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Letter to the Editor

Convergence of wavelet frame operators as the sampling density tends to infinity ☆

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ABSTRACT

In this paper, we study the convergence of wavelet frame operators defined by Riemann sums of inverse wavelet transforms. We show that as the sampling density tends to the infinity, the wavelet frame operator tends to the identity or embedding mapping in various operator norms provided the wavelet function satisfies some smoothness and decay conditions. As a consequence, we also get some spanning results of affine systems.

1. Introduction and the main result

This paper is a continuation of the work done in [22].

Let ψ be a function in $L^2(\mathbb{R}^d)$. Given $(s,t) \in \mathcal{G} := \{(s,t){:}\ s>0,\ t \in \mathbb{R}^d\}$, we define the dilation and translation operators D_s and T_t by

$$(D_s\psi)(x) = s^{-d/2}\psi\left(s^{-1}x\right) \quad \text{and} \quad (T_t\psi)(x) = \psi(x-t),$$

respectively. The wavelet transform of $f \in L^2(\mathbb{R}^d)$ with respect to ψ is defined by

$$(W_\psi f)(s,t) = \langle f, T_t D_s \psi\rangle.$$

Let $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ be such that

$$C_{\psi_1,\psi_2} := \int_0^{+\infty} \overline{\hat{\psi}_1(a\omega)}\hat{\psi}_2(a\omega)\frac{1}{a}\,da \tag{1.1}$$

is a non-zero constant for $\omega \neq 0$. Then we have

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$$f(x) = C_{\psi_1,\psi_2}^{-1} \iint_{\mathcal{G}} (W_{\psi_1} f)(a,b)(T_b D_a \psi_2)(x) \frac{da\,db}{a^{d+1}}, \tag{1.2}$$

where the convergence is in $L^2(\mathbb{R}^d)$. And in [24], the continuous wavelet transform was extended to $L^p(\mathbb{R}^d)$.

Motivated by [12,13,19,20,27,30], where the convergence of Riemann sums of the inverse windowed Fourier transform was studied, in [22,28], the authors studied the approximation of the integral in (1.2) using Riemann sums.

Set $a > 1$ and $b > 0$. We define the operator $S_{a,b;\psi_1,\psi_2}$ as

$$S_{a,b;\psi_1,\psi_2} f = \frac{b^d(a^d - 1)}{d a^{d/2} C_{\psi_1,\psi_2}} \sum_{j\in Z,k\in\mathbb{Z}^d} \langle f, T_{t_{j,k}} D_{s_j} \psi_1 \rangle T_{t_{j,k}} D_{s_j} \psi_2, \tag{1.3}$$

where $(s_j, t_{j,k}) \in E_{j,k} := [a^{j-1/2}, a^{j+1/2}) \times a^j b(k + [-1/2, 1/2)^d)$. It is easy to see that $S_{a,b;\psi_1,\psi_2} f$ can be viewed as a Riemann sum of the integral in (1.2) with respect to the Haar measure $a^{-d-1} da\,db$ on $\mathcal{G}$.

For the case of $(s_j, t_{j,k}) = (2^j, 2^j k)$ and $\psi_1 = \psi_2 = \psi$, it is well known (e.g., see [8, Chapter 9]) that if ψ satisfies some regular conditions, an orthonormal basis for $L^2(\mathbb{R}^d)$ of the form $\{2^{jd/2}\psi(2^j \cdot -k)\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is also an unconditional basis for $L^p(\mathbb{R}^d)$, $1 < p < \infty$. Chui and Shi [7] proved a sharper result in this aspect, where $\{2^{jd/2}\psi(2^j \cdot -k)\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is relaxed to be a Bessel sequence in $L^2(\mathbb{R}^d)$. We refer to [6,8] for an introduction on frames and Bessel sequences. See also [9] for the convergence of wavelet series in $L^p(\mathbb{R}^d)$.

It was shown in [22] that, for certain ψ_1 and ψ_2, $S_{a,b;\psi_1,\psi_2}$ converges to the identity operator on $L^2(\mathbb{R}^d)$ as (a, b) tends to $(1, 0)$.

