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INVARIANT THEORY AND $H^*(\mathrm{GL}_n(\mathbb{F}_p); \mathbb{F}_p)$

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One of the main problems remaining from Quillen's study of group cohomology is the determination of $H^*(\mathrm{GL}_n(\mathbb{F}_p); \mathbb{F}_p)$. Very little is known about these groups except that they stabilize to zero, that is, $H^*(\mathrm{GL}_\infty)\mathbb{F}_p); \mathbb{F}_p)=0$ in positive dimensions [11]. This fact, along with the rich variety of elementary abelian p-subgroups (p-tori) of $\mathrm{GL}_n(\mathbb{F}_p)$, makes computation very difficult. On the other hand, for $n<\infty$ these cohomology groups are important because they give universal characteristic classes for modular group representations.

Our main result, Theorem 6.5 gives families of explicit classes in $H^*(\mathrm{GL}_n(\mathbb{F}_p); \mathbb{F}_p)$ detected on certain maximal p-tori of 'block' form. Our methods involve a study of the invariant theory of these blocks, which generalizes the classical theory of Dickson [2], together with favorable properties of the transfer for the general linear groups.

Before going into more detail, we recall that in the non-modular case the situation is much clearer: a complete computation of $H^*(\mathrm{GL}_n(\mathbb{F}_q); \mathbb{F}_p)$ for p not dividing q is given in [11].

Let $M_{n,m}$ be the additive group of $n\times m$ matrices over $\mathbb{F}_p$. We consider $M_{n,m}$ as the subgroup

$$\begin{pmatrix} I_n & M_{n,m} \\ 0 & I_m \end{pmatrix}$$

of $\mathrm{GL}_{n+m}(\mathbb{F}_p)$ and propose to study the restriction map in mod p cohomology

$$H^*(\mathrm{GL}_{n+m}(\mathbb{F}_p)) \to H^*(M_{n,m})^{\mathrm{GL}_n\times \mathrm{GL}_m}$$

whose image lies in the $\mathrm{GL}_n(\mathbb{F}_p)\times\mathrm{GL}_m(\mathbb{F}_p)$ invariants since

$$\begin{pmatrix} \mathrm{GL}_n & M_{n,m} \\ 0 & \mathrm{GL}_m \end{pmatrix}$$

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is the normalizer of $M_{n,m}$ in $\mathrm{GL}_{n,m}(\mathbb{F}_p)$. If $S(M^*_{n,m})$ denotes the symmetric algebra on the dual, then

$$H^*(M_{n,m})^{\mathrm{GL}_n\times\mathrm{GL}_m}\supset S(M^*_{n,m})^{\mathrm{GL}_n\times\mathrm{GL}_m}$$

where, as usual, $M^*_{n,m}$ is concentrated in dimension 2 if $p>2$ or in dimension 1 if $p=2$. (For $p>2$ the inclusion is proper but there is equality when $p=2$.) Thus we are led to study the action of $\mathrm{GL}_n\times\mathrm{GL}_m$ on $S(M^*_{n,m})$ and the resulting invariants. The invariant sets of this action give rise to representations whose Chern classes provide elements in the cohomology of the unipotent group $U_{n+m}(\mathbb{F}_p)$ of upper unitriangular matrices. The transfers of these elements to $H^*(\mathrm{GL}_{n+m}(\mathbb{F}_p))$ give the desired classes.

The paper is organized as follows: In Section 1 we give the preliminaries on the action of $\mathrm{GL}_n\times\mathrm{GL}_m$. Invariant polynomials and generalized Dickson invariants are studied in Section 2. In particular an invariant set $\mathcal{J}$ is introduced whose corresponding invariants are easier to compute than the general case. Invariants on the level of fraction fields are determined in Section 3 by a simple application of Galois Theory. By composing Dickson invariants in an appropriate manner, a maximal set of nm algebraically independent invariants is determined in Section 4. P-tori of block form are studied in Section 5. These results are applied to the transfer in Section 6 to obtain Theorem 6.5.

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Throughout this paper all cohomology groups are taken with simple coefficients in $\mathbb{F}_p$.

