{0}, d] \gg e_z^a(-\infty, \mathbf{0}] + e_z^a(d, +\infty)$. Indeed, the semi-axis $\mathbb{R}_+ = (0, +\infty)$ can be split into countable family of intervals $I_n = [\lambda_n, \mu_n), n \geq 1$. Then $e^d(I_n) \in P(Z(\mathcal{M}))$, and since the spectral measure e^d is σ -additive, we write

$$\bigvee_{n=1}^\infty e^d(I_n) = e^d(0, +\infty) = \mathbf{s}(d) = \mathbf{1}.$$

Next, assuming that $e^d(I_n) \neq 0$, we have

$$\begin{aligned} e_z^a(\mathbf{0}, d]e^d(I_n) &= e_z^a(\mathbf{0}, de^d(I_n)]e^d(I_n) \quad (\text{by Lemma 22(ii)}) \\ &\geq e_z^a(\mathbf{0}, \lambda_n e^d(I_n)]e^d(I_n) \quad (\text{by Lemma 22(i)}) \\ &= e_z^a(\mathbf{0}, \lambda_n \mathbf{1}]e^d(I_n) \quad (\text{by Lemma 22(ii)}) \\ &\gg (e_z^a(-\infty, \mathbf{0}] + e_z^a(\lambda_n \mathbf{1}, +\infty))e^d(I_n) \quad (\text{by the assumption}) \\ &= (e_z^a(-\infty, \mathbf{0}] + e_z^a(\lambda_n e^d(I_n), +\infty))e^d(I_n) \quad (\text{by Lemma 22(ii)}) \\ &\geq (e_z^a(-\infty, \mathbf{0}] + e_z^a(d, +\infty))e^d(I_n) \quad (\text{by Lemma 22(i)}). \end{aligned}$$

This inequality implies, in particular, that for such choice of d the projection $e_z^a(\mathbf{0}, d]$ is properly infinite.

Next our goal is to construct a decreasing to zero sequence

$$\{d_n\}_{n=0}^\infty \subset Z_+(LS(\mathcal{M})), \quad d_0 \leq \mathbf{1}, \quad d_{n+1} \leq d_n/2, \quad \mathbf{s}(d_n) = \mathbf{1}$$

for all $n \in \mathbb{N}$ and two sequences $\{p_n\}_{n=0}^\infty, \{q_n\}_{n=0}^\infty$ of pairwise disjoint projections in \mathcal{M} , which satisfy the following conditions:

- (i) $p_n q_m = 0, ap_n = p_n a, aq_n = q_n a, p_n \sim q_n$ for every $n, m \geq 0$.
- (ii) $p_n \leq e_z^a(d_n, +\infty), q_n \leq e_z^a(-\infty, \varepsilon d_n]$ for every $n \geq 0, q_0 \geq e_z^a(-\infty, \mathbf{0}]$.
- (iii) $\bigvee_{n=0}^\infty p_n \vee \bigvee_{n=0}^\infty q_n = \mathbf{1}$.

For any projection $p \in P(\mathcal{M})$ there exists a unique central projection z such that pz is a finite projection and $p(\mathbf{1} - z)$ is properly infinite projection and $\mathbf{c}(p) \geq \mathbf{1} - z$ (if p is finite projection then $z = \mathbf{1}$, otherwise the assertion follows from [16, Proposition 6.3.7]). In this case, we have $\mathbf{c}(p(\mathbf{1} - z)) = \mathbf{c}(p)(\mathbf{1} - z) = \mathbf{1} - z$ [15, Proposition 5.5.3]. Let $z_0 \in P(Z(\mathcal{M}))$ be such a projection for $e_z^a(-\infty, \mathbf{0}]$ and let z_n be such a projection for $e_z^a(\mathbf{1}/n, +\infty), n \in \mathbb{N}$. Since the sequence $e_z^a(\mathbf{1}/n, +\infty)$ is non-decreasing, it follows that the sequence $\{z_n\}_{n=1}^\infty$ is non-increasing. In addition we have $\mathbf{1} = (\mathbf{1} - z_0) + \bigwedge_{n=1}^\infty z_0 z_n + z_0(\mathbf{1} - z_1) + \bigvee_{n=1}^\infty z_0(z_n - z_{n+1})$. Hence, it is sufficient to prove the assertion for reduced algebras $\mathcal{M}[\bigwedge_{n=1}^\infty z_0 z_n], \mathcal{M}[z_0(\mathbf{1} - z_1)], \mathcal{M}[z_0(z_n - z_{n+1})]$ and $\mathcal{M}(\mathbf{1} - z_0)$. It is sufficient to consider three following cases:

- (a) The projection $e_z^a(-\infty, \mathbf{0}]$ is finite and all projections $e_z^a(\lambda \mathbf{1}, +\infty)$ are the same for every $\lambda > 0$. The algebra $\mathcal{M}[\bigwedge_{n=1}^\infty z_0 z_n]$ satisfies this condition. Note that in this case the projection $e_z^a(\mathbf{0}, +\infty)$ is a supremum of non-decreasing sequence of finite projections $e_z^a(\mathbf{1}/n, +\infty)$.
- (b) The projection $e_z^a(-\infty, \mathbf{0}]$ is finite and there exists $\lambda > 0$ such that the projection $e_z^a(\lambda \mathbf{1}, +\infty)$ is properly infinite and $\mathbf{c}(e_z^a(\lambda \mathbf{1}, +\infty)) = \mathbf{1}$. Algebras $\mathcal{M}[z_0(z_n - z_{n+1})]$ (in this case $\lambda = 1/(n + 1)$) and $\mathcal{M}[z_0(\mathbf{1} - z_1)]$ (in this case $\lambda = 1$) satisfy this condition.
- (c) The projection $e_z^a(-\infty, \mathbf{0}]$ is properly infinite and $\mathbf{c}(e_z^a(-\infty, \mathbf{0})) = \mathbf{1}$. The algebra $\mathcal{M}(\mathbf{1} - z_0)$ satisfies this condition.

We would like to show that there exists an element $d_0 \in Z_+(\mathcal{M}), d_0 \leq \mathbf{1}$ such that $e_z^a(-\infty, \mathbf{0}] \ll e_z^a(d_0, +\infty) \mathbf{s}(d_0) = \mathbf{1}$.

Consider the case (a). We shall use Lemma 36. To this end, we set $b = \mathbf{1}, \mu_n = 1/n, q = e_z^a(-\infty, \mathbf{0}]$. In this case the projection $q = e_z^a(-\infty, \mathbf{0}]$ is finite and projection $e_z^a(\mathbf{0}, +\infty)$ is properly infinite and $\mathbf{c}(e_z^a(\mathbf{0}, +\infty)) = \mathbf{1}$. Hence, the assumptions of Lemma 36 hold. Thus, there exists an element $0 < d_0 \in Z(\mathcal{M})$ such that $d_0 \leq \mathbf{1}, \mathbf{s}(d_0) = \mathbf{1}$ and $e_z^a(-\infty, \mathbf{0}] \ll e_z^a(d_0, +\infty)$.

In the case (b) we set $d_0 = \min(1, \lambda)\mathbf{1}$. Then $e_z^a(-\infty, \mathbf{0}]$ is a finite projection and $e_z^a(d_0, +\infty)$ is a properly infinite projection. Hence, $e_z^a(-\infty, \mathbf{0}] \ll e_z^a(d_0, +\infty)$ and $\mathbf{s}(d_0) = \mathbf{1}$.

In the case (c) we will use Lemma 36 again. Set $b = \mathbf{1}, \mu_n = 1/n, q = e_z^a(-\infty, \mathbf{0}]$. We have that $q = e_z^a(-\infty, \mathbf{0}]$ is a properly infinite projection, $\mathbf{c}(q) = \mathbf{1}$ and $q \ll e_z^a(\mathbf{0}, +\infty)$. Hence, the assumptions of Lemma 36 hold. So, there exists an element $0 < d_0 \in Z(\mathcal{M})$ such that $d_0 \leq \mathbf{1}, \mathbf{s}(d_0) = \mathbf{1}$ and $e_z^a(-\infty, \mathbf{0}] \ll e_z^a(d_0, +\infty)$.

This completes the construction of the element d_0 . Let us now show that there exists a sequence $d_n \in Z_+(\mathcal{M})$ such that $d_n \leq d_{n-1}/2, \mathbf{s}(d_n) = \mathbf{1}$ and $e_z^a(d_n, +\infty) \gg e_z^a(d_{n-1}, +\infty)$ for every $n \in \mathbb{N}$.