In this paper, we study the convergence of $S_{a,b;\psi_1,\psi_2}$ in $\mathcal{B}(L^p(\mathbb{R}^d))$, where $\mathcal{B}(L^p(\mathbb{R}^d))$ is the space of all bounded linear operators on $L^p(\mathbb{R}^d)$, $1 < p < \infty$. We show that it tends to the identity operator for all $1 < p < \infty$ whenever ψ_1 and ψ_2 satisfy some smoothness and decay conditions.

Moreover, we also study the convergence of $S_{a,b;\psi_1,\psi_2}$ as operators from the Hardy space $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ and from $L^\infty(\mathbb{R}^d)$ to $BMO(\mathbb{R}^d)$, respectively. We show that it tends to the corresponding embedding mapping.

We use the following set of multi-index: $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha! = \alpha_1! \cdots \alpha_d!$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $(X^\alpha f)(x) = x^\alpha f(x)$, and $(\partial^\alpha f)(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(x)$ is the partial derivative in ordinary sense. For $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the greatest integer which is less than or equal to a.

Our main result is the following.

Theorem 1.1. *Set $n_0 = \lfloor d/2 \rfloor + 1$. Let β and γ be positive constants such that $d/2 < \gamma \leqslant d$ and $1/\beta + 1/(2\gamma) < 1/d$. Suppose that ψ_1 and ψ_2 are functions on $\mathbb{R}^d$ satisfying the following conditions,*

(i) $|(\partial^\alpha \psi_i)(x)| \leqslant C/(1 + |x|)^\beta$, $|\alpha| \leqslant n_0 - 1$,
(ii) $|\psi_i(x) - \sum_{|\alpha| \leqslant n_0-1} (\partial^\alpha \psi_i)(t)(x - t)^\alpha/\alpha!| \leqslant C|x - t|^\gamma$, *and*
(iii) $\int_{\mathbb{R}^d} x^\alpha \psi_i(x)\,dx = 0$, *whenever* $|\alpha| \leqslant n_0 - 1$.

Let $S_{a,b;\psi_1,\psi_2}$ be defined as in (1.3). *Then we have*

$$\lim_{(a,b)\to(1,0)} \|S_{a,b;\psi_1,\psi_2} - I\|_{L^p\to L^p} = 0, \quad 1 < p < \infty, \tag{1.4}$$

$$\lim_{(a,b)\to(1,0)} \|S_{a,b;\psi_1,\psi_2} - I\|_{L^1\to L^1_{weak}} = 0, \tag{1.5}$$

$$\lim_{(a,b)\to(1,0)} \|S_{a,b;\psi_1,\psi_2} - I\|_{H^1\to L^1} = 0, \tag{1.6}$$

$$\lim_{(a,b)\to(1,0)} \|S_{a,b;\psi_1,\psi_2} - I\|_{L^\infty\to BMO} = 0, \tag{1.7}$$

where I in the last three equations stands for the corresponding embedding mapping.

Remark 1.2. For the case of $d = 1$, the hypotheses (i), (ii) and (iii) turn out to be

(i) $|\psi_i(x)| \leqslant C/(1 + |x|)^\beta$,
(ii) $|\psi_i(x) - \psi_i(t)| \leqslant C|x - t|^\gamma$, and
(iii) $\int_{\mathbb{R}} \psi_i(x)\,dx = 0$.

In [23], Meyer asked that whether the affine system $\{T_{2^j k} D_{2^j} \psi\colon\ j, k \in \mathbb{Z}\}$ spans all $L^p(\mathbb{R})$ for $1 < p < \infty$, where $\psi = (1 - x^2)e^{-x^2/2}$ is the Mexican hat function. It is well known that it is the case for $p = 2$. But it is difficult to deal with other p-values.

In [16], the authors showed that under certain conditions, the affine system $\{T_{a^j bk} D_{a^j} \psi\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$ is a frame for a scale of Triebel–Lizorkin spaces (which includes Lebesgue, Sobolev and Hardy spaces) and the reproducing formula converges in norm as well as pointwise a.e. In particular, they show that the affine system spans $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$. Note that (a, b) has to be close to $(1, 0)$ in this case. So Meyer's problem was partially solved.

Now we see from (1.4) that we can get the same conclusion with arbitrary sampling points $\{(s_j, t_{j,k})\colon\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$.

