Physics Letters B 713 (2012) 104–109

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Quaternionic and hyper-Kähler metrics from generalized sigma models

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ARTICLE INFO

Article history:
Received 3 March 2012
Received in revised form 15 May 2012
Accepted 17 May 2012
Available online 23 May 2012
Editor: M. Cvetič

ABSTRACT

The problem of finding new metrics of interest, in the context of SUGRA, is reduced to two stages: first, solving a generalized BPS sigma model with full quaternionic structure proposed by the authors and, second, constructing the hyper-Kähler metric, or suitable deformations of this condition, taking advantage of the correspondence between the quaternionic left-regular potential and the hyper-Kähler metric of the target space. As illustration, new solutions are obtained using generalized Q-sigma model for Wess–Zumino type superpotentials. Explicit solutions analog to the Berger's sphere and Abraham–Townsend type are given and generalizations of 4-dimensional quaternionic metrics, product of complex ones, are shown and discussed.

1. Introduction

Several attempts have been proposed in order to find new metric structures for the target space in supersymmetric models, in particular the $\mathcal{N} = (4, 0)$ and the $\mathcal{N} = (4, 4)$ cases. Since each supersymmetry, beyond the first, requires the existence of complex or quaternionic structure, these attempts are expected to lead to interesting new hypercomplex geometries in the context of SUGRA and, consequently, in type IIA and type IIB superstring theories. Considerable efforts and beautiful methods and prescriptions have been developed in that direction: from the bosonic approach, monopole solutions in S^3 submanifolds [1,2] and the method of the calibrations [3] (and references therein); from the supersymmetric side, the harmonic superspace method [4,5]. However, tentatives to connect the truly BPS solutions of the nonlinear sigma model under consideration, with the corresponding geometries (metrics) showing the expected hyper-Kähler and quaternionic properties, remain lacking.

In this work, our main goal is to attack this problem by reducing it to two independent stages consisting in: (i) solving a generalized BPS sigma model with full quaternionic structure and (ii), with this information, constructing the hyper-Kähler metric using the correspondence between the quaternionic left-regular potential and the hyper-Kähler metric in the target space. This correspondence, valid for hyper-Kähler manifolds and quaternionic ones with compact substructure (also smooth departures of the hyper-Kähler condition are allowed), is based on the existence of certain geometrical mappings that, at the classical level, allow to establish the equivalence between *a model with a potential in flat spacetime* and *a free model with a suitable metric in the target space*.

The first step of our strategy consists in finding BPS solutions for a Generalized Quaternionic Lagrangian (*GQL*), as introduced in Ref. [6], which presents the important property that both the base and the target spaces live in $\mathbb{H}$ (see subsection below). As we will show in the following, the proposed *GQL* has a standard form with a potential depending on scalar (quaternionic) fields. Then, with solutions for this model at hand, we will be able to establish a correspondence with the even sector (B_0) of a supergravity theory. The link is realized by a direct mapping between the action of the free sigma model with metric $g_{\mu\nu}$ and the *GQL* action in a Minkowski space with a potential $V(q)$.

It is worth mentioning that the considered model allows, for certain choices of the coset, cohomogeneity one metric solutions, a type well studied in the context of Spin(7) manifolds – see, for instance [7].

Due to their simplicity and the clear importance they have in the context of supersymmetric nonlinear sigma models, through this Letter we will focus our study on Wess–Zumino type potentials. This will allow us to present the analysis by showing specific explicit solutions, which will put in evidence the underling quaternionic structures behind them.

2. Generalized quaternionic action

In all the considered cases, our model is defined on a quaternionic spacetime (base space) and with *a quaternionic space of scalar*

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0370-2693
http://dx.doi.org/10.1016/j.physletb.2012.05.036

fields as target. This is the correct and clear geometrical definition. Another possible terminology (not strictly mathematically accurate) is *worldvolume* and *target space*, in referring to the domain and range manifolds, respectively, of the sigma models [8].

Because of the obvious group relation $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$, we can take a standard real 4-dimensional spacetime, or a 2-dimensional complex or a 1-dimensional quaternionic manifold as our base space to the quaternionic (target) space of the (four) scalar fields.