Suppose that elements d_1, \dots, d_n have been already constructed.

We are going to use Lemma 36 again. For this we set $b = d_n, \mu_m = 1/(2m), q = e_z^a(d_n, +\infty)$. In the case (a) $q = e_z^a(d_n, +\infty)$ is finite projection. In the cases (b) and (c) $e_z^a(d_n, +\infty)$ is a properly infinite projection, $\mathbf{c}(q) = \mathbf{1}$ and $q \ll e_z^a(\mathbf{0}, +\infty)$ (we have shown this fact at the beginning

of the proof). The assumptions of Lemma 36 hold. Hence, there exists $0 < d_{n+1} \in Z(\mathcal{M})$ such that $d_{n+1} \leq d_n/2$, $\mathbf{s}(d_{n+1}) = \mathbf{1}$ and $e_z^a(d_{n+1}, +\infty) \succ e_z^a(d_n, +\infty)$.

Thus, we have constructed the sequence $\{d_n\} \subset Z_+(LS(\mathcal{M}))$ such that $d_{n+1} \leq d_n/2$ and $e_z^a(d_{n+1}, +\infty) \succ e_z^a(d_n, +\infty)$, $\mathbf{s}(d_n) = \mathbf{1}$ for every $n \in \mathbb{N}$. In addition we have $e_z^a(d_0, +\infty) \succ e_z^a(-\infty, \mathbf{0}]$.

Set $p_0 = e_z^a(d_0, +\infty)$. There exists a projection $r \in P(\mathcal{M})$ such that $e_z^a(-\infty, \mathbf{0}] \sim r < p_0$. By the assumption and using the argument at the beginning of the proof, we have $e_z^a(\mathbf{0}, \varepsilon d_0] \succ p_0$, and so, by Lemma 35, it follows that there exists a projection $q_0^1 \in \mathcal{M}$ such that $p_0 - r \sim q_0^1 < e_z^a(\mathbf{0}, \varepsilon d_0]$ and $aq_0^1 = q_0^1 a$ (in the case (a) the condition (i) of Lemma 35 is applied and in the cases (b) and (c) the condition (ii) is used). Set $q_0 := e_z^a(-\infty, \mathbf{0}] + q_0^1$. Then $q_0 \sim p_0$.

Suppose that projections $p_0, \dots, p_n; q_0, \dots, q_n$ have been constructed. Set $p_{n+1} = e_z^a(d_{n+1}, +\infty) \prod_{k=0}^n (\mathbf{1} - p_k) \prod_{k=0}^n (\mathbf{1} - q_k)$. In the case (a) all projections p_k, q_k with $k \leq n$ are finite and $e_z^a(\mathbf{0}, \varepsilon d_{n+1}]$ is a properly infinite projection which is a supremum of non-decreasing sequence of finite projections $\{e_z^a(\mathbf{1}/m, \varepsilon d_{n+1})\}_{m=1}^\infty$. Hence, $e_z^a(\mathbf{0}, \varepsilon d_n] \prod_{k=0}^n (\mathbf{1} - p_k) \prod_{k=0}^n (\mathbf{1} - q_k)$ is a properly infinite projection. It follows from Lemma 35(i) that there exists a projection $q_{n+1} \in \mathcal{M}$ such that $p_{n+1} \sim q_{n+1} < e_z^a(\mathbf{0}, \varepsilon d_{n+1}) \prod_{k=0}^n (\mathbf{1} - p_k) \prod_{k=0}^n (\mathbf{1} - q_k)$ and $aq_{n+1} = q_{n+1} a$.