Note that other spanning results have been established recently. In [1,2], Bui and Laugesen proved that for certain ψ, $\{T_k D_{A_j}\psi\colon\ j > 0,\ k \in \mathbb{Z}^d\}$ spans $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, where $\{A_j\colon\ j > 0\}$ is a sequence of expanding matrices, i.e., $\|A_j\| \to +\infty$. Note that only "half" of dilation parameters are used in this construction. And in [3,4], the same authors finally solved the Mexican hat problem.

The paper is organized as follows. In Section 2, we collect some preliminary results. And in Section 3, we give the proof of Theorem 1.1.

2. Preliminary results on Calderón–Zygmund operators

The theory of Calderón–Zygmund operators is a very useful tool in the study of singular integral operators. In this section, we reformulate some classical results on Calderón–Zygmund operators, which are used in the proof of Theorem 1.1.

There are various definitions of Calderón–Zygmund kernels. Here we follow the definition in [25].

Definition 2.1. We call $K(x, y)$ a Calderón–Zygmund kernel if there exist constants $C_K > 0$ and $0 < \delta \leqslant 1$ such that for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ with $x \neq y$, we have

$$\bigl|K(x, y)\bigr| \leqslant \frac{C_K}{|x-y|^d}, \tag{2.1}$$

$$\bigl|K(x, y) - K\bigl(x, y'\bigr)\bigr| \leqslant \frac{C_K|y-y'|^\delta}{|x-y|^{d+\delta}}, \quad \bigl|y - y'\bigr| \leqslant \frac{1}{2}|x-y|, \tag{2.2}$$

$$\bigl|K(x, y) - K\bigl(x', y\bigr)\bigr| \leqslant \frac{C_K|x-x'|^\delta}{|x-y|^{d+\delta}}, \quad \bigl|x - x'\bigr| \leqslant \frac{1}{2}|x-y|. \tag{2.3}$$

Definition 2.2. We call T a Calderón–Zygmund operator if

(i) T is a bounded operator on $L^2(\mathbb{R}^d)$,
(ii) there exists a Calderón–Zygmund kernel $K(x, y)$ such that for any compactly supported $f \in L^2(\mathbb{R}^d)$,

$$(Tf)(x) = \int_{\mathbb{R}^d} K(x, y) f(y)\,dy, \quad x \notin \text{support of } f.$$

It is well known that a Calderón–Zygmund operator is bounded from $L^1(\mathbb{R}^d)$ to the weak $L^1(\mathbb{R}^d)$. In the following we give an explicit expression of the bound, which can be proved with the standard method.

Proposition 2.3. *(See [5, Theorem 5.1.3] and [18, Theorem 8.2.1].) Let T be a Calderón–Zygmund operator with kernel satisfying* (2.1), (2.2) *and* (2.3). *Then T is a bounded operator from $L^1(\mathbb{R}^d)$ to $L^1_{weak}(\mathbb{R}^d)$. More precisely, for any $\theta > 2d^{1/2} + 1$, we have*

$$\|T\|_{L^1 \to L^1_{weak}} \leqslant 2^{d/2+2}\theta^{d/2}\|T\|_{L^2 \to L^2} + 4\theta^{-\delta} C_K C_\delta, \tag{2.4}$$

where $C_\delta = d^{\delta/2}(3/2)^{d+\delta} \int_{\mathbb{R}^d \setminus [-1,1]^d} |u|^{-d-\delta}\,du$.

Another property of Calderón–Zygmund operators which is used in this paper is that they are bounded from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. Again, we state the result without a proof.

Proposition 2.4. *Let T be a Calderón–Zygmund operator with kernel satisfying* (2.1), (2.2) *and* (2.3). *Then T is a bounded operator from $H^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ and $\|T\|_{H_1 \to L^1} \leqslant \theta^{d/2}\|T\|_{L^2 \to L^2} + \theta^{-\delta} C_K C_\delta$, $\forall \theta > 2d^{1/2} + 1$, where C_δ is the same as defined in Proposition* 2.3.

The following Marcinkiewicz interpolation theorem [32] appears in many books on harmonic analysis. Here we cite a version with an explicit estimation on the operator bound. We refer to [17, Theorem 1.3.2] and [31, p. 86] for a proof.