Consider a system of four scalar fields governed by the Generalized Quaternionic Lagrangian density (*GQL*) of the form

$$\mathcal{L} = \frac{1}{2}\overline{\Pi q}\Pi q - \frac{1}{2}\left|W'(q)\right|^2. \tag{1}$$

The *Cauchy–Fueter operator* Π is defined by

$$\Pi \equiv \widehat{i^0}\partial_0 - \widehat{i^1}\partial_1 - \widehat{i^2}\partial_2 - \widehat{i^3}\partial_3, \tag{2}$$

where $\widehat{i^0} = \mathbb{I}$ and $\widehat{i^i}$ $(i = 1, 2, 3)$ obey the standard quaternionic algebra, and $\partial_0 \equiv \partial/\partial x^0$ and $\partial_i \equiv \partial/\partial x^i$.

Throughout this work we shall adopt Einstein's convention of indices summation, with the Latin indices i and j running from 1 to 3, unless otherwise stated. As usual, *Sc* and *Vec* will denote the scalar and vector parts of the corresponding quaternionic expression. In particular, whenever convenient, we will use the notation $\Pi_0 \equiv Sc\,\Pi$ and $\boldsymbol{\Pi} \equiv Vec\,\Pi$. We also define $q_i^2 \equiv q_1^2 + q_2^2 + q_3^2$, and $W_{q_0} \equiv \Pi_{q_0} W(q)$ and $W_{q_i} \equiv \boldsymbol{\Pi}_q W(q)$, where the Cauchy–Fueter operator acting on the target space is given by

$$\Pi_q \equiv \widehat{i^0}\partial_{q_0} - \widehat{i^1}\partial_{q_1} - \widehat{i^2}\partial_{q_2} - \widehat{i^3}\partial_{q_3}. \tag{3}$$

Note that these equations for W present both the scalar and vector parts.

2.1. Wess–Zumino model

Following the lines of the complex field case treated in Ref. [9], in the present work we will specialize our generalized Q-sigma model to the case of a Wess–Zumino (*WZ*) type superpotential [10] of the form

$$W'(q) = n - q^N = n - \left(q_0 + \widehat{i^i}q_i\right)^N, \tag{4}$$

where $N \in \mathbb{Z}$, in principle $n \in \mathbb{H}$ but through this Letter we will take $n \in \mathbb{C}$ or its subgroups, and the prime indicates derivative with respect to the argument of the considered function. This standard choice is simply motivated by the fact that the *WZ* superpotential is the basic prototype for any analysis involving hypercomplex quaternionic structures in several areas of the modern theoretical physics.

Thus, the first order equation $\Pi q = \overline{W'(q)}$ for our *WZ* potential (4) reads

$$\frac{dq}{dx} = n - \bar{q}^N = n - \left(q_0 - \widehat{i^i}q_i\right)^N. \tag{5}$$

This expression, with x identified below, arises from the relation between the left-regular superpotential $W(q)$ and the BPS conditions as *quaternionic configurations with left-regular superpotentials minimize the energy of the system to the BPS bound* (see Appendix A).

The corresponding vacuum (minima) manifold is described by the set of the N-roots of the unity in the field of the quaternions, i.e. S^2 spheres,

$$v_N^k = e^{a_N^k} = \exp\left(a2\pi\frac{k-1}{N}\right), \quad k = 1, \dots, N, \tag{6}$$

being a a pure quaternion of unitary norm.

In the following we will present some solutions (orbits) for the *WZ* model above, in the simplest cases of $N = 1$ and $N = 2$. As a first approach to the problem, we will consider an ordinary (commutative) base space as spacetime equivalent. Then, we will focus on the case of a quaternionic (noncommutative) base space as spacetime equivalent.

2.2. Commutative spacetime equivalent solutions

The realization of the commutative base space as spacetime equivalent is achieved by making the identification

$$x \to \widehat{i_0}X^0 \quad (\text{i.e. } x \in \mathbb{R}). \tag{7}$$

2.2.1. Case $N = 1$ (commutative)

For $N = 1$, Eq. (5) reads

$$\frac{dq}{dx} = n - \bar{q} = n - q_0 + \widehat{i^1}q_1 + \widehat{i^2}q_2 + \widehat{i^3}q_3. \tag{8}$$

Splitting up this equation into its *Sc* and *Vec* parts, we have

$$\frac{dq_0}{dx} = n - q_0, \tag{9}$$

$$\frac{dq_i}{dx} = q_i. \tag{10}$$

This system admits the direct solution

$$q(x) = n - C_0 e^{-x} + \widehat{i^i}C_i e^x, \tag{11}$$

where C_0 and C_i $(i = 1, 2, 3)$ are integration constants.