Let now consider the cases (b) and (c). Recall that in these cases all $e_z^a(d_n, +\infty)$ are properly infinite projections. Since $\sum_{k=0}^n p_k \leq e_z^a(d_n, +\infty) \prec e_z^a(d_{n+1}, +\infty)$, by Lemma 21(ii) we obtain that $\sum_{k=0}^n p_k + \sum_{k=0}^n q_k \prec e_z^a(d_{n+1}, +\infty)$. Then $p_{n+1} = e_z^a(d_{n+1}, +\infty) \prod_{k=0}^n (\mathbf{1} - p_k) \prod_{k=0}^n (\mathbf{1} - q_k) = e_z^a(d_{n+1}, +\infty) (\mathbf{1} - \sum_{k=0}^n p_k - \sum_{k=0}^n q_k)$ is a properly infinite projection. It follows from Lemma 21(iv) that $p_{n+1} \sim e_z^a(d_{n+1}, +\infty) \prec e_z^a(\mathbf{0}, \varepsilon d_{n+1}] \sim e_z^a(\mathbf{0}, \varepsilon d_{n+1}) \prod_{k=0}^n (\mathbf{1} - p_k) \prod_{k=0}^n (\mathbf{1} - q_k)$ (we applied Lemma 21(iv) at the beginning and at the end of the chain). So, it follows from Lemma 35(ii) that there exists a projection $q_{n+1} \in P(\mathcal{M})$ such that $q_{n+1} < e_z^a(\mathbf{0}, \varepsilon d_{n+1}) \prod_{k=0}^n (\mathbf{1} - p_k) \prod_{k=0}^n (\mathbf{1} - q_k)$, $q_{n+1} \sim p_{n+1}$ and $aq_{n+1} = q_{n+1} a$.

Thus, projections p_{n+1} and q_{n+1} are constructed.

It is clear that these projections satisfy conditions (i) and (ii). To check the condition (iii) we note that $\bigvee_{k=0}^n p_k \vee \bigvee_{k=0}^n q_k \geq e_z^a(-\infty, \mathbf{0}] + e_z^a(d_n, +\infty) \uparrow \mathbf{1}$ with $n \rightarrow \infty$.

Now, we can proceed with the construction of the unitary operator $u_\varepsilon \in \mathcal{M}$ from the assertion.

Let $v_n \in \mathcal{M}$ be a partial isometry such that $v_n^* v_n = p_n$, $v_n v_n^* = q_n$, $n = 0, 1, \dots$. We set

$$u_\varepsilon = \sum_{n=0}^\infty v_n + \sum_{n=0}^\infty v_n^*$$

(here, the sums are taken in the strong operator topology).

Then, we have

$$u_\varepsilon^* u_\varepsilon = \sum_{n=0}^\infty p_n + \sum_{n=0}^\infty q_n = \mathbf{1}, \quad u_\varepsilon u_\varepsilon^* = \sum_{n=0}^\infty q_n + \sum_{n=0}^\infty p_n = \mathbf{1}.$$

Observe that

$$u_\varepsilon p_n = q_n u_\varepsilon, \quad u_\varepsilon q_n = p_n u_\varepsilon, \quad a p_n = p_n a, \quad q_n a = a q_n, \quad n \geq 0,$$

and so the element $u_\varepsilon^* a u_\varepsilon$ commutes with all the projections p_n and q_n , $n \geq 0$. Moreover, since for all $n \geq 0$, it holds

$$\begin{aligned}ap_n &= ae_z^a(d_n, +\infty)p_n \geq d_n e_z^a(d_n, +\infty)p_n = d_n p_n, \\aq_n &= ae_z^a(-\infty, \varepsilon d_n)q_n \leq \varepsilon d_n e_z^a(-\infty, \varepsilon d_n)q_n = \varepsilon d_n q_n,\end{aligned}$$

we obtain immediately for all such n 's that

$$\begin{aligned}u_\varepsilon^* a u_\varepsilon p_n &= u_\varepsilon^* a q_n u_\varepsilon \leq \varepsilon d_n u_\varepsilon^* q_n u_\varepsilon = \varepsilon d_n p_n, \\u_\varepsilon^* a u_\varepsilon q_n &= u_\varepsilon^* a p_n u_\varepsilon \geq d_n u_\varepsilon^* p_n u_\varepsilon = d_n q_n.\end{aligned}$$

In particular, $(u_\varepsilon^* a u_\varepsilon - a)p_n \leq \varepsilon d_n p_n - d_n p_n = -d_n(1 - \varepsilon)p_n \leq 0$. Taking into account that $ap_n \geq d_n p_n$, we now obtain

$$\begin{aligned}|u_\varepsilon^* a u_\varepsilon - a|p_n &= (a - u_\varepsilon^* a u_\varepsilon)p_n \geq ap_n - \varepsilon d_n p_n \\&\geq ap_n - \varepsilon ap_n = (1 - \varepsilon)ap_n \\&= (1 - \varepsilon)|a|p_n.\end{aligned}$$