Proposition 2.5 *(Marcinkiewicz interpolation theorem). If an operator T satisfies the following two conditions,*

$$\|Tf\|_{L^{p_1}_{weak}} \leqslant C_1\|f\|_{L^{p_1}}, \qquad \|Tf\|_{L^{p_2}_{weak}} \leqslant C_2\|f\|_{L^{p_2}},$$

where $1 \leqslant p_1 \leqslant p_2$, then for $0 < t < 1$, $1/p = (1-t)/p_1 + t/p_2$, we have $\|Tf\|_{L^p} \leqslant M C_1^{1-t} C_2^t \|f\|_{L^p}$, where $M = 2(p/(p-p_1) + p/(p_2-p))^{1/p}$.

3. Proof of the main result

In this section, we give the proof of Theorem 1.1. First of all, we show that $S_{a,b;\psi_1,\psi_2}$ is well defined on $L^p(\mathbb{R}^d)$.

To show that a singular integral operator is well defined on $L^p(\mathbb{R}^d)$ for all $1 < p < \infty$, a standard way is to prove that it is a bounded linear operator on $L^2(\mathbb{R}^d)$ and related to some Calderón–Zygmund kernel. For $S_{a,b;\psi_1,\psi_2}$, it suffices to show that

$$K(x,y) = \frac{b^d(a^d-1)}{da^{d/2}C_{\psi_1,\psi_2}} \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^d} (T_{t_{j,k}}D_{s_j}\psi_2)(x)\overline{(T_{t_{j,k}}D_{s_j}\psi_1)(y)} \tag{3.1}$$

is a Calderón–Zygmund kernel.

The following result can be proved with the standard method for studying convergence and basis properties of orthonormal wavelets, e.g., see [8,21].

Lemma 3.1. *Let $\psi_1(x)$ and $\psi_2(x)$ be functions defined on $\mathbb{R}^d$ such that*

$$\left|\psi_i(x)\right| \leqslant \frac{C}{(1+|x|)^{d+\varepsilon}} \quad \text{and} \quad \left|\psi_i(x)-\psi_i(y)\right| \leqslant C|x-y|^{\nu}, \quad i=1,2,$$

where ν and ε are constants, $0 < \nu \leqslant 1$ and $\varepsilon > 0$. Define

$$K(x,y) = \frac{b^d(a^d-1)}{da^{d/2}C_{\psi_1,\psi_2}} \sum_{j\in\mathbb{Z},\,k\in\mathbb{Z}^d} \omega_{j,k}(T_{t_{j,k}}D_{s_j}\psi_2)(x)\overline{(T_{t_{j,k}}D_{s_j}\psi_1)(y)}, \tag{3.2}$$

where $\omega_{j,k}$ equals to -1, 0, or 1. Then $K(x,y)$ is a Calderón–Zygmund kernel, i.e., $K(x,y)$ meets (2.1), (2.2) *and* (2.3) *with constants*

$$C_K = \frac{(a^d-1)}{da^{d/2}|C_{\psi_1,\psi_2}|}\left(\left(2+a^{1/2}\right)^d C^2 C_\varepsilon\left(\frac{a^{5d/2}}{a^d-1}+\frac{2^{d+1+\varepsilon}a^{d+\varepsilon/2}}{a^\varepsilon-1}\right) + \left(1+2^{d+\nu\eta}\right)C_{\nu,\varepsilon,\eta}\cdot\left(\frac{a^{5(d+\nu\eta)/2}}{a^{d+\nu\eta}-1}+\frac{a^{(d+\nu\eta)/2+(d+\varepsilon)(1-\eta)/2}}{a^{(d+\varepsilon)(1-\eta)-(d+\nu\eta)}-1}\right)\right)$$

and $\delta = \nu\eta$, where $\eta = \varepsilon/(2(d+\nu+\varepsilon))$.

Note that for the regular case, i.e., $(s_j, t_{j,k}) = (a^j, a^j bk)$, it was shown in [16] that $S_{a,b;\psi_1,\psi_2}$ is well defined on more general Triebel–Lizorkin spaces [15,29].