2.2.2. Case $N = 2$ (commutative)

For $N = 2$, the first order equation takes the form

$$\frac{dq}{dx} = n - \left(q_0^2 - q_i^2 - 2\widehat{i^i}q_i q_0\right), \quad n \in \mathbb{Z}. \tag{12}$$

Given the condition on the quaternionic phase and the $Vec(q)$, this case presents two minima in the field space, located at $Sc(q) = \pm 1$ (analogously to the complex field case of [9]).

Breaking Eq. (12) into its *Sc* and *Vec* parts, we obtain the system of equations

$$\frac{dq_0}{dx} = n - q_0^2 + q_i^2, \tag{13}$$

$$\frac{dq_i}{dx} = 2q_i q_0. \tag{14}$$

Let us now propose a generic equation for the curves connecting the two minima of the potential (trial orbit method [11]),

$$Aq_0^2 + B^i q_i^2 = C \quad \to \quad q_0^2 + \beta^i q_i^2 = 1 \tag{15}$$

where the reduction from five (A, B_i, C) to three (β_i) parameters is due the $N = 2$ condition. Differentiating the orbit equation with respect to x, and using the first order derivatives of the system above we obtain

$$q_0^2 + \alpha^i q_i^2 = n, \quad \alpha_i = -(1 + 2\beta_i)\ \forall i, \tag{16}$$

from which it is easily seen by simple comparison that acceptable hypercurves must fulfill the ellipticity conditions $\forall \beta_i \leqq -1/2$ $(\alpha_i \geqslant 0)$ and $n = 1$. Thus, Eq. (16) reduces to

$$q_0^2 + \alpha^i q_i^2 = 1, \quad \alpha_i \geqslant 0\ \forall i. \tag{17}$$

Imposing the orbit condition on the first order equations leads to,

$$\frac{dq_0}{dx} = q_i^2 + \alpha^i q_i^2, \tag{18}$$

$$\frac{dq_i}{dx} = \pm 2q_i\sqrt{1 - \alpha^i q_i^2}. \tag{19}$$

Leaving aside the trivial case $\alpha_i = 0\ \forall i$ (which leads to $q_0 = \pm 1$, $q_i = 0\ \forall i$), a solution to the system above presents components of the form

$$q_0(x) = \tanh(2x + c), \qquad q_i(x) = \frac{1}{\sqrt{3\alpha_i}} \operatorname{sech}(2x + c), \tag{20}$$

where a_i, and c are integration constants. Requiring consistency with the orbit under consideration results in the restriction $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 = 3$, which implies that $\alpha_3 = \alpha_1\alpha_2(3\alpha_1\alpha_2 - \alpha_1 - \alpha_2)^{-1}$, for arbitrary values of α_1 and α_2. (In particular, we can take simply $\alpha_1 = \alpha_2 = \alpha_3 = 1$.)

Collecting the results we have that our commutative $N = 2$ case solution reads

$$q(x) = \tanh(2x + c) + \hat{i}^i \frac{1}{\sqrt{3\alpha_i}} \operatorname{sech}(2x + c). \tag{21}$$

As expected, in the present case the structure of the solutions is much richer than in the complex case, as we have here the parameters α_i to play with. Applying the usual parameterization $\Lambda(x) = \tanh(2x + c)$, the solution takes the form

$$q(x) = \Lambda(x) + \hat{i}^i \frac{1}{\sqrt{3\alpha_i}} \sqrt{1 - \Lambda(x)^2}, \tag{22}$$

which looks more suitable for a geometrical analysis.

2.3. Quaternionic spacetime equivalent solutions

Let us now evaluate a quaternionic base manifold (noncommutative spacetime equivalent). This can be done by identifying the spacetime spatial coordinate x with one of the complex directions of a quaternionic manifold, namely

$$x \to \hat{i}_1 X^1 \quad \big(\text{i.e. } x \in SU(2)\big). \tag{23}$$

Therefore, we shall consider scalar fields of the target space depending on this coordinate, and we must also specialize the operator Π to the X^1 coordinate, that is

$$\Pi \equiv \hat{i}^0\partial_0 - \hat{i}^1\partial_1 - \hat{i}^2\partial_2 - \hat{i}^3\partial_3 \quad \to \quad \Pi = -\hat{i}^1\partial_1. \tag{24}$$

As a consequence of this choice, the spacetime assumes the structure $S_1 \otimes O(3) \sim S_1 \otimes SU(2)$, perfectly described by an element X of $\mathbb{H}$, in the representation that we have adopted in this work.

Now, in the present case, due to the non-trivial topology of the base manifold (spacetime equivalent), we solve the first order equations directly (no trial orbit method). Then, the two cases analyzed before ($N = 1$ and $N = 2$), take the completely different form described below.