1. Preliminaries

Suppose V, W are finite-dimensional vector spaces over $\mathbb{F}_p$. Then an action of $\mathrm{GL}(V)\times\mathrm{GL}(W)$ on the symmetric algebra $S(V\otimes W^*)$ is defined by

$$(g\times h)(v\otimes\omega)=gv\otimes h\omega \quad \text{for } g\times h\in\mathrm{GL}(V)\times\mathrm{GL}(W),\ v\otimes\omega\in V\otimes W^* \tag{1.1}$$

where $\mathrm{GL}(W)$ acts contragrediently on the dual space W^*, i.e., $h\omega(w)=\omega(h^{-1}w)$ for $w\in W$. More explicitly there is an isomorphism

$$\phi : V\otimes W^* \xrightarrow{\approx} \mathrm{Hom}(W, V)$$

given by $\phi(v\otimes\omega)(w)=\omega(w)\cdot v$ for $v\in V$, $w\in W$, $\omega\in W^*$. Thus we may identify $V\otimes W^*$ with the set $M_{n,m}$ of $n\times m$ matrices over $\mathbb{F}_p$ where n, m are the dimensions of V, W respectively. Then the corresponding action of $\mathrm{GL}_n(\mathbb{F}_p)\times\mathrm{GL}_m(\mathbb{F}_p)$ is given by

$$(g\times h)M=gMh^{-1} \quad \text{for } g\times h\in\mathrm{GL}_n\times\mathrm{GL}_m,\ M\in M_{n,m}. \tag{1.2}$$

This action extends to $S(M_{n,m})$ via the diagonal action as usual.

In Section 6 we shall consider cohomology groups. As indicated in the introduction, this entails studying the $\mathrm{GL}_n \times \mathrm{GL}_m$ invariants of $S(M^*_{n,m})$. However, if

$$\tau : \mathrm{GL}_n \times \mathrm{GL}_m \xrightarrow{\approx} \mathrm{GL}_m \times \mathrm{GL}_n$$

is the twist isomorphism, then there is a τ-equivariant isomorphism given by

$$M_{m,n} \xrightarrow[\approx]{\phi^{-1}} W \otimes V^* \xrightarrow[\approx]{\psi} (V \otimes W^*)^* \xrightarrow[\approx]{\phi^*} M^*_{n,m}$$

where

$$\psi(w \otimes \nu)(v \otimes \omega) = \omega(w)\nu(v) \quad \text{for } w \in W,\ \omega \in W^*,\ v \in V,\ \nu \in V^*.$$

Thus we are reduced to studying $\mathrm{GL}_m \times \mathrm{GL}_n$ invariants in $S(M_{n,m})$, eliminating the need for a special discussion of invariants in $S(M^*_{n,m})$.

If $\langle x_1, \dots, x_n \rangle$, $\langle y_1, \dots, y_m \rangle$ are bases for V, W^* respectively, then

$$\langle x_{ij} = x_i \otimes y_j :\ 1 \le i \le n,\ 1 \le j \le m \rangle$$

is a basis for $V \otimes W^*$. We view x_{ij} as the $n \times m$ matrix with 1 in the (i,j)th position and zeros elsewhere.

2. Invariant polynomials

If $\mathscr{I} \subset V \otimes W^*$ is a $\mathrm{GL}(V) \times \mathrm{GL}(W)$ invariant set, then we can form the *invariant polynomial*

$$P_{\mathscr{I}}(X) = \prod_{u \in \mathscr{I}} (X - u) \in S(V \otimes W^*)[X]. \tag{2.1}$$

Clearly $P_{\mathscr{I}}(X) \in R[X]$, where $R = S(V \otimes W^*)^{\mathrm{GL}(V) \times \mathrm{GL}(W)}$ and so the coefficients of $P_{\mathscr{I}}(X)$ provide a ready source of invariants.

In the case of a single vector space V of dimension n over $\mathbb{F}_p$, the algebra $S(V)^{\mathrm{GL}(V)}$ was computed by Dickson [2] in 1911. He found the result to be a polynomial algebra

$$S(V)^{\mathrm{GL}(V)} = \mathbb{F}_p[c_{n,0}, c_{n,1}, \dots, c_{n,n-1}] \tag{2.2}$$

where the $c_{n,i}$ are coefficients of the polynomial

$$P_V(X) = \sum_{v \in V} (X - v) = X^{p^n} + \sum_{i=0}^{n-1} (-1)^{n-i} c_{n,i} X^{p^i}.$$

The dimension of $c_{n,i}$ is $2(p^n - p^i)$ if $p > 2$ or $2^n - 2^i$ if $p = 2$. See [14] for a further discussion of Dickson invariants.