Similarly to the boundedness of Calderón–Zygmund operators, the convergence of $S_{a,b;\psi_1,\psi_2}$ in $\mathcal{B}(L^2(\mathbb{R}^d))$ also implies the convergence in $\mathcal{B}(L^p(\mathbb{R}^d))$. Specifically, we have the following.

Lemma 3.2. *Let ψ_1 and ψ_2 be functions on $\mathbb{R}^d$ such that*

$$\left|\psi_i(x)\right| \leqslant \frac{C}{(1+|x|)^{d+\varepsilon}} \quad \text{and} \quad \left|\psi_i(x)-\psi_i(y)\right| \leqslant C|x-y|^{\nu}, \quad i=1,2,$$

where $0 < \nu \leqslant 1$ and $\varepsilon > 0$. Let $S_{a,b;\psi_1\psi_2}$ be defined as in (1.3)*. If $S_{a,b;\psi_1,\psi_2}$ is well defined on $L^2(\mathbb{R}^d)$ and*

$$\lim_{(a,b)\to(1,0)} \|S_{a,b;\psi_1,\psi_2} - I\|_{L^2\to L^2} = 0,$$

then $S_{a,b;\psi_1,\psi_2}$ is well defined on $L^p(\mathbb{R}^d)$ and

$$\lim_{(a,b)\to(1,0)} \|S_{a,b;\psi_1,\psi_2} - I\|_{L^p\to L^p} = 0, \quad \forall 1 < p < \infty.$$

Proof. Without loss of generality, we assume that $1 < a < 2$ and $0 < b < 1$.

Let $K(x,y)$ be defined by (3.1). By Lemma 3.1, $K(x,y)$ is a Calderón–Zygmund kernel. That is, for $x \neq y$,

$$\left|K(x,y)\right| \leqslant \frac{C_K}{|x-y|^d},$$

$$\left|K(x,y)-K\left(x,y'\right)\right| \leqslant \frac{C_K|y-y'|^\delta}{|x-y|^{d+\delta}}, \quad \left|y-y'\right| \leqslant \frac{1}{2}|x-y|,$$

$$\left|K(x,y)-K\left(x',y\right)\right| \leqslant \frac{C_K|x-x'|^\delta}{|x-y|^{d+\delta}}, \quad \left|x-x'\right| \leqslant \frac{1}{2}|x-y|,$$

where

$$C_K = \frac{(a^d-1)}{da^{d/2}|C_{\psi_1,\psi_2}|}\left(\left(2+a^{1/2}\right)^d C^2 C_\varepsilon\left(\frac{a^{5d/2}}{a^d-1}+\frac{2^{d+1+\varepsilon}a^{d+\varepsilon/2}}{a^\varepsilon-1}\right)\right.$$
$$\left.+\left(1+2^{d+\nu\eta}\right)C_{\nu,\varepsilon,\eta}\cdot\left(\frac{a^{5(d+\nu\eta)/2}}{a^{d+\nu\eta}-1}+\frac{a^{(d+\nu\eta)/2+(d+\varepsilon)(1-\eta)/2}}{a^{(d+\varepsilon)(1-\eta)-(d+\nu\eta)}-1}\right)\right),$$

$\delta=\nu\eta$ and $\eta=\varepsilon/(2(d+\nu+\varepsilon))$. It is easy to see that $C_K<\infty$. Consequently, $S_{a,b;\psi_1,\psi_2}$, and therefore, $S_{a,b;\psi_1,\psi_2}-I$, are Calderón–Zygmund operators with the same kernel. Since

$$\lim_{(a,b)\to(1,0)}\|S_{a,b;\psi_1,\psi_2}-I\|_{L^2\to L^2}=0,$$

for any $0<\zeta<1$, there exist constants a_0 and b_0 such that for any $1<a<a_0$, $0<b<b_0$, we have

$$\|S_{a,b;\psi_1,\psi_2}-I\|_{L^2\to L^2}\leqslant\zeta<1.$$

By (2.4), we have

$$\|S_{a,b;\psi_1,\psi_2}-I\|_{L^1\to L^1_{weak}}\leqslant M_\delta,$$

where $M_\delta=2^{d/2+2}\theta^{d/2}+4\theta^{-\delta}C_K C_\delta$.