Analogously, for every $n \geq 0$, we have $(u_\varepsilon^* a u_\varepsilon - a)q_n \geq d_n q_n - \varepsilon d_n q_n = (1 - \varepsilon)d_n q_n \geq 0$. Therefore,

$$\begin{aligned}|u_\varepsilon^* a u_\varepsilon - a|q_n &= (u_\varepsilon^* a u_\varepsilon - a)q_n \geq (1 - \varepsilon)d_n q_n \\&\geq (1 - \varepsilon)aq_n.\end{aligned}$$

Observe that the inequalities above hold for all $n \geq 0$. If $n > 0$, then $q_n < e_z^a(0, \varepsilon d_n]$, $q_n a = a q_n$ by the construction and so $aq_n = |a|q_n$, that is, we have

$$|u_\varepsilon^* a u_\varepsilon - a|q_n \geq (1 - \varepsilon)|a|q_n.$$

A little bit more care is required when $n = 0$. In this case, recall that $q_0 = e_z^a(-\infty, \mathbf{0}] + q_0^1$, where $q_0^1 < e_z^a(\mathbf{0}, \varepsilon d_0]$. Obviously, $ae_z^a(-\infty, \mathbf{0}) \leq 0$, and so $ae_z^a(-\infty, \mathbf{0}) = -|a|e_z^a(-\infty, \mathbf{0})$. Therefore since (see above) $u_\varepsilon^* a u_\varepsilon q_0 \geq d_0 q_0$ and $aq_0 = ae_z^a(-\infty, \mathbf{0}) + aq_0^1 = -|a|e_z^a(-\infty, \mathbf{0}) + aq_0^1$, we have

$$\begin{aligned}|u_\varepsilon^* a u_\varepsilon - a|q_0 &\geq (u_\varepsilon^* a u_\varepsilon - a)q_0 \geq d_0 q_0 - aq_0^1 + |a|e_z^a(-\infty, \mathbf{0}) \\&\geq d_0 q_0^1 - \varepsilon d_0 q_0^1 + |a|e_z^a(-\infty, \mathbf{0}) = (1 - \varepsilon)d_0 q_0^1 + |a|e_z^a(-\infty, \mathbf{0}) \\&\geq (1 - \varepsilon)aq_0^1 + |a|e_z^a(-\infty, \mathbf{0}) = (1 - \varepsilon)|a|q_0^1 + |a|e_z^a(-\infty, \mathbf{0}) \\&\geq (1 - \varepsilon)(|a|q_0^1 + |a|e_z^a(-\infty, \mathbf{0})) = (1 - \varepsilon)|a|q_0.\end{aligned}$$

Collecting all preceding inequalities, we see that for every $k \geq 0$ we have

$$|u_\varepsilon^* a u_\varepsilon - a| \sum_{n=0}^k (p_n + q_n) \geq (1 - \varepsilon)|a| \sum_{n=0}^k (p_n + q_n)$$

and since $\sum_{n=0}^{\infty} (p_n + q_n) = \mathbf{1}$, we conclude

$$|u_\varepsilon^* a u_\varepsilon - a| \geq (1 - \varepsilon)|a|.$$

The assertion of the lemma now follows by observing that $|u_\varepsilon^* a u_\varepsilon - a| = |[a, u_\varepsilon]|$. \square

Proof of Theorem 1. Prior to Lemma 33 we have shown that the identity $\mathbf{1} \in \mathcal{M}$ can be written as a sum of three central projections $\mathbf{1} = p_0 + p_- + p_+$. The assertion (i) of Theorem 1 holds for the element $a p_0$ (by Lemma 30) affiliated with the algebra $\mathcal{M} p_0$. The assumptions of Lemma 37 hold for the element $(a - c_0) p_-$ in the algebra $\mathcal{M} p_-$. Hence, the assertion (ii) of Theorem 1 holds in this algebra. The assumptions of Lemma 37 hold for the element $(c_0 - a) p_+$ in the algebra $\mathcal{M} p_+$ as well (see the discussion preceding Lemma 35). Hence, the assertion (ii) of Theorem 1 holds in this algebra as well.

Next, if \mathcal{M} is finite or purely infinite σ -finite algebra, then by Lemmas 31 and 32 we have $\mathbf{1} \in P_0$, which implies $\mathbf{1} \leq p_0$. In other words, we have $p_0 = \mathbf{1}$ and by Lemma 30 the assertion (i) of theorem holds in this case. \square

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