2.3.1. Case N = 1 (noncommutative)

Taking $x \to \hat{i}_1 X^1$ in the case $N = 1$ leaves the first order equation (5) with the form

$$\begin{aligned} &\Pi q = \overline{W'(q)} = n - \bar{q}, \\ &-\hat{i}^1\partial_1[q_0 + \hat{i}_1 q_1 + \hat{i}_2 q_2 + \hat{i}_3 q_3] \\ &\quad = n - q_0 + \hat{i}^1 q_1 + \hat{i}^2 q_2 + \hat{i}^3 q_3, \end{aligned} \tag{25}$$

which can be put in the form

$$\frac{dq_0}{dX_1} = -q_1, \tag{26}$$

$$\frac{dq_1}{dX_1} = n - q_0, \tag{27}$$

$$\frac{dq_2}{dX_1} = -q_3, \tag{28}$$

$$\frac{dq_3}{dX_1} = q_2. \tag{29}$$

As the equations are not all coupled but in pairs, the resolution of the system is quite direct, and we obtain the solution

$$q_0(X_1) = n - e^{C_+ - C_-} \cosh(X_1 + C_+ + C_-), \tag{30}$$

$$q_1(X_1) = e^{C_+ - C_-} \sinh(X_1 + C_+ + C_-), \tag{31}$$

$$q_2(X_1) = C \cos(X_1 + \varphi), \tag{32}$$

$$q_3(X_1) = C \sin(X_1 + \varphi) \tag{33}$$

where C, C_+, and C_- are integration constants. Now, rewriting the hyperbolic/trigonometric functions by introducing the algebraic parameterization $\Lambda_A = \tanh(X_1 + C_+ + C_-)$, $\Lambda_B = \tan(X_1 + \varphi)$, the quaternion solution assumes a much more appropriate form for a geometrical analysis

$$q = n - \frac{e^{C_+ - C_-}}{\sqrt{1 - \Lambda_A^2}}(1 + \hat{i}_1\Lambda_A) + \frac{C}{\sqrt{1 + \Lambda_B^2}}(\hat{i}_2 + \hat{i}_3\Lambda_B). \tag{34}$$

2.3.2. Case N = 2 (noncommutative)

In this case the first order equation $\Pi q = \overline{W'(q)}$ can be split up into its *Sc* and *Vec* parts to give the system

$$\frac{dq_0}{dX_1} = -2q_1q_0, \tag{35}$$

$$\frac{dq_1}{dX_1} = n - q_0^2 + q_1^2 + q_2^2 + q_3^2, \tag{36}$$

$$\frac{dq_2}{dX_1} = -2q_3q_0, \tag{37}$$

$$\frac{dq_3}{dX_1} = +2q_2q_0. \tag{38}$$

The explicit symmetry of this system suggests that the simplest non-trivial solution is $q_1 = constant$. With that choice we have

$$n = q_0^2 - \big(q_1^2 + q_2^2 + q_3^2,\big), \tag{39}$$

$$q_0(X_1) = C_0 e^{-2X_1q_1}, \tag{40}$$

$$q_2(X_1) = C_1 \sin\big(C_0 q_1^{-1} e^{-2X_1q_1} + \varphi\big), \tag{41}$$

$$q_3(X_1) = C_1 \cos\big(C_0 q_1^{-1} e^{-2X_1q_1} + \varphi\big) \tag{42}$$

where the integration constants C_0 and C_1 must fulfill the constraint equation

$$n^2 = \big(C_0 e^{-2X_1q_1}\big)^2 - \big(q_1^2 + C_1^2\big). \tag{43}$$

In particular, we can make the convenient choice

$$C_0 = \sqrt{n^2 + 2q_1^2} \quad \text{and} \quad C_1 = q_1, \tag{44}$$

which simplifies the solution to the expression

$$\begin{aligned} q(X_1) = q_0(X_1) + q_1\big(&\hat{i}^1 + \hat{i}^2 \sin\big(q_0 q_1^{-1} + \varphi\big) \\ &+ \hat{i}^3 \cos\big(q_0 q_1^{-1} + \varphi\big)\big) \end{aligned} \tag{45}$$

where

$$q_0(X_1) = \sqrt{n^2 + 2q_1^2}\, e^{-2X_1q_1}. \tag{46}$$

This expression can be rewritten in an algebraic form using the parameterization $\Phi = \tan(q_0 q_1^{-1} + \varphi)$. We obtain

$$q(X_1) = q_0(X_1) + q_1\left[\hat{i}^1 + \frac{1}{\sqrt{1 + \Phi^2}}\big(\hat{i}^2 + \hat{i}^3\Phi\big)\right]. \tag{47}$$