In the notation of (2.1) if n or $m = 1$ and $\mathscr{I} = V \otimes W^*$ we are reduced to this classical situation. For example if $m = 1$

$$S(M_{n,1})^{\mathrm{GL}_n \times \mathrm{GL}_1} \approx S(V)^{\mathrm{GL}(V)}.$$

This is clear since GL_1 consists of scalars and thus does not affect invariance.

Remark 2.3. From linear algebra we know that the orbits of $\mathrm{GL}_n \times \mathrm{GL}_m$ acting on $M_{n,m}$ are determined by rank; that is, two matrices $M, M' \in M_{n,m}$ satisfy $M' = PMQ^{-1}$ for some $P \in \mathrm{GL}_n$, $Q \in \mathrm{GL}_m$ iff $\operatorname{rank} M = \operatorname{rank} M'$.

A particularly useful invariant set consists of the union of rank zero and rank one matrices, denoted $\mathscr{J}$. Alternately, we have

Lemma 2.4. $\mathscr{J} = \{v \otimes w : v \in V,\ w \in W^*\}$.

Proof. The right hand side is clearly an invariant set. If $v \otimes w = 0$, then $v \otimes w$ corresponds to the rank zero matrix. If $v \otimes w \in \mathscr{J}$ is non-zero, then there exist elements $(g, h) \in \mathrm{GL}_n \times \mathrm{GL}_m$ such that $gv = x_1$, $hw = y_1$. Hence $v \otimes w$ corresponds to a matrix in the orbit of the rank one matrix x_{11}. The converse is equally obvious.

Now consider the invariant polynomial

$$P_{\mathscr{J}}(X) = \prod_{v,w} (X - v \otimes w) = \prod_{v} \left(\prod_{w} X - v \otimes w \right).$$

For a fixed v we can use Dickson's result (2.2) to write

$$\prod_{w} (X - v \otimes w) = X^{p^m} + \sum_{j=0}^{m-1} (-1)^{m-j} c_{m,j}(v) X^{p^j}$$

where the $c_{m,j}(v)$ are ordinary Dickson invariants based on v.

Example 2.5. For $m = 2$, $p = 2$, $S(W^*)^{\mathrm{GL}_2} = \mathbb{F}_2[y_1, y_2]^{\mathrm{GL}_2} = \mathbb{F}_2[c_{2,0}, c_{2,1}]$ where $c_{2,0} = y_1 y_2^2 + y_1^2 y_2$, $c_{2,1} = y_1^2 + y_1 y_2 + y_2^2$. Then, for example,

$$c_{2,1}(v) = x_{11}^2 + x_{11}x_{12} + x_{12}^2 \quad \text{for } v = x_1$$

$$c_{2,1}(v) = (x_{11} + x_{21})^2 + (x_{11} + x_{21})(x_{12} + x_{22}) + (x_{12} + x_{22})^2 \quad \text{for } v = x_1 + x_2.$$

Thus

$$P_{\mathscr{J}}(X) = \prod_{v} (X^{p^m} + \sum (-1)^{m-j} c_{m,j}(v) X^{p^j}).$$

In this form the coefficients of $P_{\mathscr{J}}(X)$ are much easier to compute.

Example 2.6. Let $n = m = 2$ and $p = 2$. Then

$$\begin{aligned}
P_{\mathscr{J}}(X) &= X^{16} + z_2 X^{14} + z_3 X^{13} + \cdots + z_9 X^7,\\
z_2 &= \det, \qquad z_3 = \det',\\
z_4 &= X_1^2 + X_1 X_2 + X_2^2 + (X_1 + X_2)\det,\\
z_5 &= X_1 X_2' + X_1' X_2 + (X_1 + X_2)\det' + (X_1' + X_2')\det,\\
z_6 &= X_1^2 X_2 + X_1 X_2^2 + X_1 X_2 \det + X_1'^2 + X_1' X_2' + X_2'^2 + (X_1' + X_2')\det',\\
z_7 &= X_1 X_2 \det' + (X_1' X_2 + X_1 X_2')\det' + X_1^2 X_2' + X_1' X_2^2,\\
z_8 &= X_1 X_2'^2 + X_1'^2 X_2 + (X_1' X_2 + X_1 X_2')\det' + X_1' X_2' \det,\\
z_9 &= X_1' X_2' (X_1' + X_2' + \det')
\end{aligned}$$

where

$$\begin{aligned} \det &= x_{11}x_{22}+x_{12}x_{21}, & \det' &= \mathrm{Sq}^1(\det), \\ X_1 &= c_{2,1}(x_1) = x_{11}^2+x_{11}x_{12}+x_{12}^2, & X_1' &= \mathrm{Sq}^1(X_1), \\ X_2 &= c_{2,1}(x_2) = x_{21}^2+x_{21}x_{22}+x_{22}^2, & X_2' &= \mathrm{Sq}^1(X_2). \end{aligned}$$