By Proposition 2.5, we have

$$\|S_{a,b;\psi_1,\psi_2}f-f\|_{L^p}\leqslant MM_\delta^{2/p-1}\zeta^{2-2/p}\|f\|_{L^p},\quad 1<p<2,$$

where $M=2(p/(p-p_1)+p/(p_2-p))^{1/p}$. Hence

$$\|S_{a,b;\psi_1,\psi_2}-I\|_{L^p\to L^p}\leqslant MM_\delta^{2/p-1}\zeta^{2-2/p},\quad 1<a<a_0,\ 0<b<b_0.$$

Note that $\sup_{1<a<2,\,0<b<1}M_\delta<\infty$. We have

$$\lim_{(a,b)\to(1,0)}\|S_{a,b;\psi_1,\psi_2}-I\|_{L^p\to L^p}=0,\quad 1<p<2.$$

For the case of $2<p<\infty$, we consider the adjoint of $S_{a,b;\psi_1,\psi_2}-I$. It is easy to see that

$$\left(S^*_{a,b;\psi_1,\psi_2}g\right)(y)=\int_{\mathbb{R}^d}\overline{K(x,y)}g(x)\,dx.$$

Observe that $\overline{K(x,y)}$ is also a Calderón–Zygmund kernel. Hence $S^*_{a,b;\psi_1,\psi_2}$ is a Calderón–Zygmund operator. Since

$$\lim_{(a,b)\to(1,0)}\left\|S^*_{a,b;\psi_1,\psi_2}-I\right\|_{L^2\to L^2}=\lim_{(a,b)\to(1,0)}\|S_{a,b;\psi_1,\psi_2}-I\|_{L^2\to L^2}=0,$$

we see from the above arguments that

$$\lim_{(a,b)\to(1,0)}\left\|S^*_{a,b;\psi_1,\psi_2}-I\right\|_{L^q\to L^q}=0,\quad 1<q<2.$$

On the other hand, note that

$$\|S_{a,b;\psi_1,\psi_2}-I\|_{L^p\to L^p}=\left\|S^*_{a,b;\psi_1,\psi_2}-I\right\|_{L^q\to L^q},\quad \frac{1}{p}+\frac{1}{q}=1,$$

we have

$$\lim_{(a,b)\to(1,0)}\|S_{a,b;\psi_1,\psi_2}-I\|_{L^p\to L^p}=0,\quad 2<p<\infty.$$

This completes the proof. □

To prove the main result, it remains to show the convergence of $S_{a,b;\psi_1,\psi_2}$ in $\mathcal{B}(L^2(\mathbb{R}^d))$.

In [11], the authors introduced the Banach space $F_1(\mathbb{R}^d)$ defined by

$$F_1\left(\mathbb{R}^d\right)=\left\{f\in L^2\left(\mathbb{R}^d\right):\ W_\varphi f,W_f\varphi\in L^1(\mathcal{G})\right\},$$

where $\varphi(x) = (\partial_{x_1}^2 + \cdots + \partial_{x_d}^2)^d e^{-\pi x^2}$ is a fixed function and

$$L^p(\mathcal{G}) = \left\{\Phi:\ \|\Phi\|_{L^p(\mathcal{G})} = \left(\iint\limits_{\mathcal{G}} \left|\Phi(u,v)\right|^p \frac{du\,dv}{u^{d+1}}\right)^{1/p} < +\infty\right\}.$$

Similar to the Feichtinger's algebra S_0 studied in [10,14] for Gabor analysis, $F_1(\mathbb{R}^d)$ contains nice wavelet atoms and therefore are useful in wavelet analysis. For example, functions from $F_1(\mathbb{R}^d)$ generate wavelet frames for every well-spread time-scale sequences of the form $\{(s_j, s_j t_k):\ j,k \in \mathbb{Z}\}$. Moreover, frames generated by such functions remain frames when time-scale parameters and generating functions undergo small perturbations.

The following result shows that for wavelet functions from $F_1(\mathbb{R}^d)$, the operators $S_{a,b;\psi_1,\psi_2}$ converge to the identity operator in $\mathcal{B}(L^2(\mathbb{R}^d))$.