Another simple solution is obtained by putting $q_0 \equiv 0$, which results in

$$\frac{dq_1}{dX_1} = n + q_1^2 + q_2^2 + q_3^2,$$
$$q_2 = C_2,$$
$$q_3 = C_3. \tag{48}$$

Thus,

$$q_1(X_1) = \sqrt{n + C_2{}^2 + C_3{}^2} \tan(X_1 + C_1). \tag{49}$$

The compactness of the solution is quite evident, even more putting $\Phi = \tan(X_1 + C_1)$, which gives

$$q(X_1) = \hat{i}^1 \sqrt{n + C_2^2 + C_3^2}\,\Phi + \hat{i}^2 C_2 + \hat{i}^3 C_3. \tag{50}$$

3. *GQA* and hyper-Kähler Q-structures

In this Letter the case with torsion will not be considered. Nevertheless, it is worth noting the particular cases of interest:

i) $\mathcal{N} = (2, 0)$, $D = 4$ (or $\mathcal{N} = (4, 4)$, $D = 2$): the torsion vanishes, complex structures are annihilated by covariant derivatives and form the quaternionic algebra (hyper-Kähler geometry). In both cases, prepotentials are known in the seminal references [12,13].

ii) $\mathcal{N} = (4, 0)$, $D = 2$: the torsion is a closed 3-form; complex structures are annihilated by covariant derivatives and form the quaternionic algebra (hyper-Kähler with torsion).

Then, in the following we will be dealing with the first case, just to show the consistency of the procedure of finding BPS solutions in the context of SUGRA.

3.1. The new metrics: Quink, a Q-Kink analogue?

The general form of the metrics we are interested in, admitting a quaternionic structure (hyper-Kähler or not) is, following the notation of [14],

$$ds^2_{new} = U\,dq \cdot dq + U^{-1}(dq_0 + \omega \cdot dq)^2. \tag{51}$$

Now, as in [14] ω can be considered as an euclidean 3-vector and it must fulfill the Killing's equation as follows

$$\mathcal{L}_X E_0 = 0, \tag{52}$$

where $E_0 = U^{-1}(dq_0 + \omega \cdot dq) \equiv U^{-1}(dq_0 + \widetilde{\omega})$ is the tetrad one-form corresponding to the scalar component of the line element (51) and the Killing vector field X is given i.e. by ∂_{q_0}. Notice that, from the point of view of the Cartan's structure equations, relation (52) can interpreted as an integrability condition.

As our case does not include torsion (hyper-Kähler and Conformal hyper-Kähler target spaces), clearly, the relation between U and $\widetilde{\omega} \equiv \omega \cdot dq$, is

$$U d(U^{-1}) = \widetilde{\omega} d(\widetilde{\omega}^{-1}). \tag{53}$$

As shown in detail in [6] a connexion (harmonic map) between the potential in our *GQL* and the free sigma model can be explicitly established, leading to

$$V = \left[\det(g_{ab})\right]^{-1}, \tag{54}$$

where g_{ab} is the metric associated to the Kähler manifold (target space) of the free model. Consequently, we can write a simple relation between our *GQL* potential and the U factor of the metrics (51), namely

$$V = \frac{m^2}{4} U^{-1}, \tag{55}$$

which corroborates the important results from physical arguments given in [8,14], now obtained from a purely geometrical approach.

3.1.1. Metric for the $N = 1$ case (commutative)

Let us consider first the $N = 1$ case. The potential is related to the U factor by

$$V = \frac{1}{2}(n - \bar{q})(n - q) = \frac{m^2}{4} U^{-1}. \tag{56}$$

Then we can write

$$U = \det(g_{\alpha\beta}) = \frac{m^2/2}{n^2 - |q|^2}. \tag{57}$$

Then, the line element corresponding to the quaternionic solution (11) reads

$$ds^2 = U^{-1} C_0^2 e^{-2x} d^2x + U C_i^2 e^{2x} \sigma^i \otimes \sigma^i. \tag{58}$$

3.1.2. Metric for the $N = 2$ case (commutative)

Let us now consider $N = 2$ case. For the potential we can write

$$V = \frac{1}{2}\left(n - \bar{q}^2\right)\left(n - q^2\right) = \frac{m^2}{4} U^{-1}. \tag{59}$$