3. Galois theory

In this section we show that on the level of fraction fields any non-zero orbit essentially determines all $\mathrm{GL}_n\times\mathrm{GL}_m$ invariants. We begin by observing that since $M_{n,m}$ is an nm-dimensional vector space over $\mathbb{F}_p$ we have

$$\mathrm{GL}_n\times\mathrm{GL}_m\subset\mathrm{GL}_{nm} \quad\text{and}\quad S(M_{n,m})^{\mathrm{GL}_{nm}}\subset S(M_{n,m})^{\mathrm{GL}_n\times\mathrm{GL}_m}$$

where

$$S(M_{n,m})^{\mathrm{GL}_{nm}} = \mathbb{F}_p[c_{nm,0}, c_{nm,1},\ldots,c_{nm,nm-1}]$$

are the Dickson invariants in $\mathbb{F}_p[x_{11},\ldots,x_{nm}]$ as described in (2.2).

Let K,F be the field of fractions of $S(M_{n,m})$ and $S(M_{n,m})^{\mathrm{GL}_{nm}}$ respectively. Then K is a Galois extension of F with polynomial

$$P_{M_{n,m}}(X) = \prod_{M\in M_{n,m}} (X-M).$$

Let $\mathcal{O}$ be a non-zero orbit (2.3) and let $L=L_{\mathcal{O}}$ be the subfield of K generated over F by the coefficients of $P_{\mathcal{O}}(X)$.

Lemma 3.1. *The Galois group $G(K/L)$ is the set $T_{\mathcal{O}}$ of invertible linear transformations of $M_{n,m}$ which leave $\mathcal{O}$ setwise fixed.*

Proof. If $g\in\mathrm{GL}_{nm}$ and leaves $\mathcal{O}$ invariant, then g fixes $P_{\mathcal{O}}(X)$. Hence g fixes L. Conversely, if $g\in G(K/L)\subset\mathrm{GL}_{nm}$, then g fixes $P_{\mathcal{O}}(X)$ and hence permutes the roots of $P_{\mathcal{O}}(X)$.

Questions related to the determination of $T_{\mathcal{O}}$ have a long history dating back to Frobenius.

Lemma 3.2. *If $n\neq m$, then $T_{\mathcal{O}}=\mathrm{GL}_n\times\mathrm{GL}_m$. If $n=m$, then $T_{\mathcal{O}}$ has an additional generator given by matrix transposition.*

Proof. Marcus and Moyls [8] have shown that if $f: M_{n,m}\to M_{n,m}$ is linear and preserves rank one matrices, then it has the desired form assuming the ground field is algebraically closed. Thus $f(\cdot)=P(\cdot)Q$ for fixed $P\in\mathrm{GL}_n$, $Q\in\mathrm{GL}_m$; if $n=m$, $f(\cdot)=P(\cdot)^{\mathrm{t}}Q$ is also allowed. Later McDonald [9] observed that if f is also invertible, then the argument works for any field. Assuming f is invertible and the field is infinite Djoković [3] showed that if f preserves matrices of rank w for some $w>0$, then f preserves matrices of rank k for $k<w$. The lemma follows since passing to the algebraic closure does not affect the question of preserving rank.

Standard Galois theory proves

Theorem 3.3. *If $n \neq m$, then $L_{\mathscr{O}} = K^{\mathrm{GL}_n \times \mathrm{GL}_m}$. If $n=m$, then $K^{\mathrm{GL}_n \times \mathrm{GL}_n}$ is a degree two extension of $L_{\mathscr{O}}$.*

Thus, for $n=m$ we must find one invariant in addition to the coefficients of $P_{\mathscr{O}}$ in order to obtain a complete set of generators for

$$K^{\mathrm{GL}_n \times \mathrm{GL}_n}.$$

This is done in Section 4.

4. Algebraic independence of composite Dickson invariants

In this section we describe a type of composite Dickson invariant, which can be considered as a Dickson invariant on Dickson invariants. Roughly speaking, we form Dickson invariants on the column variables, then form Dickson invariants on the resulting variables.