Proposition 3.3. *(See [22, Theorem 4.2].) Let $\psi_1, \psi_2 \in F_1(\mathbb{R}^d)$ be such that $C_{\psi_1,\psi_2} \neq 0$. Then $S_{a,b;\psi_1,\psi_2}$ is well defined on $\mathbb{R}^d$ and*

$$\lim_{(a,b)\to(1,0)} \|S_{a,b;\psi_1,\psi_2} - I\|_{L^2\to L^2} = 0.$$

Next we give a sufficient condition for $f \in F_1(\mathbb{R}^d)$, which improves a similar result in [26, Theorem 4.1].

Lemma 3.4. *Set $n_0 = \lfloor d/2 \rfloor + 1$. Let β and γ be positive constants such that $d/2 < \gamma \leqslant d$ and $1/\beta + 1/(2\gamma) < 1/d$. Suppose that $f, g \in L^1(\mathbb{R}^d)$ satisfy the following conditions,*

(i) $|(\partial^\alpha f)(x)| \leqslant C/(1+|x|)^\beta$, $|\alpha| \leqslant n_0 - 1$,
(ii) $|f(x) - \sum_{|\alpha|\leqslant n_0-1} \frac{(\partial^\alpha f)(t)}{\alpha!}(x-t)^\alpha| \leqslant C|x-t|^\gamma$, $\gamma > d/2$,
(iii) $|x|^\gamma g(x) \in L^1(\mathbb{R}^d)$,
(iv) $\int_{\mathbb{R}^d} x^\alpha g(x)\,dx = 0$ *whenever* $|\alpha| \leqslant n_0 - 1$.

Then $W_g f \in L^1(\mathcal{G})$.

Proof. First, it is easy to check that $\int_1^\infty \frac{ds}{s^{d+1}} \int_{\mathbb{R}^d} |W_g f(s,t)|\,dt < \infty$. On the other hand,

$$\begin{aligned}
\int_0^1 \frac{ds}{s^{d+1}} \int_{\mathbb{R}^d} \left|W_g f(s,t)\right| dt &= \int_0^1 \frac{ds}{s^{d+1}} \int_{\mathbb{R}^d} dt \left| \int_{\mathbb{R}^d} \left(f(x) - \sum_{|\alpha|\leqslant n_0-1} \frac{(\partial^\alpha f)(t)}{\alpha!}(x-t)^\alpha \right) s^{-d/2} \overline{g\left(\frac{x-t}{s}\right)}\, dx \right| \\
&\leqslant C^\delta \int_0^1 \frac{ds}{s^{d+1}} \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} \left| f(x) - \sum_{|\alpha|\leqslant n_0-1} \frac{(\partial^\alpha f)(t)}{\alpha!}(x-t)^\alpha \right|^{1-\delta} |x-t|^{\delta\gamma} s^{-d/2} \left| g\left(\frac{x-t}{s}\right)\right| dx \\
&\leqslant C^\delta \int_0^1 \frac{ds}{s^{d+1}} \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} \left|f(x)\right|^{1-\delta} \cdot |x-t|^{\delta\gamma} s^{-d/2} \left| g\left(\frac{x-t}{s}\right)\right| dx \\
&\quad + C^\delta \int_0^1 \frac{ds}{s^{d+1}} \int_{\mathbb{R}^d} dt \int_{\mathbb{R}^d} \left| \sum_{|\alpha|\leqslant n_0-1} \frac{(\partial^\alpha f)(t)}{\alpha!}(x-t)^\alpha \right|^{1-\delta} |x-t|^{\delta\gamma} s^{-d/2} \left| g\left(\frac{x-t}{s}\right)\right| dx \\
&= C^\delta \int_0^1 s^{\delta\gamma - d/2 - 1}\, ds \left\| |f|^{1-\delta} \right\|_1 \left\| |t|^{\delta\gamma} g(t) \right\|_1 \\
&\quad + C^\delta \sum_{|\alpha|\leqslant n_0-1} \int_0^1 s^{\delta\gamma + |\alpha|(1-\delta) - d/2 - 1}\, ds \left\| \left|\partial^\alpha f(t)\right|^{1-\delta} \right\|_1 \left\| |x|^{\delta\gamma + (1-\delta)|\alpha|} g(x) \right\|_1.
\end{aligned}$$

Since $1/\beta + 1/(2\gamma) < 1/d$, we have $(1 - d/(2\gamma))\beta > d$. Hence there exists some constant δ such that $d/(2\gamma) < \delta < 1$ and $(1-\delta)\beta > d$. Since $\delta\gamma > d/2$, it is easy to see that $\int_0^1 \frac{ds}{s^{d+1}} \int_{\mathbb{R}^d} |W_g f(s,t)|\,dt < \infty$. This completes the proof. $\square$

The following is an immediate consequence.