Therefore, we have the relation

$$U = \det(g_{\alpha\beta}) = \frac{m^2/2}{n^2 - |q|^4}. \tag{60}$$

Similarly to the $N = 1$ case, we can write the line element corresponding to the quaternionic solution (21) calculated above,

$$ds^2 = 4\,\mathrm{sech}^4(2x + c)\left[U^{-1} dx^2 + U \frac{\sigma^i \otimes \sigma^i}{3\alpha_i} \sinh^2(2x + c)\right]. \tag{61}$$

The analysis of this expression is extremely simplified introducing the relation $\Lambda \equiv \tanh(2x + c)$ as in (22). Such definition transforms the hyperbolic/trigonometric expressions in algebraic ones leading to the line element

$$ds^2 = 4\left(1 - \Lambda^2\right)^2\left[U^{-1} dx^2 + U \frac{\sigma^i \otimes \sigma^i}{3\alpha_i} \frac{\Lambda^2}{(1 - \Lambda^2)}\right]. \tag{62}$$

3.2. Generalization of the Berger's sphere and comparison with other solutions

Let us now analyze the 3-dimensional (compact) part of the metrics obtained above, $ds_3^2 = U\,dq \cdot dq$. In order to make explicit the $S_1 \otimes S_2$ structure, we introduce the usual left angle-variables representative forms of the compact submanifold, and some constant coefficients $\mathring{q}_i$. Two main consequences immediately arise:

i) If $\mathring{q}_1 = \mathring{q}_2 = \mathring{q}_3$, that is, if the compact part of the metric takes the form

$$ds^2_{3(N=1)} = U C_i^2 e^{2x} \mathring{q}_1{}^2 \left[d\theta^2 + \sin^2\theta\, d\varphi^2 + (d\psi + \cos\theta\, d\varphi)^2\right], \tag{63}$$

$$ds^2_{3(N=2)} = 4U\left(1 - \Lambda^2\right)\Lambda^2 \mathring{q}_1{}^2 \left[d\theta^2 + \sin^2\theta\, d\varphi^2 + (d\psi + \cos\theta\, d\varphi)^2\right] \tag{64}$$

then the solution is a generalization similar to the Abraham–Townsend (Q-Kink) solution [15], and to the multicenter Gibbons–Hawking [16] solution, which are hyper-Kähler or conformally hyper-Kähler target manifolds.

Table 1
Comparison with solutions from Abraham–Townsend (AT) model.

AT		GQSM (commutative)	
One center	Two centers	$N=1$	$N=2$
$U=2m\frac{1}{\lvert q-q_0\rvert}$	$U=2m[\frac{1}{\lvert q-q_0\rvert}+\frac{1}{\lvert q+q_0\rvert}]$	$U=\frac{m^2/2}{(n^2-\lvert q\rvert^2)}$	$U=\frac{m^2/2}{(n^2-\lvert q\rvert^4)}$
$q_0(x)=const.$, (angular character: compact)		$q_0(x)=n-C_0e^{-x}$	$q_0(x)=\tanh(\frac{m\lvert\eta\rvert}{8\mu}(2x+c))$
$q_i(x)=\Phi\tanh(\frac{m\lvert\eta\rvert}{8\mu}(x-x_0))$, $\Phi=const.$		$q_i(x)=C_ie^x$	$q_i(x)=\mathring{q}_i\,\mathrm{sech}(\frac{m\lvert\eta\rvert}{8\mu}(2x+c))$, $\mathring{q}_i=1/\sqrt{3\alpha_i}$ ($\alpha_i\geqq 0\ \forall i$) and $1/\alpha_1+1/\alpha_2+1/\alpha_3=3$

ii) If $\mathring{q}_1=\mathring{q}_2\neq\mathring{q}_3$, then the metric corresponds to a *generalization of the Berger's sphere* (deformation of the $O(3)\approx SU(2)$ structure), which is a smooth deviation from the pure hyper-Kähler condition,

$$ds^2_{3(N=1)}=UC_i^2e^{2x}\mathring{q}_1\big[\mathring{q}_1^{\,2}\big(d\theta^2+\sin^2\theta\,d\varphi^2\big)+\mathring{q}_3^{\,2}(d\psi+\cos\theta\,d\varphi)^2\big],\tag{65}$$

$$ds^2_{3(N=2)}=4U\big(1-\Lambda^2\big)\Lambda^2\big[\mathring{q}_1^{\,2}\big(d\theta^2+\sin^2\theta\,d\varphi^2\big)+\mathring{q}_3^{\,2}(d\psi+\cos\theta\,d\varphi)^2\big].\tag{66}$$

We can now establish a comparison between our solutions and the well known results from other works [15,16] – see Table 1.