More precisely, for each j, $0 \leq j \leq m-1$ consider the polynomial

$$Q_j(X) = \prod_{v \in V} (X - c_{m,j}(v))$$

which is $\mathrm{GL}_n \times \mathrm{GL}_m$ invariant since the $c_{m,j}(v)$ are $1 \times \mathrm{GL}_m$ invariant. Let $c_{n,i}(c_{m,j})$ denote the coefficient of X_{p^i} in $Q_j(X)$. Then we have

Theorem 4.1. *The invariants $c_{n,i}(c_{m,j})$, $0 \leq i \leq n-1$, $0 \leq j \leq m-1$ are algebraically independent, i.e., they form a polynomial subalgebra of*

$$S(M_{n,m})^{\mathrm{GL}_n \times \mathrm{GL}_m}$$

and a transcendence basis for K.

The proof uses an ordering of the $\{x_{i,j}\}$. Set

$$x_{i,j} < x_{i',j'}$$

if $j<j'$ or if $j=j'$ and $i<i'$. We shall also need the lexiographical ordering on monomials in the $x_{i,j}$'s. If $m = x_{i,j} \cdots x_{i_r,j_r}$ and $m' = x_{i'_1,j'_1} \cdots x_{i'_r,j'_r}$ are two monomials in which the $x_{i,j}$'s are arranged in non-decreasing order, then

$$m < m' \quad \text{if } x_{i_u,j_u} \leq x_{i'_u,j'_u}$$

for $1 \leq u \leq r$. Here we allow $x_{i,j}=1$ and specify $1 \leq x_{i',j'}$ for all i',j'. This ordering is multiplicative in the sense that if $m_i \le n_i$ $(i=1,2)$, then $m_1 m_2 \le n_1 n_2$.

Let $q=p-1$ and set $y_{i,j}=(x_{i,j})^q$.

Lemma 4.2.

$$c_{n,j}(c_{m,j}) \equiv \pm(y_{11}^{p^{m-1}} y_{12}^{p^{m-2}} \cdots y_{1,m-j}^{p^j})^{p^{n-1}} (y_{21}^{p^{m-1}} y_{22}^{p^{m-2}} \cdots y_{2,m-j}^{p^j})^{p^{n-2}} \cdots$$
$$\cdots (y_{n-i,1}^{p^{m-1}} y_{n-i,2}^{p^{m-2}} \cdots y_{n-i,m-j}^{p^j})^{p^i}$$

modulo larger terms.

Proof. From the definition of $c_{n,j}(c_{m,j})$ as a composite it suffices to show

$$c_{n,i} \equiv \pm y_1^{p^{n-1}} y_2^{p^{n-2}} \cdots y_{n-i}^{p^i}$$

modulo larger terms, where $y_i = x_i^q$ and $0 \le i \le n-1$. This result is clear for $n=1$ since $c_{1,0} = \pm y_1$, $c_{1,1}=1$. According to [14, Proposition 1.3] we have

$$c_{n,i} = c_{n-1,i-1}^p - c_{n-1,i}[f_{n-1}(x_n)]^q$$

where $f_n(X) = P_V(X)$ is the polynomial of (2.2). Now if $i>0$, then

$$c_{n,i} \equiv c_{n-1,i-1}^p \equiv \pm(y_1^{p^{n-2}} \cdots y_{n-i}^{p^{i-1}})^p$$

by induction. Hence, $c_{n,i} \equiv \pm y_1^{p^{n-1}} \cdots y_{n-i}^{p^i}$ as required. If $i=0$, then

$$c_{n,0} \equiv -c_{n-1,0}[\pm c_{n,0} x_n]^q = \pm c_{n-1,0}^p x_n^q \equiv \pm y_1^{p^n} \cdots y_{n-1}^p y_n$$

by induction.

Proof of Theorem 4.1. Since the terms on the right of the formulas of Lemma 4.2 are clearly algebraically independent, the result follows from the multiplicative property of the order filtration.

Remarks. (1) It follows from Theorem 4.1 that $S(M_{n,m})^{\mathrm{GL}_n \times \mathrm{GL}_m}$ is a polynomial algebra on the $c_{n,i}(c_{m,j})$ only in case $n=1$ or $m=1$. For example $\sum_{v \in V} c_{m,m-1}(v)$ is a non-trivial $\mathrm{GL}_n \times \mathrm{GL}_m$ invariant not in the polynomial algebra generated by the $c_{n,i}(c_{m,j})$.