Theorem 3.5. *Set $n_0 = \lfloor d/2 \rfloor + 1$. Let β and γ be positive constants such that $d/2 < \gamma \leqslant d$ and $1/\beta + 1/(2\gamma) < 1/d$. Suppose that f satisfies the following conditions,*

(i) $|(\partial^\alpha f)(x)| \leqslant C/(1+|x|)^\beta$, $|\alpha| \leqslant n_0 - 1$,
(ii) $|f(x) - \sum_{|\alpha| \leqslant n_0 - 1} \frac{(\partial^\alpha f)(t)}{\alpha!}(x-t)^\alpha| \leqslant C|x-t|^\gamma$,
(iii) $\int_{\mathbb{R}^d} x^\alpha f(x)\,dx = 0$ *whenever* $|\alpha| \leqslant n_0 - 1$.

Then $f \in F_1(\mathbb{R}^d)$.

Proof. Since $d/2 < \gamma \leqslant d$ and $1/\beta + 1/(2\gamma) < 1/d$, we have $\beta > 2d$. Hence $\beta - \gamma > d$. Consequently, the hypotheses imply that $|x|^\gamma f(x) \in L^1(\mathbb{R}^d)$. Now the conclusion follows by Lemma 3.4. $\square$

We are now ready to prove the main result.

Proof of Theorem 1.1. As in the proof of Lemma 3.2, define $K(x,y)$ by (3.1). Then $S_{a,b;\psi_1,\psi_2} - I$ is a Calderón–Zygmund operator with kernel $K(x,y)$.

First, we prove (1.4). By Lemma 3.2, we only need to prove the convergence in $\mathcal{B}(L^2(\mathbb{R}^d))$, for which it suffices to prove that $\psi_1, \psi_2 \in F_1(\mathbb{R}^d)$, thanks to Proposition 3.3. On the other hand, we see from Theorem 3.5 that $\psi_1, \psi_2 \in F_1(\mathbb{R}^d)$. This completes the proof of (1.4).

Next we prove (1.5). By Proposition 2.3, we have

$$\|S_{a,b;\psi_1,\psi_2} - I\|_{L^1 \to L^1_{weak}} \leqslant 2^{d/2+2}\theta^{d/2}\|S_{a,b;\psi_1,\psi_2} - I\|_{L^2 \to L^2} + \frac{M_1}{\theta^\delta},$$

where $\theta > 2d^{1/2} + 1$ is an arbitrary constant and

$$M_1 = \sup_{1<a<2,\,0<b<1} \frac{4C_K d^{\delta/2} 3^{d+\delta}}{2^{d+\delta}} \int\limits_{\mathbb{R}^d \setminus [-1,1]^d} \frac{du}{|u|^{d+\delta}} < \infty.$$

For any $\varepsilon > 0$, we can find some $\theta_0 > 2d^{1/2} + 1$ such that

$$\frac{M_1}{\theta_0^\delta} < \frac{\varepsilon}{2}.$$

On the other hand, we see from (1.4) that there exist some $1 < a_0 < 2$ and $0 < b_0 < 1$ such that

$$\|S_{a,b;\psi_1,\psi_2} - I\|_{L^2 \to L^2} < \frac{\varepsilon}{2^{d/2+3}\theta_0^{d/2}}, \quad 1 < a < a_0,\ 0 < b < b_0.$$

Hence $\|S_{a,b;\psi_1,\psi_2} - I\|_{L^1 \to L^1_{weak}} < \varepsilon$, $1 < a < a_0$, $0 < b < b_0$. This proves (1.5).

By Proposition 2.4, (1.6) can be proved similarly to (1.5). And finally, (1.7) is a consequence of (1.6) since the dual of H^1 and L^1 are BMO and L^∞, respectively. $\square$

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