Finally, note that in the cases of the commutative spacetime equivalent, the obtained metrics are generalizations of Majumdar–Papapetrou solutions.

4. More examples: Generalization of standard metrics factorization

It is well known, in the context of SUGRA, the importance of the metrics that can be factorized as product of lower dimensional ones. The main reason is the claim that the appearing of these type of metrics (in particular the Bertotti–Robinson's (*BR*) and generalizations) in supergravity theories, indicates that the theory is fully renormalizable. The proof of the non-renormalization theorem for the *BR* background was almost trivial due to conformal flatness of this type of metrics, and because the Maxwell field is constant. These properties are not present in the general case of metrics admitting super-covariantly constant spinors. In General Relativity these solutions are known as the conformal-stationary class of Einstein–Maxwell fields, with conformally flat 3-dimensional spaces. Some generalizations of this class of metrics have been found by Neugebauer, Perjes, Israel and Wilson [17]: the flat space Laplacian in x. However, the analysis of these subjects is out of the scope of this Letter, and will be discussed elsewhere.

The product type metrics are 4-dimensional but composed by two 2-dimensional ones, in general, of Kähler type. In particular, we have found two metrics showing this structure in general form, as described below.

4.1. Case $N=1$ (noncommutative)

For this case we have obtained a metric solution of the form

$$ds^2=\frac{e^{2(C_+-C_-)}}{1-\Lambda_A^2}\big(U^{-1}\Lambda_A^2\,dx_1^2+U\sigma^1\otimes\sigma^1\big)+\frac{C_1^2}{1+\Lambda_B^2}U\big(\Lambda_B^2\sigma^2\otimes\sigma^2+\sigma^3\otimes\sigma^3\big).\tag{67}$$

This metric 'product' is the result of the geometrical structure of the quaternion solution (30). Precisely, if we make the choice $n=0$, $C_+=C_-$ and $i2C_+=\varphi$, the quaternion solution (34) can be written as the result of a product of the form $\mathbb{C}\otimes\mathbb{C}\to\mathbb{H}$, as follows

$$q=\big[\mathbb{I}_2+f\sigma^2\big]z,\tag{68}$$

where the complex number $z\in\mathbb{C}\subset\mathbb{H}$ is defined as

$$z\equiv\cosh(x_1+C_++C_-)+\sinh(x_1+C_++C_-)\sigma^1,\tag{69}$$

and the mapping f over the $\mathbb{C}$ field is given by

$$f:\mathbb{C}(x_1+C_++C_-)\to\mathbb{C}\big(i(x_1+C_++C_-)\big).\tag{70}$$

This is the reason why in this case the metric can be interpreted as product of two complex Kähler metrics with the structure $\mathbb{C}\otimes\mathbb{C}\to\mathbb{H}$.

4.2. Case $N=2$ (noncommutative)

For this case we have obtained a metric solution of the form

$$ds^2=4q_0^2q_1^2\left(U^{-1}\,dx_1^2+\frac{U}{q_0^2}\sigma^1\otimes\sigma^1\right)+\frac{U}{1+\Phi^2}\big(\sigma^2\otimes\sigma^2+\Phi^2\sigma^3\otimes\sigma^3\big).\tag{71}$$

This metric 'product' is the result of the geometrical structure of the quaternionic solution (47). Notice that now, in a sharp contrast with the previous case, the solution cannot be written directly as the result of a product of the form $\mathbb{C}\otimes\mathbb{C}\to\mathbb{H}$, or as the action of an ideal over a complex field $\in\mathbb{H}$.

5. Concluding remarks

In the present Letter we have proposed a new method for finding BPS solutions in the context of SUGRA. This method, in sharp contrast with the methods of the calibrations [1], or the monopole method introduced in [2,3], allows to find suitable quaternionic and hyper-Kähler geometries with the required properties appearing in the bosonic sector of supergravity theories based in the susy extensions of the nonlinear sigma models. The solutions found have the particularity of being BPS and are not just generalizations of the well known solutions, but new distinct ones.