(2) If $n=m$, then $c_{n,i}(c_{n,j})$ and $c_{n,j}(c_{n,i})$ provide distinct invariants (if $i \ne j$) of the same dimension, thus supplying extra invariants to generate

$$K^{\mathrm{GL}_n \times \mathrm{GL}_m}$$

as promised in Section 2. This is illustrated in Example 2.6 by the decomposition of z_6 as the sum of two invariants

$$z_6 = c_{2,0}(c_{2,1}) + c_{2,1}(c_{2,0}).$$

5. P-tori and weak closure

As in the introduction we consider the additive group $M_{n,m}$ of $n \times m$ matrices over $\mathbb{F}_p$ as the subgroup

$$\begin{pmatrix} I_n & M_{n,m} \\ 0 & I_m \end{pmatrix}$$

of $\mathrm{GL}_{n+m}(\mathbb{F}_p)$. Then $M_{n,m}$ is an elementary abelian p-subgroup or *p-torus* for short. Let $H = M_{n,m}$, $G = \mathrm{GL}_{n+m}$, then $N_G H$, the normalizer of H in G, is easily seen to be the subgroup

$$\begin{pmatrix} \mathrm{GL}_n & M_{n,m} \\ 0 & \mathrm{GL}_m \end{pmatrix}.$$

The Weyl group $W_G H = N_G H/H = \mathrm{GL}_n \times \mathrm{GL}_m$ acts on H by conjugation

$$(g_1, g_2)h(g_1^{-1}, g_2^{-1}) = g_1 h g_2^{-1}, \quad (g_1, g_2) \in W_G H,\ h \in H$$

which is precisely the action defined in (1.2).

Let $U_{n+m} = U_{n+m}(\mathbb{F}_p)$ be the unipotent subgroup of GL_{n+m}, i.e., the upper triangular matrices with 1's on the diagonal. Then U_{n+m} is a p-Sylow subgroup containing $M_{n,m}$. Generators for U_{n+m} are given by the matrices $u_{i,j}$, $1 \le i < j \le n+m$, which agree with I_{n+m} except for a single non-zero entry, namely 1 in the (i,j)th position. Then $u_{i,j}^p = I_{n+m}$ and commutators $(a,b) = aba^{-1}b^{-1}$ are given by

$$(u_{ij}, u_{kl}) = \begin{cases} u_{il} & \text{if } j = k, \\ u_{kj}^{-1} & \text{if } i = l, \\ I_{n+m} & \text{otherwise.} \end{cases}$$

Using these relations one easily checks

Lemma 5.1. *The p-tori $M_{n,m}$ are maximal in GL_{n+m}.*

One also has

Proposition 5.2. *Among the maximal p-tori of GL_k, the largest rank is $k^2/4$ if k is even and $(k^2-1)/4$ if k is odd. These ranks are attained by $M_{k/2,k/2}$ and $M_{(k-1)/2,(k+1)/2}$ respectively.*

Proof. The result is clearly true for $k = 1, 2$. Let $E_k \subset U_k$ be the subgroup generated by $\{u_{1,j}, u_{j,k}\}$, i.e., the union of the top row and right hand column subgroups. Then E_k is normal and there is an extension

$$E_k \to U_k \to U_{k-2}.$$

E_k is an extra-special p-group of order p^{2k-3}; by inspection the rank of the largest

p-torus of E_k is $k-1$. By induction, the rank of the largest p-torus of U_{k-2} is $(k-2)^2/4$ or $((k-2)^2-1)/4$ according as k is even or odd. Consequently the largest possible rank for a p-torus in U_k is

$$(k-1)+(k-2)^2/4=k^2/4$$

if k is even and

$$(k-1)+((k-2)^2-1)/4=(k^2-1)/4$$

if k is odd.

This completes the proof since $M_{k/2,k/2}$ and $M_{(k-1)/2,(k+1)/2}$ have these ranks respectively.

Remarks. (1) It follows from a result of Quillen [12] that the Krull dimension of $H^*(\mathrm{GL}_k(\mathbb{F}_p))$ is $k^2/4$ if k is even and $(k^2-1)/4$ is odd.

(2) There are maximal p-tori in GL_{m+n} not congugate to one of the $M_{n,m}$. For example, $\mathbb{F}\langle u_{12}u_{34}, u_{13}u_{24}, u_{14}\rangle \subset \mathrm{GL}_4(\mathbb{F}_2)$.

In order to facilitate computation of transfer in group cohomology we shall need the following result. Let $K=M_{n,m}$, $H=U_{n+m}$, $G=\mathrm{GL}_{n+m}$.

Lemma 5.3. *$gKg^{-1}\subset H$ implies $gKg^{-1}=K$ for $g\in G$.*

Proof. Let $B\subset G$ be the subgroup of upper triangular matrices, then $B\subset N_GK$ and so $gKg^{-1}\subset K$ iff $\tilde{g}K\tilde{g}^{-1}\subset H$ for $\tilde{g}\in HgB$. However $H\setminus G/B=\Sigma_{n+m}$ by Bruhat's decomposition [1]. The action of $\sigma\in\Sigma_{n+m}$ is given by $\sigma x_{ij}\sigma^{-1}=x_{\sigma(i),\sigma(j)}$. Thus $\sigma K\sigma^{-1}\subset H$ implies $\sigma\in\Sigma_n\times\Sigma_m$ and so $\sigma K\sigma^{-1}=K$.

Remark 5.4. In the situation of Lemma 5.3, K is said to be weakly closed in H. In case K is also a p-torus this condition forces a drastic simplification in the double coset formula for the transfer as we shall observe in Proposition 6.2. These notions were used by Kahn and the second author in [5] and by Mui in [10]. Kuhn has also studied this condition in [4]. Finally S. Mitchell has pointed out that Lemma 5.3 holds generally for K the nilpotent radical of a parabolic subgroup in a Chevalley group in characteristic p with unipotent group H.

6. Characteristic classes for $H^*(\mathrm{GL}_{n+m}(\mathbb{F}_p))$

In this section we construct characteristic classes for $H^*(\mathrm{GL}_{n+m}(\mathbb{F}_p))$ related to the invariant theory of the previous sections.

Let $\mathrm{GL}_{n,m}$ be the parabolic subgroup

$$\mathrm{GL}_{n,m}=M_{n,m}\rtimes(\mathrm{GL}_n\times\mathrm{GL}_m)$$

which is the normalizer of $M_{n,m}$ in GL_{n+m} described in Section 5. For each $\mathrm{GL}_n \times \mathrm{GL}_m$ invariant set $\mathcal{I} \subset M_{n,m}^*$ set

$$\gamma_{\mathcal{I}} = \prod_{u \in \mathcal{I}} (-u) \colon M_{n,m} \to \mathbb{F}_p^N, \quad N = |\mathcal{I}|.$$

The action of $\mathrm{GL}_n \times \mathrm{GL}_m$ on $\mathcal{I}$ defines a $\gamma_{\mathcal{I}}$-equivariant permutation representation

$$\sigma_{\mathcal{I}} \colon \mathrm{GL}_n \times \mathrm{GL}_m \to \Sigma_N$$

Combining these maps we obtain a representation of $\mathrm{GL}_{n,m}$

$$\varrho_{\mathcal{I}} \colon M_{n,m} \rtimes (\mathrm{GL}_n \times \mathrm{GL}_m) \xrightarrow{\gamma_{\mathcal{I}} \rtimes \sigma_{\mathcal{I}}} \mathbb{F}_p^N \rtimes \Sigma_N \longrightarrow U(1)^N \rtimes \Sigma_N \subset U(N)$$

where $U(k)$ is the unitary group and $\mathbb{F}_p \approx \mathbb{Z}/p \to U(1)$ is the standard inclusion (if $p=2$, $U(k)$ can be replaced by the orthogonal group $\mathrm{O}(k)$).

Restricting to $M_{n,m}$, it is clear that the total Chern class (resp. Stiefel-Whitney class if $p=2$) of $\varrho_{\mathcal{I}}$ satisfies

$$\varrho_{\mathcal{I}}^*(C) = \sum_{u \in \mathcal{I}} (1-u) = P_{\mathcal{I}}(1)$$

in $S(M_{n,m}^*)$ where $P_{\mathcal{I}}(X)$ is the invariant polynomial (2.1) associated with $\mathcal{I}$. Thus we have proved

Proposition 6.1. *For any $\mathrm{GL}_n \times \mathrm{GL}_m$ invariant set $\mathcal{I} \subset M_{n,m}^*$, the coefficients of $P_{\mathcal{I}}(X)$ are Chern classes of $\varrho_{\mathcal{I}}$ belonging to the image of the restriction*

$$i^* \colon H^*(\mathrm{GL}_{n,m}) \to H^*(M_{n,m})^{\mathrm{GL}_n \times \mathrm{GL}_m}$$

where $i \colon M_{n,m} \to \mathrm{GL}_{n,m}$ denotes inclusion.