While the focus of the present work was to present the method, that is, the geometrical link between genuine BPS quaternionic solutions of the *GQL* and the target space metric of the bosonic sector of the SUGRA theory, it is important to note the clear existence of a relation between our solutions and the Bianchi *IX* generalizations of hyper-Kähler metrics with Taub-NUT structures, and also the possible close connexion of our method with the (unconstrained) Harmonic Superspace Formalism [18]. The proper analysis of these points requires the evaluation of the full version of the supergravity theory (bosonic and fermionic sectors), and will be addressed in future work.

Acknowledgements

We would like to thank CNPq and PROCAD/CAPES for partial financial support.

Appendix A. Energy, left-regular $W(q)$ and BPS conditions

Similarly to the complex case of Ref. [9], we can rewrite the energy in a convenient fashion, in order to make explicit the relation between the BPS conditions and the corresponding gradient and potential terms of the Hamiltonian. Namely, we have

$$E = \frac{1}{2}\int dx\left[\left(\frac{dq_0}{dx} + W_{q_i}\right)\overline{\left(\frac{dq_0}{dx} + W_{q_i}\right)} + \left(\frac{dq_i}{dx} + W_{q_0}\right)\overline{\left(\frac{dq_i}{dx} + W_{q_0}\right)}\right] - \int dx\, Sc\left(\frac{dq_0}{dx}\overline{W_{q_i}} + \frac{dq_i}{dx}\overline{W_{q_0}}\right). \tag{72}$$

Here, consistently with the 1-dimensional spatial coordinate, we retain the scalar (commuting) part of the Π operator and of the quaternionic position X introducing the usual (commutative) x coordinate: $\Pi_0 \to \frac{d}{dx}$, $X_0 \to x$.

For quaternionic field solutions obeying

$$\frac{dq_0}{dx} = -W_{q_i}, \tag{73}$$

$$\frac{dq_i}{dx} = -W_{q_0}, \tag{74}$$

expression (72) is minimized to the Bogomol'nyi bound, and the energy is given by the superpotential

$$E_{\mathrm{BPS}} = \int dX\left(|W_{q_i}|^2 + |W_{q_0}|^2\right). \tag{75}$$

Analogously to the complex case, where Cauchy–Riemann conditions arise, the above equations solve the quaternionic equation of motion if we impose the Fueter-harmonic condition on $W(q_0, q_i)$, that is

$$0 = \Box\overline{W(q_0,q_i)} = \partial_{q_0}^2\overline{W} + \partial_{q_1}^2\overline{W} + \partial_{q_2}^2\overline{W} + \partial_{q_3}^2\overline{W} = \Box W(q_0,q_i) - 2\Pi_q\left[Sc\left(W(q_0,q_i)\right)\right]. \tag{76}$$

That is, the Fueter-harmonic condition implies the (left) holomorphy of W

$$0 = \Pi_q\overline{W(q_0,q_i)} = \overline{\Pi}_q W(q_0,q_i) - 2\Pi_q\left[Sc\left(W(q_0,q_i)\right)\right]. \tag{77}$$

Therefore, we can generalize previous results concerning the complex fields [9], to a quaternionic function $\mathcal{W} = \widetilde{W} + W$, such that the Cauchy–Fueter left regularity is satisfied

$$\widetilde{W}_{q_0} = -W_{q_i}, \qquad \widetilde{W}_{q_i} = -W_{q_0}. \tag{78}$$

It is worth mentioning here that all the above expressions involving analytical properties of the functions in the quaternionic field, reduce to their corresponding analogous expressions in the complex field case, when we retain only two of the quaternionic variables, namely

$$W(q_0, q_i) \to W(q_0, q_1) = W_0 + iW_1. \tag{79}$$

In this case, Eq. (77) reduces to the Cauchy–Riemann conditions

$$(\partial_{q_0} + i\partial_{q_1})(W_0 + iW_1) - 2i\partial_{q_1}W_0 = 0,$$
$$(\partial_{q_0}W_0 - \partial_{q_1}W_1) + i(\partial_{q_0}W_1 - \partial_{q_1}W_0) = 0. \tag{80}$$

Taking into account the above statements, the energy can be put in a more general fashion

$$E = \frac{1}{2}\int dX\left[(\widetilde{W}_{q_0} + W_{q_i})\overline{(\widetilde{W}_{q_0} + W_{q_i})} + (\widetilde{W}_{q_i} + W_{q_0})\overline{(\widetilde{W}_{q_i} + W_{q_0})}\right] - \int dX\, Sc(\widetilde{W}_{q_0}\overline{W_{q_i}} + \widetilde{W}_{q_i}\overline{W_{q_0}}), \tag{81}$$

generalizing Eq. (72).

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