In fact, these classes come from $H^*(\mathrm{GL}_{n+m})$. This is our main result. To prove it we invoke the following general fact

Proposition 6.2 [4, Proposition 2.3]. *Let $K \lhd H \subset G$ be finite groups with K a p-torus. Suppose K is weakly closed in H ($gKg^{-1} \subset H$ implies $gKg^{-1} = K$ for $g \in G$, see Remark 5.4). Then the double coset formula for transfer and restriction simplifies to the commutative diagram*

$$\begin{tikzcd} H^*(H) \arrow[r, "i^*"] \arrow[d, "\mathrm{tr}"'] & H^*(K)^{W_H(K)} \arrow[d, "\mathrm{tr}"] \\ H^*(G) \arrow[r, "i^*"] & H^*(K)^{W_G(K)} \end{tikzcd}$$

Corollary 6.3. *If $W_H(K)$ is a p-Sylow subgroup of $W_G(K)$, then*

$$H^*(K)^{W_G(K)} \cap \mathrm{Im}(H^*(H) \to H^*(K)^{W_H(K)}) = \mathrm{Im}(H^*(G) \to H^*(K)^{W_G(K)}).$$

Proof. The inclusion $\supset$ is clear. Now assume x is an element of the left hand side. Then $\mathrm{tr}(x)=kx$ where $k=[W_G(K):W_H(K)]$ is prime to p. Hence x is also an element of the right hand side by the proposition.

By Lemma 5.3 the conditions of Proposition 6.2 are satisfied for $K=M_{n,m}$, $H=U_{n,m}$, $G=\mathrm{GL}_{n+m}$. Hence

Corollary 6.4.

$$\begin{aligned} H^*(M_{n,m})^{\mathrm{GL}_n\times\mathrm{GL}_m} &\cap \mathrm{Im}(H^*(U_{n+m})\to H^*(M_{n,m})^{U_n\times U_m}) \\ &= \mathrm{Im}(H^*(\mathrm{GL}_{n+m})\to H^*(M_{n,m})^{\mathrm{GL}_n\times\mathrm{GL}_m}). \end{aligned}$$

Proof. $W_G(K)=\mathrm{GL}_n\times\mathrm{GL}_m$, $W_H(K)=U_n\times U_m$.

Combining Proposition 6.1 and Corollary 6.4 we have our main result

Theorem 6.5. *For each $\mathrm{GL}_n\times\mathrm{GL}_m$ invariant set $\mathscr{I}\subset M^*_{n,m}$, let $\bar{\varrho}_{\mathscr{I}}$ denote the restriction of $\varrho_{\mathscr{I}}$ to U_{n+m}. Then $k\cdot\mathrm{tr}(\bar{\varrho}^*_{\mathscr{I}}(C))$ are classes in $H^*(\mathrm{GL}_{n+m})$ which restrict to the coefficients of $P_{\mathscr{I}}(X)$ in*

$$S(M^*_{n,m})^{\mathrm{GL}_n\times\mathrm{GL}_m}\subset H^*(M_{n,m}),\quad k=[\mathrm{GL}_n\times\mathrm{GL}_m:U_m\times U_m]^{-1}\in\mathbb{F}_p^*.$$

Corollary 6.6. *If n or $m=1$, then the restriction map*

$$H^*(\mathrm{GL}_{n+m})\xrightarrow{i^*}H^*(M_{n,m})^{\mathrm{GL}_n\times\mathrm{GL}_m}$$

surjects onto the polynomial algebra $S(M_{n,m})^{\mathrm{GL}_n\times\mathrm{GL}_m}$ of ordinary Dickson invariants discussed in Section 2.

Remarks. (1) For a discussion of Chern classes of the regular representation see [7].
(2) Work of Maazen [6] shows

$$H^k(\mathrm{GL}_n)=0\begin{cases}\text{for } k<[n/2] & \text{if } p=2,\\ \text{for } k<n & \text{if } p>2.\end{cases}$$

If $p=2$, the class

$$\det_n=\det(x_{ij})\in H^*(M_{n,n})^{\mathrm{GL}_n\times\mathrm{GL}_n}$$

is a natural candidate for a non-trivial class of lowest dimension coming from $H^*(\mathrm{GL}_{2n})$. Example 2.6 shows $\det_2$ is, in fact, in the image of restriction. We do not know if this is true in general, however.

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